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Some Applications of Quasi-boundedness for Excessive Measures

by

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ABSTRACT

Let ξ and m be excessive measures for a right Markov process X and let Q_ξ and Q_m be the associated stationary Kuznetsov processes. We show that if ξ and m are harmonic, then $Q_\xi \ll Q_m$ if and only if ξ is quasi-bounded by m in the sense that $\xi = \sum_k \xi_k$ where each term in the sum is an excessive measure dominated by m. This result allows us to describe the Lebesgue decomposition of Q_ξ relative to Q_m and to give an explicit formula for the Radon-Nikodym derivative dQ_ξ/dQ_m in case $Q_\xi \ll Q_m$. As a second application of quasi-boundedness for excessive measures, we obtain a general form of a theorem of Ü. Kuran, in which regularity for the Dirichlet problem is characterized by the quasi-boundedness of a suitable excessive measure.

1. Introduction

Let $X = (X_t, P^x)$ be a Borel right Markov process and let ξ be an excessive measure for X. Associated with X and ξ is a stationary Markov process (Y_t, Q_ξ) with random birth and death times. This process has the same transition mechanism as X and $Q_\xi(Y_t \in \cdot\,) = \xi$ for all $t \in \mathbb{R}$. We refer to Q_ξ as the Kuznetsov measure associated with X and ξ. Let Exc denote the convex cone of excessive measures of X. The "cone map" $\xi \mapsto Q_\xi$ provides a powerful tool in the study of the potential theory of Exc.

Our goal in this paper is to explore certain aspects of the notion *quasi-bounded* as it applies to Exc. The class of quasi-bounded harmonic functions was introduced by Parreau [**P51**] in his study of Riemann surfaces. The concept was extended to superharmonic functions by Arsove and Leutwiler [**AL74**] and played a fundamental role in their development [**AL80**] of an abstract potential theory. See also Doob [**Do84**]. Following [**AL80**] we say that $\xi \in$ Exc is quasi-bounded by $m \in$ Exc provided ξ admits a series representation $\xi = \sum_n \xi_n$, where $\xi_n \in$ Exc and $\xi_n \leq m$ for all n.

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Suppose that μ and ν are initial distributions for X. It is well-known that $P^\mu \ll P^\nu$ if and only if $\mu \ll \nu$. The analogous problem for Kuznetsov measures is a bit more involved, and it turns out that quasi-boundedness plays an important role in its solution. We shall examine the absolute continuity question for a given pair of Kuznetsov measures, and when $Q_\xi \ll Q_m$ we shall exhibit an explicit expression for the Radon-Nikodym derivative dQ_ξ/dQ_m. As a corollary we obtain a Fatou-type limit theorem for the Radon-Nikodym densities of harmonic excessive measures.

We shall also explore the connection between quasi-boundedness and regularity (in the sense of the Dirichlet problem). Let D be a bounded domain in $\mathbb{R}^d$. For simplicity assume $d \geq 3$. Fix $x \in \partial D$ and let $u(y) = |y-x|^{2-d}$ be the Green potential with pole at x. Kuran **[Kr79]** has shown that x is regular for D in the sense of the Dirichlet problem if and only if the restriction of u to D is quasi-bounded by 1 on D. We will show that this regularity criterion, when stated in terms of excessive measures, holds in complete generality. The proof, which involves Kuznetsov measures, is new even in the Newtonian case studied by Kuran.

The connection between quasi-boundedness and absolute continuity is discussed in sections 2 and 3. The extension of Kuran's regularity criterion is the content of section 4. In the rest of this section we recall some of the basic facts and notation concerning Kuznetsov measures.

Throughout the paper $X = (X_t, P^x)$ will be a right Markov process with Borel semigroup (P_t) and Lusin state space $(E, \mathcal{E})$. Let $\Delta \notin E$ be the cemetery for X. It is convenient to realize X and (Y, Q_m) as coordinate processes on canonical path spaces Ω and W respectively. Let W denote the space of paths $w: \mathbb{R} \to E \cup \{\Delta\}$ that are E-valued and right continuous on an open interval $]\alpha(w), \beta(w)[\subset \mathbb{R}$, and that take the value Δ outside of this interval. (W contains the "dead" path $[\Delta]: t \to \Delta$ for which $\alpha([\Delta]) = +\infty$, $\beta([\Delta]) = -\infty$.) Let $(Y_t: t \in \mathbb{R})$ denote the coordinate process on W, with associated σ-fields

$$\mathcal{G}_t^\circ = \sigma\{Y_s: s \leq t\}, \qquad \mathcal{G}^\circ = \sigma\{Y_s: s \in \mathbb{R}\}.$$

A family of shift operators is defined on W by

$$(\theta_t w)(s) = \begin{cases} w(t+s), & s > 0, t \in \mathbb{R}, \\ \Delta, & s \leq 0, t \in \mathbb{R}. \end{cases}$$

Let $\Omega = \{w \in W: \alpha(w) = 0, \ \ Y_{\alpha+}(w) \ \text{exists in} \ E\} \cup \{[\Delta]\}$, and for $t \geq 0$ let X_t, $\mathcal{F}^\circ$, $\mathcal{F}_t^\circ$ denote the restrictions to Ω of Y_{t+}, $\mathcal{G}^\circ$, $\mathcal{G}_t^\circ$ respectively. Since (P_t) is a Borel right semigroup, there is a Borel measurable family $\{P^x, x \in E\}$ of probability measures on $(\Omega, \mathcal{F}^\circ)$ such that $X = (\Omega, \mathcal{F}^\circ, \mathcal{F}_{t+}^\circ, X_t, \theta_t, P^x)$ is a strong Markov realization of (P_t). Of course, P^x is the law of X started at x.

Recall that Exc denotes the class of excessive measures for X: $m \in \text{Exc}$ if and only if m is a σ-finite measure on E such that $mP_t \leq m$ for all $t > 0$. By our right hypotheses on (P_t) and a theorem of Kuznetsov **[Kz74]**, given $m \in \text{Exc}$ there is a unique measure Q_m on $(W, \mathcal{G}^\circ)$ such that $Q_m([\Delta]) = 0$,

$$Q_m(Y_t \in A) = m(A), \qquad \forall t \in \mathbb{R}, A \in \mathcal{E}, \tag{1.1}$$

and

$$Q_m(F \circ \theta_T \mid \mathcal{G}_{T+}^\circ) = P^{Y_T}(F), \quad Q_m\text{-a.s. on } \{\alpha < T < \beta\}, \tag{1.2}$$

for each $F \in p\mathcal{F}^\circ$ and $(\mathcal{G}^\circ_{t+})$-stopping time T. Implicit in (1.2) is the assertion that Q_m restricted to $\mathcal{G}^\circ_{T+}$ is σ-finite on $\{\alpha < T < \beta\}$.) It follows that Q_m is a σ-finite measure on $\mathcal{G}^\circ$ and that Q_m is invariant with respect to the shift operators σ_t, $t \in \mathbb{R}$, defined on W by

$$(\sigma_t w)(s) = w(t+s), \qquad s \in \mathbb{R}.$$

(As opposed to θ_t, no truncation of the path is induced by σ_t.) Let $\mathcal{G}^m_t$ denote $\mathcal{G}^\circ_t$ augmented by the null sets in the Q_m-completion $\mathcal{G}^m$ of $\mathcal{G}^\circ$. The filtration $(\mathcal{G}^m_t)_{t\in\mathbb{R}}$ is right continuous. We say that a set $A \in \mathcal{G}^m$ is *m-invariant* (and write $A \in \mathcal{I}^m$) provided $Q_m(A \Delta \sigma_t^{-1} A) = 0$ for all $t \in \mathbb{R}$.

A set $B \in \mathcal{E}$ is m-polar provided $Q_m(Y_t \in B \text{ for some } t \in \mathbb{R}) = 0$. Equivalently, $P^x(T_B < \infty) = 0$ for m a.e. $x \in E$, where T_B is the first hitting time of B.

The potential kernel of X is $U := \int_0^\infty P_t\,dt$. If μ is a measure on E then the measure μU is excessive provided it is σ-finite. Such excessive measures are called potentials, and the class of all potentials is denoted Pot. We say that $\xi \in$ Exc is harmonic (and write $\xi \in$ Har) provided ξ strongly dominates no nonzero potential. (Recall that $\eta \in$ Exc is strongly dominated by ξ if there exists $\gamma \in$ Exc such that $\eta + \gamma = \xi$. The symbol $\prec$ is used to indicate this order relation.) There is a Riesz decomposition: Exc $=$ Pot $\oplus$ Har, whose probabilistic significance is as follows. Let $m = \mu U + \gamma$ be the decomposition of $m \in$ Exc into potential and harmonic components. Then $Q_m = Q_{\mu U} + Q_\gamma$ and

(1.3) $$Q_{\mu U} = Q_m(\,\cdot\,; W_p), \qquad Q_\gamma = Q_m(\,\cdot\,; W_p^c),$$

where W_p is, in essence, the set of paths in $w \in W$ such that $\alpha(w) \in \mathbb{R}$ and the right limit $Y_{\alpha+}(w)$ exists in E in the original topology and in the Ray topology. For a precise definition of W_p see [**G90**, p. 57]. For our purposes it is enough to know that for any given strictly positive bounded Borel function h on E, W_p can be constructed so that (1.3) holds whenever $m(h) < \infty$. Moreover,

(1.4) $$W_p \in \mathcal{G}^\circ_{\alpha+} \text{ and } \sigma_t^{-1} W_p = W_p \text{ for all } t \in \mathbb{R},$$

(1.5) $$Y_{\alpha+}(w) \text{ exists in } E \text{ for all } w \in W_p,$$

(1.6) If $m(h) < \infty$ then for Q_m a.e. $w \in W$, $b_t w \in W_p$, $\forall t \in]\alpha(w), \beta(w)[$.

Here b_t is the birthing map defined by $b_t w(s) = w(s)$ if $s > t$, $= \Delta$ otherwise.

Finally, certain arguments will depend on the reverse filtration $\hat{\mathcal{G}}^\circ_t := \sigma\{Y_s : s > t\}$, its completion $\hat{\mathcal{G}}^m_t$ ($m \in$ Exc), and the associated notion of (co-)predictability. Given $m \in$ Exc, the Q_m-copredictable σ-algebra on $W \times \mathbb{R}$ is generated by the Q_m-evanescent processes together with the $(\hat{\mathcal{G}}^\circ_t)$-adapted processes that are right continuous and vanish on $[\![\beta, \infty[\![$. (Notice the time reversal implicit in this definition.) A random variable $S : W \to \mathbb{R}$ is a $(\hat{\mathcal{G}}^m_t)$-stopping time provided $S \le \beta$ and $\{S > t\} \in \hat{\mathcal{G}}^m_t$ for all $t \in \mathbb{R}$. Such a time is copredictable if and only if $]\!]-\infty, S]\!]$ is a copredictable set. For full details on these matters see [**Fi87**].

2. Quasi-boundedness and the Lebesgue decomposition of Kuznetsov measures

We adopt the convention that the letters ξ, η, γ, ρ, and m, with or without affixes, denote elements of Exc.

Recall from the introduction that ξ is quasi-bounded by m provided $\xi = \sum_n \xi_n$, where $\xi_n \leq m$ for all $n \in \mathbb{N}$. We write Qbd(m) for the elements of Exc that are quasi-bounded by m. The class Qbd(m) is a convex cone which is solid in the natural order of measures; see **[FG91b]**

The following simple observation will be used repeatedly in the sequel. Let $J \geq 0$ be $\mathcal{G}_\alpha^m \cap \mathcal{I}^m$-measurable. Suppose that the measure

$$\xi(A) := Q_m(J; Y_t \in A), \quad A \in \mathcal{E},$$

is σ-finite. (Note that ξ doesn't depend on $t \in \mathbb{R}$ since J is m-invariant.) Then $\xi \in$ Exc and

$$Q_\xi = J \cdot Q_m.$$

Conversely, we have the following

(2.1) Lemma. *If $Q_\xi \ll Q_m$, and if $J \in \mathcal{G}^m$ is any version of the Radon-Nikodym derivative dQ_ξ/dQ_m, then J is $\mathcal{G}_\alpha^m \cap \mathcal{I}^m$-measurable.*

The proof of this lemma is deferred to the end of this section. Note that we make no assertion concerning the σ-finiteness of $Q_m|\mathcal{G}_\alpha^m$ or $Q_\xi|\mathcal{G}_\alpha^m$. In fact, if m has a harmonic component then typically $Q_m|\mathcal{G}_\alpha^m$ is not a σ-finite measure.

As a simple corollary of the lemma we obtain

(2.2) Proposition. *Given $\xi, m \in$ Exc let $Q_\xi = Z \cdot Q_m + Q'$ be the Lebesgue decomposition of Q_ξ with respect to Q_m. Then both $Z \cdot Q_m$ and Q' are Kuznetsov measures. That is, there exist $\eta, \xi' \in$ Exc such that $Z \cdot Q_m = Q_\eta$ and $Q' = Q_{\xi'}$. In particular, $\xi = \eta + \xi'$.*

Proof. By Lemma (2.1) there exists $J \in \mathcal{G}_\alpha^{m+\xi} \cap \mathcal{I}^{m+\xi} \subset (\mathcal{G}_\alpha^m \cap \mathcal{I}^m) \cap (\mathcal{G}_\alpha^\xi \cap \mathcal{I}^\xi)$ with $0 \leq J \leq 1$ such that $Q_\xi = J \cdot (Q_\xi + Q_m)$. Then

$$Q_\xi = J(1-J)^{-1} 1_{\{J<1\}} Q_m + 1_{\{J=1\}} Q_\xi$$

is the Lebesgue decomposition of Q_ξ with respect to Q_m. In view of the discussion preceding Lemma (2.1), the proposition follow upon taking $Z = J(1-J)^{-1} 1_{\{J<1\}}$ and $Q' = 1_{\{J=1\}} Q_\xi$. □

From **[FG91b]** we know that Exc admits a Riesz decomposition relative to Qbd(m): Exc $=$ Qbd$(m) \oplus$ Qbd$(m)^\perp$, where Qbd$(m)^\perp$ denotes the class of excessive measures that strongly dominate no nonzero element of Qbd(m). In what follows we write $\xi \perp$ Qbd(m) to indicate $\xi \in$ Qbd$(m)^\perp$. As usual, $\mu \perp \nu$ means that the measures μ and ν are mutually singular.

(2.3) Proposition.
(i) $Q_\xi \ll Q_m \Rightarrow \xi \in$ Qbd(m).
(ii) $\xi \perp$ Qbd$(m) \Rightarrow Q_\xi \perp Q_m$.

Proof. (i) If $Q_\xi \ll Q_m$ then by Lemma (2.1) we have $Q_\xi = J \cdot Q_m$, with $J \in p\mathcal{G}_\alpha^m \cap \mathcal{I}^m$. Clearly $J < \infty$ a.e. Q_m. For $n \in \mathbb{N}$ put $J_n = J\, 1_{\{n-1 \leq J < n\}}$ so that $J_n \cdot Q_m = Q_{\xi_n}$ for some excessive measure ξ_n. Evidently $\xi_n \leq n \cdot m$ and $\xi = \sum_n \xi_n$. It follows that $\xi \in$ Qbd(m).

(ii) Let $Q_\xi = Q_\eta + Q_{\xi'}$ be the Lebesgue decomposition of Q_ξ with respect to Q_m as in Proposition (2.2). Then $Q_\eta \ll Q_m$, so $\eta \in \mathrm{Qbd}(m)$ by point (i). This forces $\eta = 0$ since $\xi = \eta + \xi'$ and $\xi \perp \mathrm{Qbd}(m)$. Consequently $Q_\xi = Q_{\xi'} \perp Q_m$. □

The reader will have no trouble producing examples demonstrating that the reverse implications in (2.3) are not valid in general. However they are valid if we insist that ξ be harmonic. A bit more generally, recall from [**FG91b**] that given $m \in \mathrm{Exc}$, an excessive measure ξ is *m-subtractive* provided $\xi \ll m$ and $[\xi \le \eta, \eta \ll m] \Rightarrow \xi \prec \eta$. We write Sub($m$) for the class of m-subtractive elements of Exc. The class Sub(m) is a convex cone which is solid in the strong order; see [**FG91b**, (2.2)]. A complete characterization of Sub(m) can be found in [**FG91b**]; for our purposes it is enough to know that Sub(m) contains all excessive measures of the form $\eta + \mu U$, where $\eta \ll m$, η is harmonic, $\mu U \ll m$, and μ is carried by an m-polar set. (Conversely, if X has no holding points, then every element of Sub(m) takes this form.)

(2.4) Proposition. *Let ξ be m-subtractive. Then*
(i) $\xi \in \mathrm{Qbd}(m) \Rightarrow Q_\xi \ll Q_m$,
(ii) $Q_\xi \perp Q_m \Rightarrow \xi \perp \mathrm{Qbd}(m)$.

Proof. (i) Since $\xi \in \mathrm{Qbd}(m)$ we can write $\xi = \sum_n \xi_n$ where $\xi_n \le m$ for all n. Clearly $\xi_n \prec \xi$, so $\xi_n \in \mathrm{Sub}(m)$; thus $\xi_n \le m$ forces $\xi_n \prec m$. It follows that $Q_{\xi_n} \le Q_m$, and this yields the assertion since $Q_\xi = \sum_n Q_{\xi_n}$.

(ii) Decompose ξ as $\eta + \gamma$, where $\eta \in \mathrm{Qbd}(m)$ and $\gamma \perp \mathrm{Qbd}(m)$. Then $\eta \in \mathrm{Sub}(m)$ since $\eta \prec \xi \in \mathrm{Sub}(m)$. Therefore by part (i) we have $Q_\eta \ll Q_m$. Since $Q_\xi = Q_\eta + Q_\gamma$, this is compatible with the hypothesis $Q_\xi \perp Q_m$ only if $Q_\eta = 0$. Thus $\eta = 0$ and $\xi = \gamma \perp \mathrm{Qbd}(m)$. □

(2.5) Corollary. *Fix $\xi \in \mathrm{Sub}(m)$. Let $\xi = \xi_q + \xi_s$ be the Riesz decomposition of ξ relative to $\mathrm{Qbd}(m)$, so that $\xi_q \in \mathrm{Qbd}(m)$ and $\xi_s \perp \mathrm{Qbd}(m)$. Let $Q_\xi = Z \cdot Q_m + Q'$ be the Lebesgue decomposition of Q_ξ relative to Q_m. Then $Z \cdot Q_m = Q_{\xi_q}$ and $Q' = Q_{\xi_s}$.*

The final result of this section is a general characterization of the absolute continuity of Kuznetsov measures. Recall the Riesz decomposition $\mathrm{Exc} = \mathrm{Pot} \oplus \mathrm{Har}$.

(2.6) Theorem. *Write $\xi = \nu U + \eta$ and $m = \mu U + \gamma$, where η and γ are harmonic. The following statements are equivalent:*
(i) $Q_\xi \ll Q_m$.
(ii) $Q_\eta \ll Q_\gamma$ *and* $Q_{\nu U} \ll Q_{\mu U}$.
(iii) $\eta \in \mathrm{Qbd}(\gamma)$ *and* $\nu \ll \mu$.

Proof. The equivalence of (i) and (ii) follows immediately from (1.3). Observe that any harmonic element of Exc is m-subtractive provided it is absolutely continuous with respect to m. Thus $Q_\eta \ll Q_\gamma$ if and only if $\eta \in \mathrm{Qbd}(\gamma)$, by (2.3) and (2.4). Moreover, by [**G90**, (6.20)],

$$\nu = Q_{\nu U}(Y_{\alpha+} \in \cdot\,; 0 < \alpha < 1, W_p), \tag{2.7}$$

and the analogous formula holds when ν is replaced by μ. Thus $Q_{\nu U} \ll Q_{\mu U}$ implies $\nu \ll \mu$. Conversely, if $\nu \ll \mu$ then $P^\nu \ll P^\mu$ which in turn implies $Q_{\nu U} \ll Q_{\mu U}$ because of the formula

$$Q_\xi(F; W_p) = Q_{\nu U}(F) = \int_{\mathbb{R}} P^\nu(1_\Omega \cdot F \circ \sigma_t)\, dt, \qquad F \in \mathcal{G}^\circ, \tag{2.8}$$

and its companion for $Q_{\mu U}$, which follow on combining (6.11) and (6.19) in **[G90]**. □

Proof of Lemma (2.1).

Let ξ and m be given such that $Q_\xi \ll Q_m$, and let J be any $\mathcal{G}^m$-measurable version of dQ_ξ/dQ_m. Then $J = J\circ\sigma_t$ a.s. Q_m for all $t \in \mathbb{R}$ because of the (σ_t)-invariance of Q_ξ and Q_m. So we need only show that J is $\mathcal{G}^m_\alpha$-measurable. Fix $t \in \mathbb{R}$ and a strictly positive Borel function g on E such that $\xi(g) < \infty$. Then

$$Q_m(J\,g(Y_t)) = Q_\xi(g(Y_t)) = \xi(g) < \infty.$$

Notice that the trace σ-algebra $\mathcal{G}^\circ \cap \{\alpha < t < \beta\}$ is generated by products of the form $1_{\{\alpha<t<\beta\}}F \cdot G\circ\theta_t$, where $F \in p\mathcal{G}^\circ_t$ and $G \in p\mathcal{F}^\circ$. Moreover, by the Markov property of Q_ξ,

$$\begin{aligned} Q_m(Jg(Y_t)F \cdot G\circ\theta_t) &= Q_\xi(g(Y_t)F \cdot G\circ\theta_t) \\ &= Q_\xi(g(Y_t)F \cdot P^{Y_t}(G)) \\ &= Q_m(Jg(Y_t)F \cdot P^{Y_t}(G)). \end{aligned}$$

Since $\{g(Y_t) > 0\} = \{\alpha < t < \beta\}$, it follows that

$$J = Q_m(J|\mathcal{G}^\circ_t \cap \{\alpha < t < \beta\}), \quad \text{a.s. } Q_m \text{ on } \{\alpha < t < \beta\},$$

so $J|\{\alpha < t < \beta\}$ is $\mathcal{G}^m_t \cap \{\alpha < t < \beta\}$-measurable. But

$$1_{\{\alpha<t\}}J = 1_{\{\alpha<t\}} \limsup_{q\downarrow t, q \text{ rational}} 1_{\{\alpha<q<\beta\}}J + 1_{\{\beta\le t\}}J,$$

and this is $\mathcal{G}^m_t$-measurable because $1_{\{\alpha<q<\beta\}}J \in \mathcal{G}^m_q$, $\mathcal{G}^m_{t+} = \mathcal{G}^m_t$, and $\mathcal{G}^m \cap \{\beta \le t\} = \mathcal{G}^m_t \cap \{\beta \le t\}$. Since $\mathcal{G}^m_{\alpha+} = \mathcal{G}^m_\alpha$, it follows that $J \in \mathcal{G}^m_\alpha$ as claimed. □

3. Radon-Nikodym derivative; Fatou theorem

Our object in this section is to compute the Radon-Nikodym derivative dQ_ξ/dQ_m in case $Q_\xi \ll Q_m$. Before stating the result we need some additional notation. Suppose that $\eta \ll \gamma$. By **[FG91a**, (2.15)**]** we know that there is a Borel version $u \ge 0$ of the Radon-Nikodym derivative $d\eta/d\gamma$ such that a.e. Q_γ, $t \mapsto u(Y_t)$ is finite-valued and right continuous on $]\alpha, \beta[$ and has a finite right limit at $t = \alpha$.

Given ξ and m, let $\xi = \eta + \nu U$ and $m = \gamma + \mu U$ be the Riesz decompositions of ξ and m into harmonic and potential components. If $Q_\xi \ll Q_m$, then by (2.6) we have $Q_\eta \ll Q_\gamma$ and $\nu \ll \mu$. In particular, $\eta \ll \gamma$. Let u be the "fine" version of the density $d\eta/d\gamma$ described in the last paragraph, and let f be any Borel version of the Radon-Nikodym derivative $d\nu/d\mu$.

(3.1) Theorem. *If $Q_\xi \ll Q_m$, then a version J of the Radon-Nikodym derivative dQ_ξ/dQ_m is given a.e. Q_m by the formula*

$$J := \begin{cases} \lim_{t\downarrow\alpha} u(Y_t), & \text{on } W_p^c, \\ f(Y_{\alpha+}), & \text{on } W_p, \end{cases} \tag{3.2}$$

the existence of the limit Q_m-a.e. on W_p^c being part of the assertion.

Because of (1.3), the second case in (3.2) follows from (2.8): if $F \in p\mathcal{G}^\circ$ then

$$\begin{aligned} Q_\xi(F; W_p) = Q_{\nu U}(F) &= \int_{\mathbb{R}} P^\nu(1_\Omega \cdot F\circ\sigma_t)\,dt \\ &= \int_{\mathbb{R}} P^\mu(f(X_0)1_\Omega \cdot F\circ\sigma_t)\,dt \\ &= \int_{\mathbb{R}} P^\mu(1_\Omega \cdot (f(Y_{\alpha+})F)\circ\sigma_t)\,dt \\ &= Q_{\mu U}(f(Y_{\alpha+})F) = Q_m(f(Y_{\alpha+})F; W_p), \end{aligned}$$

as required.

The rest of this section is devoted to proving the first case of (3.2).

(3.3) Lemma. *If m is harmonic then α is Q_m-copredictable.*

Remark. By adapting an argument of Azéma [**A72**, pp. 480–81] it can be shown that α is Q_m-copredictable if and only if $m = \mu U + \gamma$ where μ is carried by an m-polar set and γ is harmonic.

Proof of (3.3). By [**Fi87**, (6.3),(6.41)] (with $\xi = m$) there is an optional copredictable homogeneous random measure κ carried by $[\alpha, \beta[\cap \mathbb{R}$ such that

$$Q_m(F) = Q_m \int_{\mathbb{R}} F\circ b_t\,\kappa(dt) + Q_m(F; \alpha = -\infty), \quad F \in p\mathcal{G}^m. \tag{3.4}$$

Let $\ell = (\ell_t)_{t\in\mathbb{R}}$ denote the copredictable projection (relative to Q_m) of the process $1_{]\!]\alpha,\beta[\![}$. Then by [**Fi87**, pp. 437–39], ℓ is Q_m-optional,

$$]\!]\alpha, \beta[\![\subset \{\ell > 0\} \subset [\![\alpha, \beta[\![$$

up to Q_m evanescence, and for Q_m a.e. w and all $t \in \mathbb{R}$

$$\ell_t(w) > 0 \Rightarrow b_t w \in W_p.$$

The Q_m-copredictability of ℓ entails that $\ell_t\circ b_t = \ell_t$ for all $t < \beta$ a.e. Q_m. Thus, by (3.4) and the assumption that m is harmonic,

$$\begin{aligned} Q_m(\ell_\alpha > 0, \alpha > -\infty) &= Q_m(\ell_\alpha > 0, \alpha > -\infty; W_p^c) \\ &= Q_m \int_{\mathbb{R}} 1_{\{\ell>0\}}(t) 1_{W_p^c}(b_t)\,\kappa(dt) = 0. \end{aligned} \tag{3.5}$$

But the $(\hat{\mathcal{G}}_t^m)$-stopping time

$$S = \begin{cases} \alpha, & \text{on } \{\ell_\alpha = 0, \alpha \in \mathbb{R}\}, \\ -\infty, & \text{otherwise,} \end{cases}$$

is a Q_m-copredictable time since

$$]\!] -\infty, S]\!] =]\!] -\infty, S[\![\cup ([\![\alpha,\beta[\![\cap \{\ell = 0\})$$

is evidently Q_m-copredictable. By (3.5), $\alpha = S$ a.e. Q_m, and the lemma is proved. □

Proof of (3.1). In view of the earlier discussion we need only prove the first part of (3.2). Because of (1.3) we have $dQ_\xi/dQ_m = dQ_\eta/dQ_\gamma$ on W_p^c, which set carries both Q_η and Q_γ. Therefore it suffices to prove that the Radon-Nikodym derivative $J = dQ_\eta/dQ_\gamma$ is equal to $\lim_{t\downarrow\alpha} u(Y_t)$, where u is the "fine" version of the density $d\eta/d\gamma$ described before the statement of (3.1). Of course, both the asserted equality and the existence of the limit hold only a.e. $Q_\gamma = 1_{W_p^c} Q_m$. Recall from Lemma (2.1) that J is $\mathcal{G}_\alpha^\gamma \cap \mathcal{I}^\gamma$-measurable.

Let $g > 0$ be a Borel function on E such that $\eta(g) < \infty$ and $\gamma(g) = 1$, and define a probability measure $P := g(Y_0) \cdot Q_\gamma$ on $(W, \mathcal{G}^\circ)$. Then $P(J) < \infty$ and P is carried by $\{\alpha < 0 < \beta\}$. We claim that

$$(3.6) \qquad P(J|\hat{\mathcal{G}}_t^\circ) = u(Y_t), \qquad \text{a.s. } P \text{ on } \{\alpha < t < \beta\}, \quad \forall t < 0.$$

In fact if $H \in p\hat{\mathcal{G}}_t^\circ$ then

$$Q_\eta(H; \alpha < t) = Q_\gamma(H \cdot u(Y_t); \alpha < t).$$

Consequently, if $t < 0$, then

$$\begin{aligned} P(JH; \alpha < t < \beta) &= Q_\gamma(g(Y_0)JH; \alpha < t < \beta) \\ &= Q_\eta(g(Y_0)H; \alpha < t < \beta) \\ &= Q_\gamma(g(Y_0)H\, u(Y_t)) = P(H\, u(Y_t); \alpha < t < \beta). \end{aligned}$$

This proves (3.6). Now γ is harmonic so by Lemma (3.3) there is a decreasing sequence (S_n) of countably-valued $(\hat{\mathcal{G}}_t^\gamma)$-stopping times such that $\alpha < S_n$ for all n and $S_n \downarrow \alpha$ a.s. P. Since the S_n are countably-valued, (3.6) remains valid with t replaced by S_n. Passing to the limit we obtain

$$(3.7) \qquad \lim_{t\downarrow\alpha} u(Y_t) = \lim_n u(Y_{S_n}) = P(J| \vee_n \hat{\mathcal{G}}_{S_n}^\circ),$$

a.s. P, hence a.e. Q_γ on $\{\alpha < 0 < \beta\}$. But $\vee_n \hat{\mathcal{G}}_{S_n}^\circ = \mathcal{G}^\circ$ up to Q_γ-null sets, so

$$(3.8) \qquad J = \lim_{t\downarrow\alpha} u(Y_t), \qquad \text{a.s. } Q_\gamma \text{ on } \{\alpha < 0 < \beta\}.$$

But both sides of (3.8) are γ-invariant, so by composing with σ_q (q rational) we see that the equality in (3.8) holds a.e. Q_γ on $\cup_q\{\alpha < q < \beta\}$, hence a.s. Q_γ. □

Remark. The limit in (3.8) coincides a.e. Q_γ with the essential limit inferior $J' := \operatorname{ess\,lim\,inf}_{t\downarrow\alpha} u(Y_t)$ which has the advantage of being $\mathcal{G}_{\alpha+}^\circ$-measurable and perfectly invariant: $J'(\sigma_t w) = J'(w)$ for all $t \in \mathbb{R}$ and $w \in W$; cf. [**FG91a**, (2.7)]. (The "essential" here refers to Lebesgue measure on $\mathbb{R}$.) Substituting this essential limit inferior for the limit in (3.2), we see that if $Q_\xi \ll Q_m$ then the derivative dQ_ξ/dQ_m can always be taken

$\mathcal{G}^\circ_{\alpha+}$-measurable and perfectly invariant, since the other ingredients in (3.2) already enjoy these properties.

We can make the first part of (3.2) look more like a Fatou-type theorem by recalling Dynkin's decomposition [**Dy**] of a given excessive measure into minimal components. Let us fix a bounded strictly positive function $q \in \mathcal{E}$ such that $\xi(q) + m(q) < \infty$. Recall that γ is *minimal* provided the only strong minorants (in Exc) of γ are constant multiples of γ. Let M denote the class of minimal elements γ such that $\gamma(q) = 1$, and give M the σ-algebra $\mathcal{M}$ generated by the maps $\gamma \mapsto \gamma(f)$, $f \in p\mathcal{E}$. By [**Dy**], there is a uniquely determined measure π_ξ on $(M, \mathcal{M})$ such that

$$(3.9) \qquad Q_\xi(F) = \int_M Q_\gamma(F)\,\pi_\xi(d\gamma), \quad F \in \mathcal{G}^\circ.$$

In particular, $\pi_\xi(M) = \xi(q) < \infty$. Of course the analog of (3.9) holds with ξ replaced by m.

(3.10) Lemma. $Q_\xi \ll Q_m \iff \pi_\xi \ll \pi_m$.

Proof. The implication $\Leftarrow$ is clear. Conversely, suppose that $Q_\xi \ll Q_m$. By the Remark above there is a $\mathcal{G}^\circ_{\alpha+}$-measurable version J of dQ_ξ/dQ_m which is invariant (i.e., $J(\sigma_t w) = J(w)$ identically). It follows from the discussion just before Lemma (2.1) that if $\gamma \in M$ then Q_γ is trivial on $\mathcal{G}^\circ_{\alpha+}$-measurable invariant sets. Thus there is an $\mathcal{M}$-measurable map $\psi : M \to [0,\infty]$ such that $J = \psi(\gamma)$ a.e. Q_γ for all $\gamma \in M$; indeed $\psi(\gamma) = Q_\gamma(J \cdot q(Y_0))$. Then for $F \in \mathcal{G}^\circ$,

$$(3.11) \qquad Q_\xi(F) = Q_m(JF) = \int_M Q_\gamma(JF)\,\pi_m(d\gamma) = \int_M Q_\gamma(F)\psi(\gamma)\,\pi_m(d\gamma).$$

The uniqueness of π_ξ now forces $\pi_\xi(d\gamma) = \psi(\gamma)\pi_m(d\gamma)$. □

Referring again to [**Dy**], there is a $\mathcal{G}^\circ$-measurable, invariant map $\Gamma : W \to M$ such that $\Gamma \in \mathcal{G}^\xi_\alpha/\mathcal{M}$ and the representing measure π_ξ is the image of Q_ξ under Γ. Combining (3.1) and (3.10) we arrive at the following

(3.12) Proposition. *Suppose $m \in$ Har and $Q_\xi \ll Q_m$. Then*

$$\lim_{t \downarrow \alpha} \frac{d\xi}{dm}(Y_t) = \frac{d\pi_\xi}{d\pi_m}(\Gamma), \qquad \text{a.e. } Q_m,$$

where $d\xi/dm$ is understood to be the "fine" version of the Radon-Nikodym density.

When m is dissipative the random element Γ can be identified with a right limit $Y^r_{\alpha+}$, the superscript r indicating that the limit is taken in a certain Ray-Knight compactification of E. This compactification, which amounts to the construction of a Martin entrance boundary, is dicussed in detail in [**GG83,GG84**]. A closely related limit interpretation of Γ can be found in [**Dy**].

Time reversal provides a link between the results of the last two sections and the theory of Föllmer measures for supermartingales, as developed in [**Fo, AJ, AF**]. Suppose $\xi \ll m$ and let u be the "fine" version of the density $d\xi/dm$ discussed above (3.1). Then under Q_m the process $(u(Y_t))_{\alpha<t<\beta}$ is a right continuous supermartingale relative to the reverse filtration $(\hat{\mathcal{G}}_t^m)_{t\in\mathbb{R}}$ in the sense that for all $s > t$,

$$Q_m(u(Y_t)|\hat{\mathcal{G}}_s) \le u(Y_s), \quad \text{a.e. } Q_m \text{ on } \{\alpha < s < \beta\}.$$

By martingale theory the left continuous modification

$$Z_t := \lim_{s\uparrow t} u(Y_s), \quad t < \beta,$$

is likewise a supermartingale relative to the filtration $(\hat{\mathcal{G}}_{t-}^m)$, where $\hat{\mathcal{G}}_{t-}^m = \cap_{s<t}\hat{\mathcal{G}}_s^m$. It can be shown that if S is a Q_m "co-optional" time and if $F \in p\hat{\mathcal{G}}_{S-}^m$ then

$$Q_m(FZ_S; -\infty < S < \beta) = Q_\xi(F; -\infty < S < \beta).$$

In other words Q_ξ is the Föllmer measure associated with the supermartingale $(Z_t, \hat{\mathcal{G}}_{t-}^m, Q_m)$.

4. Quasi-boundedness and regularity

In this section we prove a general version of the regularity criterion of Kuran mentioned in the introduction.

Throughout this section we work with a finely open Borel set D. Let $T = \inf\{t > 0 : X_t \notin D\}$ denote the exit time from D, and let $\text{reg}(D^c) := \{x \in E : P^x(T = 0) = 1\}$ denote the corresponding set of regular points. The natural state space for the process (X, T)—X killed at time T— is $E_D := E \setminus \text{reg}(D^c)$. Clearly $D \subset E_D$, and $E_D \setminus D = D^c \setminus \text{reg}(D^c)$ is semipolar for X and polar for (X, T). Thus we can restrict (X, T) to D to obtain a right process which we denote X^D. Now X^D need not be a Borel right process, but it does satisfy the hypothesis (6.2) in [**G90**] which ensures the existence of Kuznetsov measures associated with X^D. Writing $\text{Exc}(X^D)$ for the class of excessive measures of X^D, we note that if $\xi \in \text{Exc}$ then $\xi(E_D \setminus D) = 0$ and $\xi|_D$ is an element of $\text{Exc}(X^D)$. We shall write $\text{Qbd}(m|_D)$ for the class of X^D-excessive measures that are quasi-bounded by $m|_D$.

Here is our extension of Kuran's theorem. We fix $m \in \text{Exc}$ and $x \in E$ such that $\epsilon_x U$ is σ-finite, hence excessive for X. If $m = \gamma + \mu U$ is the Riesz decomposition of m into harmonic and potential parts, then $\mathcal{N}(m)$ denotes the class of sets $B \in \mathcal{E}$ that are both m-polar and μ-null.

(4.1) Theorem. *(i) Assume $x \in D^c$ and $\{x\} \in \mathcal{N}(m)$. If $\epsilon_x U|_D \in \text{Qbd}(m|_D)$, then $x \in \text{reg}(D^c)$.*
(ii) Conversely, suppose $x \in E$ is such that $\epsilon_x U \ll m$. If $x \in \text{reg}(D^c)$ then $\epsilon_x U|_D \in \text{Qbd}(m|_D)$.

Proof. (i) If $\rho \in \text{Exc}(X^D)$ then Q_ρ^D denotes the Kuznetsov measure associated with X^D and ρ. When $\rho = \xi|_D$ for some $\xi \in \text{Exc}$ we write Q_ξ^D instead of $Q_{\xi|D}^D$. We can (and do) assume that Q_ρ^D has been constructed on the sample space W used in previous sections. In case $\xi \in \text{Exc}$ the paper [**G88**] contains a useful relationship between Q_ξ and Q_ξ^D which forms the basis of our argument. Set $\tau := \inf\{t > \alpha : Y_t \notin D\}$, and note that $\tau|_\Omega = T$. Let G denote the set of left endpoints, strictly between α and β, of the intervals contiguous to the right closed (in $]\alpha, \beta[$) random set $\{t \in]\alpha, \beta[: Y_t \notin D\}$, together with α if $\alpha < \tau$.

Note that $\tau\circ b_t = \inf\{s > t : Y_s \notin D\}$ on $\{\alpha \le t\}$. Combining Theorems (4.9) and (5.11) in [**G88**] we have

$$Q^D_\xi(F) = Q_\xi \sum_{t\in G} F\circ k_\tau \circ b_t, \qquad F \in p\mathcal{G}^\circ. \tag{4.2}$$

Here k_t is the usual killing operator: $k_t w(s) = w(s)$ if $s < t$, $= \Delta$ otherwise.

We abbreviate $\lambda = \epsilon_x U$ and $\pi = \epsilon_x U|_D$. Let V denote the potential kernel of X^D and let $\eta + \nu V$ be the Riesz decomposition of π into harmonic and potential components taken relative to X^D. By hypothesis $\pi \in \mathrm{Qbd}(m|_D)$, hence η and νV are also quasi-bounded by $m|_D$. In particular, $Q^D_\eta \ll Q^D_m$ by (2.4)(i).

Recall that $x \in D^c$. We are going to show that if $x \notin \mathrm{reg}(D^c)$ then

$$Q^D_\eta(Y_{\alpha+} = x, \alpha < \tau; W_p) = \infty \tag{4.3}$$

while

$$Q^D_m(Y_{\alpha+} = x, \alpha < \tau; W_p) = 0. \tag{4.4}$$

This will contradict $Q^D_\eta \ll Q^D_m$, allowing us to conclude that x must lie in $\mathrm{reg}(D^c)$ after all.

If $\alpha \in G$ then $\alpha < \tau$ and consequently

$$\{\alpha \in G\} \cap (k_\tau \circ b_\alpha)^{-1}[\{Y_{\alpha+} = x, \alpha < \tau\} \cap W_p] = \{Y_{\alpha+} = x, \alpha \in G\} \cap W_p.$$

Thus, using (4.2) with $\xi = \lambda = \epsilon_x U$ and dropping all terms but the one corresponding to $t = \alpha$, we have

$$Q^D_\pi(Y_{\alpha+} = x, \alpha < \tau; W_p) \ge Q_\lambda(Y_{\alpha+} = x, \alpha \in G; W_p) \tag{4.5}$$

By (2.8) the second term in (4.5) equals

$$Q_\lambda(Y_{\alpha+} = x, \alpha < \tau) = \int_{\mathbb{R}} P^x(T > 0)\, dt = \int_{\mathbb{R}} 1\, dt = \infty, \tag{4.6}$$

since $x \notin \mathrm{reg}(D^c)$. Also, by (2.8) applied to X^D,

$$Q^D_{\nu V}(Y_{\alpha+} = x) = \int_{\mathbb{R}} P^\nu(X_0 = x, T > 0)\, dt = 0, \tag{4.7}$$

since ν is carried by D and $x \in D^c$. Combining this with (4.5) and (4.6) we see that (4.3) holds. On the other hand, by by another application of (4.2),

$$\begin{aligned} Q^D_m&(Y_{\alpha+} = x, \alpha < \tau, \alpha \in G; W_p) \\ &\le Q_m \sum_{t\in G} 1_{\{Y_{t+}=x\}} 1_{W_p}(b_t) \\ &\le Q_{\mu U}(Y_{\alpha+} = x) + Q_m \sum_{t\in G, t>\alpha} 1_{\{Y_t = x\}}, \end{aligned} \tag{4.8}$$

which vanishes since $\{x\} \in \mathcal{N}(m)$ by hypothesis. This yields (4.4) and the desired contradiction.

(ii) We will prove that $Q^D_\lambda \ll Q^D_m$, which implies the stated result because of (2.3)(i). Let $(A, \hat{P}^y)$ be an exit system for X relative to the homogeneous optional set $\{t > 0 : X_t \notin D\}$. Thus A is an additive functional of X, and $(\hat{P}^y; y \in E)$ is a measurable family of σ-finite measures on $(\Omega, \mathcal{F}^\circ)$ such that for any $\rho \in \mathrm{Exc}$

$$Q_\rho \sum_{t \in G, t > \alpha} F_t \circ \theta_t = \int_{\mathbb{R}} dt \int_E \nu^\rho_A(dy) \hat{P}^y(F_t), \tag{4.9}$$

for all $\mathcal{B}(\mathbb{R}) \otimes \mathcal{F}^\circ$-measurable maps $(t, \omega) \mapsto F_t(\omega)$ of $\mathbb{R} \times \Omega$ into $[0, \infty]$. Here ν^ρ_A, the characteristic measure of A relative to ρ, is defined by

$$\nu^\rho_A(f) = \lim_{t \downarrow 0} t^{-1} P^\rho \int_0^t f(X_s)\, dA_s. \tag{4.10}$$

See [**FM86**, §6] or [**G90**, §11] for details. Given $F \in \mathcal{G}^\circ$ let $\psi(t, y) = \hat{P}^y(((F \circ \sigma_{-t})|_\Omega) \circ k_T)$. Using (2.8) we see that the hypothesis $x \in \mathrm{reg}(D^c)$ implies that $Q_\lambda(\alpha \in G) = Q_\lambda(\alpha < \tau) = 0$. Thus by (4.2), (4.9), and the identities $b_t = \sigma_{-t} \circ \theta_t$ and $k_\tau \circ \sigma_{-t} = \sigma_{-t} \circ k_\tau$,

$$\begin{aligned} Q^D_\lambda(F) &= Q_\lambda \sum_{t \in G} F \circ k_\tau \circ b_t = Q_\lambda \sum_{t \in G, t > \alpha} F \circ k_\tau \circ \sigma_{-t} \circ \theta_t \\ &= Q_\lambda \sum_{t \in G, t > \alpha} ((F \circ \sigma_{-t})|_\Omega) \circ k_T \circ \theta_t = \int_{\mathbb{R}} dt \int_E \nu^\lambda_A(dy)\, \psi(t, y). \end{aligned} \tag{4.11}$$

A similar computation yields

$$Q^D_m(F) \geq \int_{\mathbb{R}} dt \int_E \nu^m_A(dy)\, \psi(t, y). \tag{4.12}$$

But it follows easily from the definition (4.10) of characteristic measure that $\nu^\lambda_A \ll \nu^m_A$ since $\lambda = \epsilon_x U \ll m$ by hypothesis. Putting this together with (4.11) and (4.12) we see that $Q^D_\lambda \ll Q^D_m$, as desired. □

Remarks. (a) Let X be uniform motion to the right on the real line, and take $m = \epsilon_0 U$ and $D =]0, 1[$. Then $\epsilon_0 U|_D = m|_D$ is trivially quasi-bounded by $m|_D$, but $0 \notin \mathrm{reg}(D^c)$. This shows that the hypothesis $\{x\} \in \mathcal{N}(m)$ cannot, in general, be eliminated from (4.1)(i), nor can it be replaced by the weaker condition "$\{x\}$ is m-polar." On the other hand "$\{x\} \in \mathcal{N}(m)$" is not necessary for the validity of (4.1)(ii). For example, $1 \in \mathrm{reg}(D^c)$ and $\epsilon_1 U|_D = 0$, but $\{1\} \notin \mathcal{N}(m)$.

(b) The side conditions imposed in Theorem (4.1) are certainly satisfied in the Newtonian context, and in this context there is a natural isomorphism between excessive measures and positive superharmonic functions. Consequently Theorem (4.1) contains Kuran's criterion as a special case.

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