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A complete differential formalism for stochastic calculus in manifolds

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This essay will contain no surprises for the experienced practitioner of stochastic calculus in manifolds, but may be found useful by others wishing to do certain calculations, relating say to Brownian motion on a Riemannian manifold. We take the view that the stochastic calculus has two main roles – to construct new processes from old and to determine martingales. The formalism of Itô and Stratonovich differentials for real-valued processes provides a flexible, complete and highly effective way to handle these two roles. We know of no account of a similarly effective formalism for processes taking their values in a differentiable manifold. But certainly everything we do below, based as it is on the notion of a horizontal lift, is at least implicit in the quite large literature on stochastic calculus in manifolds. One of our main contentions is that, in dealing with processes in manifolds, one cannot and should not try to avoid differentials of processes in vector bundles. For example, if one wants to form a line integral along a semimartingale in a manifold, the natural integrand is a process in the cotangent space and one knows from the real case that the difference between Stratonovich and Itô integrals involves the differential of the integrand. Such differentials may be handled effectively by horizontal lift, but we have made it a principle to make the horizontal lift disappear from our formulae whenever possible. This is done by using covariant stochastic differentials. Our second main contention is that, once these covariant differentials are introduced, the formalism becomes as flexible and as complete as the real-valued case.

We begin by reviewing the basic elements of stochastic calculus in $\mathbf{R}$, to fix notation and make it clear what we are extending to manifolds. Then we show how the notions of semimartingale and Stratonovich differential extend to manifolds. Next we consider a vector bundle with connection and the associated notions of horizontal lift of a semimartingale and parallel translation. Section 4 specializes to the case where a connection is given on the tangent bundle of our manifold: this permits the definition of Itô differentials, we introduce a Doob-Meyer decomposition and show it transforms under a change of measure by the familiar Girsanov formula. So far, everything is well known and is covered in greater depth in Emery's book [Em]; in particular we refer to [Em] for the existence of the stochastic development.

Beginning in Section 5, we introduce notions of covariant stochastic differential for semimartingales in a vector bundle with connection. These notions may also be found in Elworthy's book [El]. The calculus of these differentials, which Elworthy does not pursue, is remarkably simple, perhaps trivial, but consequently easy to use and illuminating in applications. The fundamental Stratonovich to Itô conversion formulae are given in Section 6. In Section 7 we discuss covariant stochastic differential equations over a given semimartingale in a manifold. These equations arise naturally when one considers smooth variations of the base semimartingale in a parameter, as for example in the study of stochastic flows or the Malliavin calculus. Also, changes of connection are seen to correspond to a particular class of linear covariant equations, and this leads to simple formulae relating parallel translation and covariant differentials corresponding to two different connections.

The final three sections present applications of the differential formalism. In Section 8 we derive a generalized Feynman-Kač formula for heat semigroups on sections of a vector bundle. The fact that we can work with general connections allows a unification of previous results: in particular we include the Cameron-Martin formula. The ability to switch connections readily in the differential formalism is exploited in a technical lemma. This application is pursued in [N]. In Sections 9 and 10 we give simple 'stochastic calculus' proofs of some known results on the effect of mappings on martingales and on Brownian motion, including a factorization result of Elworthy and Kendall.

1. Review of stochastic calculus in R

We work on a probability space $(\Omega, \mathcal{F}, \mathbf{P})$ equipped with a filtration $(\mathcal{F}_t)_{t\geq 0}$ satisfying the usual conditions. We shall consider only *continuous* semimartingales, which from now on are called simply semimartingales, continuity being understood. The same goes for martingales and processes of finite variation.

Let x_t be a semimartingale, y_t a continuous adapted process and let σ, τ be random times with $\sigma \leq \tau$. Then, as $N \to \infty$,

$$\sum_{k=[2^N\sigma]}^{[2^N\tau]-1} y\left(\frac{k}{2^N}\right)\left(x\left(\frac{k+1}{2^N}\right) - x\left(\frac{k}{2^N}\right)\right) \to \int_\sigma^\tau y_s dx_s.$$

The convergence is in probability, in particular a subsequence converges almost surely. The limit is the *Itô integral.* The natural class of integrands for the Itô integral is in fact larger, including all locally bounded previsible process, but the Riemann sum approximation fails in general. Let

$$z_t = z_0 + \int_0^t y_s dx_s,$$

then z_t is itself a semimartingale. We write

$$dz_t = y_t dx_t.$$

Every semimartingale x_t has a unique *Doob-Meyer decomposition*

$$x_t = x_0 + x_t^m + x_t^f,$$

where x_t^m is a local martingale starting from 0, and x_t^f is a process of (locally) finite variation starting from 0. It is sometimes convenient to write the decomposition in differential notation

$$dx_t = dx_t^m + dx_t^f.$$

The Itô integral respects the Doob-Meyer decomposition: z_t has decomposition

$$dz_t = y_t dx_t^m + y_t dx_t^f.$$

This is the principal merit of the Itô integral.

Suppose now that y_t is a semimartingale; then

$$\sum_{k=[2^N\sigma]}^{[2^N\tau]-1} \frac{1}{2}\left(y\left(\frac{k}{2^N}\right) + y\left(\frac{k+1}{2^N}\right)\right)\left(x\left(\frac{k+1}{2^N}\right) - x\left(\frac{k}{2^N}\right)\right) \to \int_\sigma^\tau y_s \partial x_s.$$

This limit is the *Stratonovich integral.* Also

$$\sum_{k=[2^N\sigma]}^{[2^N\tau]-1} \left(y\left(\frac{k}{2^N}\right) - y\left(\frac{k+1}{2^N}\right)\right)\left(x\left(\frac{k+1}{2^N}\right) - x\left(\frac{k}{2^N}\right)\right) \to \int_\sigma^\tau \partial x_s \partial y_s.$$

We call this limit the *quadratic integral.* Clearly

$$\int_\sigma^\tau y_s \partial x_s = \int_\sigma^\tau y_s dx_s + \frac{1}{2}\int_\sigma^\tau \partial x_s \partial y_s. \tag{1}$$

If either x_t or y_t are of finite variation, then, by the Cauchy-Schwarz inequality, the quadratic integral vanishes and the Itô and Stratonovich integrals agree. In general the process

$$q_t = \int_0^t \partial x_s \partial y_s$$

is of finite variation. We write

$$\partial q_t = \partial x_t \partial y_t.$$

Equivalently

$$dq_t = dx_t dy_t.$$

For semimartingales x_t, x'_t and previsible processes y_t, y'_t it *is* true that

$$(y_t dx_t)(y'_t dx'_t) = (y_t y'_t) dx_t dx'_t. \tag{2}$$

The Doob-Meyer decomposition changes under an absolutely continuous change of probability measure according to the *Girsanov formula.* Suppose that $\tilde{\mathbf{P}}$ is absolutely continuous with respect to $\mathbf{P}$ on every $\mathcal{F}_t$ with continuous density martingale ρ_t. Then, in an obvious notation,

$$dx_t^{m,\tilde{\mathbf{P}}} = dx_t^{m,\mathbf{P}} - \frac{\partial x_t \partial \rho_t}{\rho_t}\,. \tag{3}$$

A process x_t in $\mathbf{R}^n$ is a semimartingale if all its components are semimartingales. For a function on $\mathbf{R}^n$ we have the *chain rule*

$$\partial(f(x_t)) = df(\partial x_t),$$

that is to say, for all random times σ and τ with $\sigma \leq \tau$,

$$f(x_\tau) = f(x_\sigma) + \int_\sigma^\tau \frac{\partial f}{\partial x^i}(x_s)\partial x_s^i.$$

This is the principal merit of the Stratonovich integral.

We often use stochastic differential equations to define and analyse semimartingales. Given a semimartingale x_t we may consider the *Stratonovich equation*

$$\partial y_t = V(y_t)\partial x_t.$$

If V is smooth, then given a stopping time σ and an $\mathcal{F}_\sigma$-measurable initial value y_σ, there is a unique solution up to explosion. That is to say there is a stopping time $\zeta > \sigma$ and a semimartingale $(y_t : \sigma \leq t < \zeta)$ such that, for all stopping times τ with $\sigma \leq \tau < \zeta$,

$$y_\tau - y_\sigma = \int_\sigma^\tau V(y_s)\partial x_s,$$

and, as $t \uparrow \zeta$, y_t leaves all compact sets. If V is linear then $\zeta = \infty$ almost surely. If we write $y_t = y_{t\sigma}(y_\sigma)$, $\zeta = \zeta_\sigma(y_\sigma)$ to indicate the dependence on the starting time and place, then, for $\sigma \leq \tau \leq t < \zeta_\sigma(y_\sigma)$ we have almost surely

$$\begin{aligned} \zeta_\tau(y_{\tau\sigma}(y_\sigma)) &= \zeta_\sigma(y_\sigma), \\ y_{t\tau}(y_{\tau\sigma}(y_\sigma)) &= y_{t\sigma}(y_\sigma). \end{aligned} \tag{4}$$

All this carries over to the *Itô equation*

$$dy_t = V(y_t)dx_t$$

and this equation has the additional property that, if V has bounded derivative, then $\zeta = \infty$ almost surely.

2. Stochastic calculus in a differentiable manifold: semimartingales and Stratonovich differentials

The notions of semimartingale and Stratonovich differential in $\mathbf{R}^n$ are invariant under diffeomorphism, so they extend to a differentiable manifold.

Let x_t be a semimartingale in $\mathbf{R}^n$ and let f be a function on $\mathbf{R}^n$; then $f(x_t)$ is again a semimartingale and the chain rule applies

$$\partial(f(x_t)) = df(\partial x_t).$$

Let M be a (metrizable) differentiable manifold and let x_t be a process with values in M. We say that x_t is a *semimartingale* if $f(x_t)$ is a semimartingale in $\mathbf{R}$ for all smooth real-valued functions f on M.

The chain rule shows that the Stratonovich differential transforms under a change of coordinates as a tangent vector. This suggests that we define the *Stratonovich differential* of a semimartingale x_t in M by

$$(\partial x_t)^i = \partial(x_t^i), \tag{5}$$

where $x = (x^1, \ldots, x^n)$ is any chart around x_t. From this symbolic definition we deduce the definitions of *bona fide* mathematical objects – the Stratonovich and quadratic integrals. Let α_t be a semimartingale in T^*M over x_t, that is, such that $\pi\alpha_t = x_t$. Let β_t be a previsible process in $T^*M \otimes T^*M$ over x_t. Fix a countable atlas of coordinate charts for M, with domains $D_1, D_2, \ldots$. There is a sequence of random times

$$0 = \tau_0 < \tau_1 < \cdots < \tau_k \to \infty$$

such that, for all k and ω, there is an ℓ such that $x_t(\omega) \in D_\ell$ whilst $\tau_k(\omega) \leq t \leq \tau_{k+1}(\omega)$. For random times σ and τ with $\tau_k \leq \sigma \leq \tau \leq \tau_{k+1}$ we define the *Stratonovich* and *quadratic integrals* by

$$\int_\sigma^\tau \alpha_s(\partial x_s) = \int_\sigma^\tau (\alpha_s)_i \partial(x_s^i),$$
$$\int_\sigma^\tau \beta_s(\partial x_s, \partial x_s) = \int_\sigma^\tau (\beta_s)_{ij} \partial(x_s^i)\partial(x_s^j),$$

where coordinates are taken in the chart on D_ℓ for any suitable ℓ, it makes no difference which. The obvious extensions of these integrals to general σ and τ do not depend on the sequence of random times, nor the atlas used.

The discrete approximations to the Stratonovich and quadratic integrals on $\mathbf{R}$ give us discrete approximations, converging in probability, to the integrals on M. This is useful for simulation. It shows also that $(\int_\sigma^\tau \alpha_s(\partial x_s))(\omega)$ depends only on $(\alpha_s(\omega) : \sigma(\omega) \leq s \leq$

$\tau(\omega)$) and similarly for the quadratic integral. It shows moreover, because of the time symmetry of the discrete approximations, that if α_t happens to be a semimartingale in some reverse-time filtration then the reverse-time integral is the same as the original.

The above argument with random times τ_k shows also that the following local differential formulae do correspond to identities between stochastic integrals. Firstly there is a natural extension of (2):

$$\alpha_t(\partial x_t)\alpha_t(\partial x_t) = (\alpha_t)_i\partial(x_t^i)(\alpha_t)_j\partial(x_t^j) = (\alpha_t)_i(\alpha_t)_j\partial(x_t^i)\partial(x_t^j) = (\alpha_t\otimes\alpha_t)(\partial x_t, \partial x_t). \quad (6)$$

Secondly, the chain rule extends to semimartingales in M:

$$\partial(f(x_t)) = \frac{\partial f}{\partial x^i}(x_t)\partial(x_t^i) = df(\partial x_t). \quad (7)$$

We can consider stochastic differential equations on manifolds. Suppose given a semimartingale x_t in M and a smooth section V of the bundle $TN \otimes T^*M$ over $N \times M$, where N is another manifold. Then a semimartingale y_t in N is a solution of the stochastic differential equation

$$\partial y_t = V(y_t, x_t)\partial x_t \quad (8)$$

if, for all 1-forms α on N, we have

$$\alpha(\partial y_t) = (\alpha \circ V)(y_t, x_t)\partial x_t.$$

It is an obvious guess that, given any starting point $y_0 \in N$, there is a unique solution to (8) up to explosion. Given global charts for M and N, this is immediate from the result in $\mathbf{R}^n$. It is also true in general: see Emery [Em].

3. Horizontal lift of a semimartingale to a principal fibre bundle, parallel translation

Let F be a vector bundle over M with fibres isomorphic to a finite-dimensional vector space E. Let U be the principal bundle $GL(E, F)$ and let ∇ be a connection on F.

Let x_t be a semimartingale in M. We will show that, *for each initial frame $u_0 \in U$ over x_0, there is a unique semimartingale u_t in U over x_t such that, for all smooth sections f of F*

$$\partial(u_t^{-1}f(x_t)) = u_t^{-1}\nabla f(\partial x_t). \quad (9)$$

The semimartingale u_t is called the *horizontal lift of x_t starting from u_0*.

Fix a countable atlas of local trivializations of U. There is a sequence of stopping times

$$0 = \tau_0 < \tau_1 < \ldots < \tau_k \to \infty$$

such that, for all k, x_t remains within the domain of one trivialization between τ_k and τ_{k+1}. Condition on whatever trivialization this may be. Suppose we are given a stopping time σ with $\tau_k \leq \sigma < \tau_{k+1}$ and some $\mathcal{F}_\sigma$-measurable $u_\sigma \in U$ over x_σ. In the trivialization, $\nabla = D + A$ for some $A \in \Gamma(\operatorname{End} E \otimes T^*M)$, so letting f run through a basis for E in (9) we obtain a linear equation in $\operatorname{End} E$

$$\partial(u_t^{-1}) = u_t^{-1} A(\partial x_t). \tag{10}$$

Equivalently

$$\partial(u_t) = -A(\partial x_t) u_t. \tag{11}$$

This equation has a unique solution starting from u_σ at time σ, which remains in $GL(E)$ up to time τ_{k+1}. Moving out of the trivialization we have shown there is a unique semimartingale u_t in $GL(E,F)$ starting at u_σ at time σ, satisfying (9) up to τ_{k+1}. Now piece together a solution to (9) for all time in the obvious way, using (4) to check that this depends neither on the atlas nor on the stopping times τ_k.

If we write $u_{ts}(u_s)$ for the solution of (9) starting from u_s at time s (as in (4)), then by linearity of equation (11)

$$\tau_{ts} := u_{ts}(u_s) u_s^{-1}, \quad t \geq s$$

does not depend on u_s. For $t \leq s$ we set $\tau_{ts} = \tau_{st}^{-1}$. The map $\tau_{ts} : F_{x_s} \to F_{x_t}$ is called *parallel translation*.

Once we have a linear isomorphism $u : E \to F$, we get isomorphisms $E^* \to F^*$, End $E \to$ End F in an obvious way. We denote these all by u; thus for $V \in$ End F we have the possible confusion $u^{-1}V = u^{-1}Vu$: on the left u^{-1} acts as an isomorphism to End E, the right side is a composition of linear maps.

4. Stochastic calculus in a manifold with connection, stochastic development, Itô differentials, Doob-Meyer decomposition and Girsanov formula

Let M be a manifold with connection ∇, then we can apply the preceding to the principal bundle $GL(M)$ of linear frames in TM. Let x_t be a semimartingale in M and let u_t be its horizontal lift in $GL(M)$ starting from u_0. We can define a semimartingale in $\mathbf{R}^n$ by the equation

$$\partial \bar{x}_t = u_t^{-1} \partial x_t. \tag{12}$$

Then x_t satisfies

$$\partial x_t = u_t \partial \bar{x}_t. \tag{13}$$

Conversely, suppose we are given a semimartingale $\bar{x}_t$ in $\mathbf{R}^n$. We admit the following well-known result (see [Em], Theorem 8.30): *for each initial frame* $u_0 \in GL(M)$ *there is a unique semimartingale* u_t *in* $GL(M)$ *defined up to explosion time* ζ *which is the horizontal lift of a semimartingale* $x_t, t < \zeta$, *satisfying* (13). We call x_t the *stochastic development of* $\bar{x}_t$ *starting from* x_0.

Define the *Itô differential* of x_t by

$$dx_t = u_t d\bar{x}_t. \tag{14}$$

This definition is, like (5), symbolic. From (14) we deduce the definition of the *Itô integral*: for a locally bounded previsible process α_t in T^*M over x_t define

$$\int_\sigma^\tau \alpha_s(dx_s) = \int_\sigma^\tau \alpha_s(u_s d\bar{x}_s). \tag{15}$$

The Doob-Meyer decomposition extends to semimartingales in M, but only at the level of differentials, not at the level of processes. Suppose $\bar{x}_t$ has Doob-Meyer decomposition

$$d\bar{x}_t = d\bar{x}_t^m + d\bar{x}_t^f.$$

Define the *martingale* and *finite variation differentials* of x_t by

$$d^m x_t = u_t d\bar{x}_t^m, \qquad d^f x_t = u_t d\bar{x}_t^f. \tag{16}$$

These are again symbolic definitions. The differentials exist as objects against which one can integrate a locally bounded previsible process α_t in T^*M over x_t:

$$\int_\sigma^\tau \alpha_s(d^m x_s) = \int_\sigma^\tau \alpha_s(u_s d\bar{x}_s^m).$$

(In fact the martingale differential does not depend on the choice of connection.) We have a *Doob-Meyer decomposition*

$$dx_t = d^m x_t + d^f x_t. \tag{17}$$

If $d^f x_t = 0$ we call x_t a *martingale*. It is clear that if $d^m x_t = 0$, then x_t has finite variation. The Itô integral respects the Doob-Meyer decomposition. If

$$dz_t = \alpha_t(dx_t)$$

then z_t has Doob-Meyer decomposition

$$dz_t = \alpha_t(d^m x_t) + \alpha_t(d^f x_t).$$

The Doob-Meyer decomposition changes under an absolutely continuous change of probability measure according to the *Girsanov formula.* If $\tilde{\mathbf{P}}$ is absolutely continuous with respect to $\mathbf{P}$ on each $\mathcal{F}_t$ with continuous density martingale ρ_t, then, in an obvious notation,

$$d^{m,\tilde{\mathbf{P}}} x_t = d^{m,\mathbf{P}} x_t - \frac{\partial x_t \partial \rho_t}{\rho_t} \, . \tag{18}$$

5. Covariant stochastic calculus in a vector bundle: Stratonovich and Itô differentials, Doob-Meyer decomposition and Girsanov formula

We return to the general set-up of Section 3. Let v_t be a semimartingale in F over x_t. Recall that u_t is a horizontal lift of x_t. We define the *covariant Stratonovich differential*

$$Dv_t = u_t \partial (u_t^{-1} v_t). \tag{19}$$

Thus Dv_t is the vertical projection of the total Stratonovich differential ∂v_t. This is a symbolic definition by which we are led to define various covariant Stratonovich and quadratic integrals. Suppose V_t is a semimartingale in $F \otimes T^*M$ over x_t, then

$$Dv_t = V_t(\partial x_t)$$

means

$$\partial(u_t^{-1} v_t) = u_t^{-1} V_t(\partial x_t),$$

or in full

$$u_\tau^{-1} v_\tau = u_\sigma^{-1} v_\sigma + \int_\sigma^\tau u_s^{-1} V_s(\partial x_s).$$

For a semimartingale W_t in $F \otimes F^*$, and for locally bounded previsible processes A_t in $F \otimes T^*M \otimes T^*M$, B_t in $F \otimes F^* \otimes T^*M$ and $C_t \in F \otimes F^* \otimes F^*$, all over x_t, it will now be obvious how to interpret the following equations of differentials

$$\begin{aligned} Dw_t &= W_t(Dv_t), \\ Da_t &= A_t(\partial x_t, \partial x_t), \\ Db_t &= B_t(Dv_t, \partial x_t), \\ Dc_t &= C_t(Dv_t, Dv_t). \end{aligned}$$

Covariant quadratic integrals give rise to processes of finite covariant variation.

There is an extension of the chain rule. Let $f \in C^\infty(F, F)$. (This notation implies $\pi \circ f = \pi$.) Then, by the chain rule (7),

$$u_t \partial(u_t^{-1} f(v_t)) = u_t \partial(\bar{f}(u_t, \bar{v}_t)) = u_t \{ d\bar{f}(\partial u_t, \bar{v}_t) + D\bar{f}(u_t, \bar{v}_t) \partial \bar{v}_t \}$$

where $\bar{f}(u,\bar{v}) = u^{-1}f(u\bar{v})$ and $\bar{v}_t = u_t^{-1}v_t$, where $d\bar{f}$ denotes the differential as a function of u and $D\bar{f}$ the differential as a function of $\bar{v}$. But ∂u_t is the horizontal lift of ∂x_t, so we get the *covariant chain rule*

$$D(f(v_t)) = \nabla f(v_t)\partial x_t + Df(v_t)Dv_t, \tag{20}$$

where Df is the derivative in the fibre.

We note three special cases. For a section f of F we get

$$D(f(x_t)) = \nabla f(\partial x_t) \tag{21}$$

and for semimartingales φ_t in F^* and v_t in F over x_t we have a product rule

$$D(\varphi_t(v_t)) = D\varphi_t(v_t) + \varphi_t(Dv_t). \tag{22}$$

If there is a metric on F with which ∇ is compatible, then for semimartingales v_t and w_t in F over x_t

$$\partial\langle v_t, w_t\rangle = \langle Dv_t, w_t\rangle + \langle v_t, Dw_t\rangle;$$

in particular for $e \in E$,

$$\partial|u_t e|^2 = 2\langle D(u_t e), u_t e\rangle = 0,$$

showing that, if u_0 is an isometry, then the horizontal lift remains an isometry for all time.

The notion of parallel translation is invariant under time reversal. By this we mean that if $(x_s : 0 \leq s \leq t)$ happens to be a semimartingale for some reverse-time filtration and if we write $\hat{\tau}_{0t}$ for the corresponding reverse-time parallel translation map $F_{x_t} \to F_{x_0}$, then

$$\hat{\tau}_{0t} = \tau_{0t}.$$

To see this, recall that $(\tau_{0s} : 0 \leq s \leq t)$ is characterized by the fact $\tau_{00} = \mathrm{id}$, together with the covariant chain rule:

$$\tau_{0t}f(x_t) - f(x_0) = \int_0^t (\tau_{0s}\nabla f)(\partial x_s), \qquad f \in \Gamma(F).$$

Discrete approximation shows that

$$\tau_{t0}\int_0^t (\tau_{0s}\nabla f)(\partial x_s) = \int_0^t (\tau_{ts}\nabla f)(\hat{\partial} x_s),$$

the right hand side being a reverse-time Stratonovich integral. Hence, for all $f \in \Gamma(F)$,

$$f(x_t) - \tau_{t0}f(x_0) = \int_0^t (\tau_{ts}\nabla f)(\hat{\partial} x_s),$$

showing that $\hat{\tau}_{t0} = \tau_{t0}$ as claimed.

We define the *covariant Itô differential* of a semimartingale v_t in F over x_t by

$$\mathsf{D}v_t = u_t d(u_t^{-1} v_t). \tag{23}$$

Thus the covariant Itô equation

$$\mathsf{D}w_t = V_t(\mathsf{D}v_t),$$

where V_t is a previsible process in $F \otimes F^*$ over x_t, is to be interpreted as

$$u_\tau^{-1} w_\tau = u_\sigma^{-1} w_\sigma + \int_\sigma^\tau u_s^{-1} V_s(u_s d(u_s^{-1} v_s)).$$

There is a covariant Doob-Meyer decomposition for semimartingales v_t in F over x_t. Suppose $\bar{v}_t = u_t^{-1} v_t$ has Doob-Meyer decomposition $\bar{v}_t = \bar{v}_0 + \bar{v}_t^m + \bar{v}_t^f$. Set $v_t^m = u_t \bar{v}_t^m$ and $v_t^f = u_t \bar{v}_t^f$; then

$$\mathsf{D}v_t = \mathsf{D}v_t^m + \mathsf{D}v_t^f \tag{24}$$

and we call this the *covariant Doob-Meyer decomposition* of v_t. If $v_t^f = 0$ we call v_t a *local covariant martingale* and if $v_t^m = 0$ we say v_t has *finite covariant variation.*

The covariant Itô integral respects the covariant Doob-Meyer decomposition: if

$$\mathsf{D}w_t = V_t(\mathsf{D}v_t),$$

then w_t has decomposition

$$\mathsf{D}w_t = V_t(\mathsf{D}v_t^m) + V_t(\mathsf{D}v_t^f).$$

There is a covariant *Girsanov formula.* Suppose $\tilde{\mathsf{P}}$ is absolutely continuous with respect to P on each $\mathcal{F}_t$ with density martingale ρ_t. Then

$$\mathsf{D}v_t^{m,\tilde{\mathsf{P}}} = \mathsf{D}v_t^{m,\mathsf{P}} - \frac{Dv_t \partial \rho_t}{\rho_t}\,. \tag{25}$$

6. Covariant stochastic calculus in a vector bundle continued: Stratonovich to Itô conversion and the Itô formula

The set-up remains the same as for the last section, except now we insist that F has the form $TM \oplus F'$ and that ∇ respects this direct sum. Thus we include also the set-up of Section 4.

We need to tidy away one more Itô integral: for a semimartingale x_t in M and a locally bounded previsible process V_t in $F \otimes T^*M$ over x_t, there is the *covariant Itô integral*

$$\mathbf{D}w_t = V_t(dx_t).$$

According to (14) and (23), this means

$$u_\tau^{-1}v_\tau = u_\sigma^{-1}v_\sigma + \int_\sigma^\tau u_s^{-1}V_s(u_s d(u_s^{-1}x_s)).$$

As usual, this respects the Doob-Meyer decomposition and w_t has decomposition

$$\mathbf{D}w_t = V_t(d^m x_t) + V_t(d^f x_t).$$

Up to this point, two sorts of stochastic calculus on manifold, Stratonovich and Itô, have been developed in parallel. We have seen that the Stratonovich calculus is well adapted to the geometry in that it obeys the chain rule (7). On the other hand, the Itô calculus is well adapted to the probability in that it preserves the Doob-Meyer decomposition. To get a useful theory the geometry and probability must be tied together: we need to know how to move between Stratonovich and Itô. Thus the central result in stochastic calculus is the *Stratonovich to Itô conversion formula.* For semimartingales x_t in M, v_t in F, V_t in $F \otimes T^*M$ and W_t in $F \otimes F^*$, all over x_t, we have

$$V_t(\partial x_t) = V_t(dx_t) + \tfrac{1}{2}DV_t(\partial x_t), \tag{26}$$

$$W_t(Dv_t) = W_t(\mathbf{D}v_t) + \tfrac{1}{2}DW_t(Dv_t). \tag{27}$$

We deduce (26) from (1):

$$\begin{aligned} u_t^{-1}V_t(\partial x_t) &= (u_t^{-1}V_t)\partial \bar{x}_t \\ &= (u_t^{-1}V_t)d\bar{x}_t + \tfrac{1}{2}\partial(u_t^{-1}V_t)\partial\bar{x}_t \\ &= u_t^{-1}V_t(dx_t) + \tfrac{1}{2}u_t^{-1}DV_t(\partial x_t). \end{aligned}$$

The proof of (27) is similar. Notice that, even in the simplest case where α_t is a semimartingale in T^*M over x_t we have

$$\alpha_t(\partial x_t) = \alpha_t(dx_t) + \tfrac{1}{2}D\alpha_t(\partial x_t).$$

Without covariant stochastic differentials, this central formula cannot be expressed in such a simple way.

We make two deductions from the conversion formula: the first is the covariant Itô formula and the second the conversion rule for stochastic differential equations.

Let $f \in C^\infty(F,F)$ and let v_t be a semimartingale in F over x_t. We already have the covariant chain rule (20)

$$D(f(v_t)) = \nabla f(v_t)\partial x_t + Df(v_t)Dv_t,$$

where Df is the derivative in the fibre

$$\mathbf{D}(f(v_t)) = \nabla f(v_t)dx_t + Df(v_t)\mathbf{D}v_t + \tfrac{1}{2}D(\nabla f(v_t))\partial x_t + \tfrac{1}{2}D(Df(v_t))Dv_t.$$

Now apply the chain rule to $D(\nabla f(v_t))$ and $D(Df(v_t))$ to obtain the *covariant Itô formula*

$$\begin{aligned}\mathbf{D}(f(v_t)) &= \nabla f(v_t)dx_t + Df(v_t)\mathbf{D}v_t \\ &\quad + \tfrac{1}{2}\nabla^2 f(v_t)(\partial x_t, \partial x_t) + \nabla Df(v_t)(Dv_t, \partial x_t) + \tfrac{1}{2}D^2 f(v_t)(Dv_t, Dv_t). \qquad (28)\end{aligned}$$

We note two special cases. If f is a section of F, then

$$\mathbf{D}(f(x_t)) = \nabla f(dx_t) + \tfrac{1}{2}\nabla^2 f(\partial x_t, \partial x_t). \qquad (29)$$

If v_t in F and φ_t in F^* are semimartingales over x_t, then

$$d(\varphi_t(v_t)) = \mathbf{D}\varphi_t(v_t) + \varphi_t(\mathbf{D}v_t) + D\varphi_t(Dv_t). \qquad (30)$$

7. Covariant stochastic differential equations

Suppose we are given semimartingales x_t in M and v_t in F over x_t, also, coefficients $V \in C^\infty(F, F \otimes T^*M)$ and $W \in C^\infty(F, F \otimes F^*)$. The covariant stochastic differential equation

$$Dw_t = V(w_t)\partial x_t + W(w_t)Dv_t \qquad (31)$$

is simply a concise way of writing the ordinary stochastic differential equation

$$\partial \bar{w}_t = \bar{V}(u_t, \bar{w}_t)\partial \bar{x}_t + \bar{W}(u_t, \bar{w}_t)\partial \bar{v}_t,$$

where we have chosen some $u_0 \in U$ over x_0, u_t is the horizontal lift of x_t starting from u_0, $\bar{w}_t = u_t^{-1}w_t$, $\bar{V}(u,\bar{w}) = u^{-1}V(u\bar{w})u$, $\partial\bar{x}_t = u_t^{-1}\partial x_t$, $\bar{W}(u,\bar{v}) = u^{-1}W(u\bar{w})u$ and $\bar{v}_t = u_t^{-1}v_t$. There is therefore no separate theory of covariant stochastic differential equations! If V and W are smooth then there is a unique solution up to explosion.

If V and W are affine in v_t, then $\bar{V}$ and $\bar{W}$ will be affine in $\bar{v}_t$, so there will be no explosion. Another case where we can prevent explosion is when there is a metric on F compatible with ∇; then, as we saw in Section 5, the horizontal lift remains bounded, so the covariant Itô equation

$$\mathbf{D}w_t = V(w_t)dx_t + W(w_t)\mathbf{D}v_t$$

does not explode provided the fibre derivatives DV and DW are bounded. We shall see that this case in fact provides a criterion of non-explosion for the general Stratonovich equation (31).

One can always transform a covariant Stratonovich equation into a covariant Itô equation by the Stratonovich to Itô conversion rule (26), (27). Equation (31) becomes

$$\mathrm{D}w_t = V(w_t)dx_t + W(w_t)\mathrm{D}v_t + \tfrac{1}{2}D(V(w_t))\partial x_t + \tfrac{1}{2}D(W(w_t))Dv_t.$$

By the covariant chain rule

$$\begin{aligned} D(V(w_t)) &= \nabla V(w_t)\partial x_t + DV(w_t)Dw_t \\ &= \nabla V(w_t)\partial x_t + DV(w_t)(V(v_t)\partial x_t + W(w_t)Dw_t). \end{aligned}$$

Hence the Itô form of (31) is

$$\begin{aligned} Dw_t &= V(w_t)dx_t + W(w_t)\mathrm{D}v_t \\ &\quad + \tfrac{1}{2}(\nabla V(w_t) + DV(w_t)V(w_t))(\partial x_t, \partial x_t) + \tfrac{1}{2}DV(w_t)W(w_t)(Dv_t, \partial x_t) \\ &\quad + \tfrac{1}{2}(\nabla W(w_t) + DW(w_t)V(w_t))(\partial x_t, Dv_t) + \tfrac{1}{2}DW(w_t)W(w_t)(Dv_t, Dv_t). \quad (32) \end{aligned}$$

For $A \in \Gamma(\mathrm{End}\, F \otimes T^*M)$ consider now the *linear equation* in End F over x_t

$$Dz_t = -A(\partial x_t)z_t.$$

Write $z_t^s, t \geq s$, for the unique solution starting from the identity at time s. Notice that z_t^{-1} satisfies

$$D(z_t^{-1}) = (z_t^{-1})A(\partial x_t)$$

so z_t^{-1} cannot explode, that is, z_t remains invertible. Set $\zeta_{st} = \tau_{st}(z_t^s)^{-1}$. By the covariant chain rule, for $f \in \Gamma(F)$,

$$D(z_t^{-1}f(x_t)) = z_t^{-1}(\nabla + A)f(\partial x_t),$$

that is

$$\partial(\zeta_{ot}f(x_t)) = \zeta_{0t}(\nabla + A)f(\partial x_t).$$

This shows that ζ_{st} is simply the parallel translation map corresponding to the new connection $\nabla + A$. Moreover, if D^A is the covariant Stratonovich differential corresponding to $\nabla + A$, then

$$\zeta_{t0}D^A v_t = \partial(\zeta_{0t}v_t) = \partial(\tau_{0t}z_t^{-1}v_t) = \tau_{t0}D(z_t^{-1}v_t) = \tau_{t0}z_t^{-1}(D + A(\partial x_t))v_t$$

so that

$$D^A = D + A(\partial x_t).$$

Given a vector bundle F over a compact manifold we can always construct a metric on F and a connection ∇^0 compatible with the metric. (See for example [DFN], Lemma 25.1.4.) Any given connection ∇ may then be written as $\nabla^0 + A$ for some $A \in \Gamma(\mathrm{End}\, F \otimes F^*M)$. Thus the covariant Stratonovich equation (31) may be written

$$(D^0 + A(\partial x_t))w_t = V(w_t)\partial x_t + W(w_t)(D^0 + A(\partial x_t))v_t.$$

This may then be written in Itô form (see (32)) so as to obtain a criterion for the non-explosion of w_t. In certain cases one also gets L^p estimates this way (see the Lemma below).

8. Probabilistic interpretation of heat semigroups

Let M be a compact Riemannian manifold and let F be a vector bundle over M. There is a natural class of linear second order differential operators on the space of smooth sections $\Gamma(F)$: call $\mathcal{L}$ a *Laplacian* if in any (and hence every) coordinate system

$$\mathcal{L} - \tfrac{1}{2}g^{ij}(x)\frac{\partial^2}{\partial x^i \partial x^j}$$

is a first order differential operator (where $g^{ij}(x) = \langle dx^i, dx^j\rangle$, the inverse of the metric tensor).

There is a more constructive way of describing the class of Laplacians. Let ∇ be any connection on F and let $V \in \Gamma(\mathrm{End}\ F)$; write ∇ also for the Levi-Civita connection on T^*M and for the product of these connections on $F \otimes T^*M$, then

$$\mathcal{L} = \tfrac{1}{2} tr \nabla^2 + V \tag{33}$$

is a Laplacian, and moreover any Laplacian can be written in this form.

There are three ingredients to the decomposition (33), each of which may be chosen arbitrarily: the metric g defining the trace, the connection ∇ and the potential V. To each of these three there corresponds naturally a stochastic process and together these processes may be used to write down a path integral formula for the heat semigroup $e^{t\mathcal{L}}$. The process corresponding to the metric is Brownian motion. Given a starting point $x_0 \in M$, this is the unique semimartingale x_t with values in M such that

x_t is a martingale

and for a section b of $T^*M \otimes T^*M$

$$b(\partial x_t, \partial x_t) = trb(x_t)\partial t.$$

The process corresponding to the connection is the parallel translation $\tau_{0t} : F_{x_t} \to F_{x_0}$ along x_s, $0 \le s \le t$, characterized by

$$\partial(\tau_{ot} f(x_t)) = \tau_{ot} \nabla f(x_t).$$

Finally define a stochastic exponential e_t in End S over x_t by the linear covariant equation

$$De_t = e_t V(x_t)\partial t, \qquad e_0 = id.$$

The following result is a generalized Feynman-Kač formula.

Theorem

For all $f \in \Gamma(f)$ we have

$$(e^{t\mathcal{L}} f)(x_0) = \mathsf{E}[\tau_{0t} e_t f(x_t)]. \tag{34}$$

Proof. Write $P_t = e^{t\mathcal{L}}$. Fix $T > 0$ and $f \in \Gamma(F)$. We compute the covariant Itô differential

$$\begin{aligned} \mathsf{D}(e_t P_{T-t} f(x_t)) &= (\mathsf{D}e_t) P_{T-t} f(x_t) + e_t \mathsf{D}(P_{T-t} f(x_t)) \\ &= e_t V(x_t) P_{T-t} f(x_t) dt - e_t \mathcal{L} P_{T-t} f(x_t) dt \\ &\quad + e_t \left\{ \nabla P_{T-t} f(dx_t) + \tfrac{1}{2} \nabla^2 P_{T-t} f(\partial x_t, \partial x_t) \right\} \\ &= e_t \nabla P_{T-t} f(dx_t). \end{aligned}$$

This shows that $M_t = \tau_{0t} e_t P_{T-t} f(x_t)$ is a local martingale. The following lemma implies that $\sup_{t \le T} |M_t|$ is integrable, so M_t is a martingale and $\mathsf{E}(M_T) = \mathsf{E}(M_0)$, proving the theorem. □

Lemma

Let $U \in \Gamma(\text{End } F \otimes T^*M)$ and $V \in \Gamma(\text{End } F)$. Then for any $z_0 \in F_{x_0}$, the covariant linear stochastic differential equation in F over x_t

$$Dz_t = (U(x_t)\partial x_t + V(x_t)\partial t) z_t$$

has a unique solution starting from z_0. Moreover, for any metric on F we have

$$\mathsf{E}\left(\sup_{0 \le s \le t} |z_s|^p \right) < \infty$$

for all $1 \le p < \infty$ and $0 \le t < \infty$.

Proof. Existence and uniqueness of the solution are standard. Fix a metric on F and an inner product on E. We deal first with the case where the connection ∇ is compatible with the metric on F. Let u_0 be a linear isometry from E to F_{x_0} and let u_t be the horizontal lift of x_t in $GL(E,F)$ starting from u_0. Then u_t remains an isometry for all $0 \leq t < \infty$. Set $\bar{z}_t = u_t^{-1} z_t$ and $\partial \bar{x}_t = u_t^{-1} \partial x_t$, then $|\bar{z}_t| = |z_t|$ and $\bar{x}_t$ is a Brownian motion in $\mathbf{R}^n$. Moreover $\bar{z}_t$ satisfies the stochastic differential equation in E

$$\partial \bar{z}_t = (\bar{U}(u_t)\partial \bar{x}_t + \bar{V}(u_t)\partial t)\bar{z}_t, \quad \bar{z}_0 = u_0^{-1} z_0,$$

where $\bar{U}(u) = u^{-1}U(x)$ and $\bar{V}(u) = u^{-1}V(x)$. By compactness, $\bar{U}$ and $\bar{V}$ are bounded on $O(E,F)$, so by the usual combination of Burkhölder-Davis-Gundy inequalities and Gronwall's Lemma,

$$\mathsf{E}\left(\sup_{0\leq s\leq t} |\bar{z}_s|^p\right) < \infty$$

for all $0 \leq t < \infty$ and $1 \leq p < \infty$, as required. Now any connection on F may be written as $\nabla + A$ where ∇ is compatible with the metric and $A \in \Gamma(\text{End } F \otimes T^*M)$. Write D^A for the covariant Stratonovich differential corresponding to $\nabla + A$. Then

$$D^A z_t = (U(x_t)\partial x_t + V(x_t)\partial t)z_t$$

if and only if

$$Dz_t = ((U - A)(x_t)\partial x_t + V(x_t)\partial t)z_t.$$

The lemma follows. □

Let Δ be the Laplace-Beltrami operator and let X be a vector field on M. Then $\mathcal{L} = \frac{1}{2}\Delta + X$ is a Laplacian so (34) must provide a formula for the semigroup of Brownian motion with drift X. We are not used to this being a corollary of the Feynman-Kač formula! So let us examine the details. In this case the vector bundle is the trivial line bundle $M \times \mathbf{R}$. Denote by ξ the 1-form dual to X by the metric; then for functions f, g on M

$$[\mathcal{L}, f]g = (\tfrac{1}{2}\Delta f)g + (df, (d+\xi)g).$$

This identifies the connection corresponding to $\mathcal{L}$ as $\nabla = d + \xi$. Using the Levi-Civita connection on TM we then have

$$\tfrac{1}{2} tr \nabla^2 f = \tfrac{1}{2}\Delta f + Xf + (\text{div} X + |X|^2).f$$

so the potential is $V = -\frac{1}{2}(\text{div} X + |X|^2)$.

The covariant chain rule gives us a way of recovering the horizontal lift u_t: we must have, for any function f,

$$\partial(u_t^{-1} f(x_t)) = u_t^{-1} \nabla f(x_t) \partial x_t.$$

Hence u_t^{-1} satisfies

$$\partial(u_t^{-1}) = u_t^{-1}\xi(\partial x_t)$$

which implies

$$u_t^{-1} = u_0^{-1}\exp\{\int_0^t \xi(\partial x_t)\}.$$

The action of u_t on End F is trivial, so the covariant equation

$$De_t = e_t V(x_t)\partial t$$

has solution

$$e_t = \exp\{-\tfrac{1}{2}\int_0^t (\mathrm{div}X + |X|^2)(x_s)ds\}.$$

Hence (34) reads

$$(e^{t\mathcal{L}}f)(x_0) = \mathsf{E}[\exp\{\int_0^t \langle X(x_s), dx_s\rangle - \tfrac{1}{2}\int_0^t |X|^2(x_s)ds\}f(x_t)],$$

as we should have known!

9. Infinitesimal analysis of the image of a semimartingale under a map

This section and the following section show how one can use the differential formalism, especially the Stratonovich to Itô conversion formula, to identify local martingales and Brownian motion. The main point is that the formal steps taken and the resulting differential formulae are simple. Some justification of these steps is left to the reader.

Suppose we are given two manifolds M and N and a smooth map $f : M \to N$. Let x_t be a semimartingale in M and set $y_t = f(x_t)$. Then the Stratonovich differentials are related by the tangent map

$$\partial y_t = f_*(\partial x_t) \tag{35}$$

and for $b \in \Gamma(T^*N \otimes T^*N)$ the quadratic variation satisfies

$$b(\partial y_t, \partial y_t) = (f^*b)(\partial x_t, \partial x_t)\,. \tag{36}$$

In order to discuss Itô differentials and martingales we must first fix connections on M and N. The relation between the map f and these two connections is described by the fundamental form of f which we shall briefly introduce. We refer to Vilms [V] for a fuller account of this and other aspects of the geometry needed below. We can regard TN as a vector bundle over M, the fibre at x being $T_{f(x)}N$. This bundle is given the pull-back connection by f of the connection on N, so that parallel translation in TN is the same whether it is regarded as a bundle over M or over N. This applies to translation along semimartingales as well as along smooth curves.

The tangent map f_* can thus be regarded as a section of $TN \otimes T^*M$ over M, and the covariant derivative $\beta(f) = \nabla(f_*)$ of this section is the *fundamental form* of f. Thus $\beta(f)(x)$ is a bilinear form on T_xM with values in $T_{f(x)}N$. We compute the covariant Stratonovich differential

$$D(f_*(x_t)) = \nabla(f_*)(\partial x_t) = \beta(f)(\cdot, \partial x_t).$$

We apply the conversion rule to (35) to obtain

$$dy_t = f_*(dx_t) + \tfrac{1}{2}D(f_*(x_t))\partial x_t$$

so

$$dy_t = f_*(dx_t) + \tfrac{1}{2}\beta(f)(\partial x_t, \partial x_t). \tag{37}$$

Some well known results are easily deduced from (36) and (37). Since Itô integrals preserve local martingales, we see that f preserves local martingales if and only if

$$\beta(f)(\partial x_t, \partial x_t) = 0$$

for all local martingales x_t. By appropriate choices of x_t we see this is equivalent to the vanishing of the symmetric part of $\beta(f)$. Now suppose M is Riemannian with its Levi-Civita connection and that x_t is Brownian motion, then

$$\beta(f)(\partial x_t, \partial x_t) = \operatorname{trace} \beta(f)(x_t)\partial t$$

so $f(x_t)$ is a local martingale if and only if $\operatorname{trace} \beta(f) = 0$. Suppose moreover that N is Riemannian with Levi-Civita connection, then using Lévy's characterization of Brownian motion, we see that $f(x_t)$ is also Brownian motion if and only if $\operatorname{trace} \beta(f) = 0$ and f_* restricted to $(\ker f_*)^\perp$ is an isometry, that is, f is a Riemannian submersion.

10. Factorization of Brownian motion in a Riemannian submersion with totally geodesic fibres

Let $\pi : M \to B$ be a Riemannian submersion. Set $V = \ker \pi_*$ and $H = (\ker \pi_*)^\perp$ so that $TM = H \oplus V$ and $\pi_*|_H$ is an isometry. Vilms [V] states that the fundamental form $\beta(\pi)$ vanishes on $H \times H$. Suppose that the fibres $M_x = \pi^{-1}(x)$ are totally geodesic, that is to say every geodesic of M_x is also a geodesic of M. Vilms shows that this is equivalent to the vanishing of $\beta(\pi)$ on $V \times V$ and also to the vanishing of the fundamental forms of the inclusions $M_x \to M$. As discussed in [EK], π is naturally associated to a principal bundle $U \to B$ whose structure group is the group of isometries of a typical fibre F. The elements of U_x may be regarded as isometries $u : F \to M_x$. Given a semimartingale x_t in B and an initial isometry $u_0 : F \to M_{x_0}$ there is a unique semimartingale u_t in U over x_t such that $\partial u_t(z) \in H$ for all $z \in F$. We call u_t the horizontal lift of x_t.

W.S. Kendall suggested that we should be able to give a 'stochastic calculus' proof of a result he obtained with K.D. Elworthy in [EK], namely that *if x_t is Brownian motion in B with horizontal lift u_t and if z_t is an independent Brownian motion in F, then $y_t := u_t(z_t)$ is Brownian motion in M.* We shall sketch a proof based on Lévy's characterization of Brownian motion using the formalism developed above. Elworthy and Kendall on the other hand use a geodesic form of Itô's formula to compute the required generators. There is a third proof by Liao [L] which is entirely analytic.

Write $\partial y_t = \partial y_t^H \oplus \partial y_t^V$ in the decomposition $H \oplus V$. From $\pi(y_t) = x_t$ and $y_t = u_t(z_t)$ we deduce $\pi_*(\partial y_t^H) = \partial x_t$ and $\partial y_t^V = (u_t)_* \partial z_t$, $\partial u_t(z)$ being horizontal. So we have

$$\partial y_t = (\pi_*\big|_H)^{-1} \partial x_t \oplus (u_t)_* \partial z_t \,.$$

Both $\pi_*\big|_H$ and $(u_t)_*$ are isometries and this is an orthogonal decomposition, so y_t does have the quadratic variation of Brownian motion. In fact we can show

$$dy_t = (\pi_*\big|_H)^{-1} dx_t \oplus (u_t)_* dz_t$$

so y_t is also a local martingale and is thus Brownian motion. For the first component

$$\pi_*(\partial y_t) = \pi_*(dy_t) + \tfrac{1}{2} D(\pi_*(y_t))\partial y_t$$

and

$$D(\pi_*(y_t))\partial y_t = \beta(\pi)(\partial y_t, \partial y_t) = 0$$

because $\beta(\pi)$ vanishes on $H \times H$ and $V \times V$ and x_t and z_t are independent. For the second component

$$(u_t)_* \partial z_t = (u_t)_* dz_t + \tfrac{1}{2} D((u_t)_*(z_t))\partial z_t$$

and

$$D((u_t)_*(z_t))\partial z_t = D(u_t)_*(\partial z_t) + \beta(u_t)(\partial z_t, \partial z_t) = 0$$

where the first term vanishes because u_t is driven by ∂x_t which is independent of z_t, and the second term vanishes because $u_t : F \to M$ is totally geodesic.

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