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AN ADDITIONAL REMARK ON UNITARY EVOLUTIONS IN FOCK SPACE

by

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This is a continuation of our discussions in [1]. Adopting the notations of Section 2 in [1] express the Hilbert space $h = h_0 \otimes (\mathbb{C}e_{-\infty} \oplus k \oplus \mathbb{C}e_{\infty})$ as a vector space of elements of the form

$$f = \begin{pmatrix} f_- \\ f_0 \\ f_+ \end{pmatrix}, f_\pm \in h_0, f_0 \in h_0 \otimes k.$$

Any bounded operator L in h can now be expressed as a 3×3 matrix of appropriate operators. Let U be a unitary operator in $h_0 \otimes k, \ell$ a bounded operator from h_0 into $h_0 \otimes k$ and let H be a bounded selfadjoint operator in h_0. Define the operator $L = L(U, \ell, H)$ in h by

$$L = \begin{pmatrix} 0 & -\ell^* & -iH - \frac{1}{2}\ell^*\ell \\ 0 & U - 1 & U\ell \\ 0 & 0 & 0 \end{pmatrix}. \tag{1}$$

Then $L \in \mathcal{I}(h)$, i.e., $Lf \otimes e_{-\infty} = L^*f \otimes e_\infty \equiv 0$ and furthermore

$$L^bL + L^b + L = LL^b + L^b + L = 0,$$

the superscript b indicating the involution described in [1]. Thus there exists a unitary operator valued adapted process U_L satisfying

$$U_L(0) = 1, dU_L = (d\Lambda_L)U_L \tag{2}$$

in the Hilbert space $h_0 \otimes \Gamma(L^2(\mathbb{R}_+) \otimes k)$, Γ indicating the boson Fock second quantization. Then for any $X \in \mathcal{B}(h_0)$, putting $j_t(X) = U_L^* X U_L$ we get

$$dj_t(X) = U_L^*(t) d\Lambda_{\theta(X)} U_L(t) \tag{3}$$

where

$$\theta(X) = L^b X + XL + L^b XL = \begin{pmatrix} 0 & \ell^* U^* X U - X\ell^* & \mathcal{L}(X) \\ 0 & U^* X U - X & U^* X U \ell - \ell X \\ 0 & 0 & 0 \end{pmatrix}, \quad (4)$$

$$\mathcal{L}(X) = i[H, X] - \frac{1}{2}(\ell^* \ell X + X \ell^* \ell - 2\ell^* X \ell). \quad (5)$$

$\{j_t, t \geq 0\}$ defined by (3)-(5) is an Evans-Hudson flow whose vacuum expectation $\mathbb{E}_0$ is given by

$$\mathbb{E}_0 j_t(X) = e^{t\mathcal{L}}(X), X \in \mathcal{B}(h_0).$$

Now consider the special case when $k = L^2(\Omega, \mathcal{F}, P)$ is a separable probability space and $h_1 = h_0 \otimes k = L^2(P, h_0)$, the Hilbert space of norm square integrable h_0-valued maps on $(\Omega, \mathcal{F}, P)$. Suppose the operators U, ℓ in (1) are of the form

$$(Uf)(\omega) = U(\omega) f(\omega), (\ell u)(\omega) = \ell(\omega) u$$

where $U(\cdot)$ is a h_0 -unitary operator valued map and $\ell(\cdot)$ is a $\mathcal{B}(h_0)$ - valued map on $(\Omega, \mathcal{F}, P)$. If $U(\omega)\ell(\omega) = M(\omega)$ then (5) assumes the form

$$\mathcal{L}(X) = i[H, X] - \frac{1}{2}\int_\Omega [M(\omega)^* M(\omega) X + X M(\omega)^* M(\omega) - 2M(\omega)^* X M(\omega)] dP(\omega) \quad (6)$$

Suppose $h_0 = L^2(\mathcal{X}, \mathcal{S}, \mu)$ where $(\mathcal{X}, \mathcal{S}, \mu)$ is a σ-finite separable measurable space and for any $\phi \in L^\infty(\mu)$

$$U(\omega)^* \phi U(\omega) = \phi \circ T(\omega)$$

where $T(\omega)$ is a μ-measure class preserving transformation on $\mathcal{X}$ for each $\omega \in \Omega$ and $L^\infty(\mu)$ is viewed as the abelian $\star$ subalgebra of $\mathcal{B}(h_0)$. Furthermore let $\ell(\omega)$ be multiplication by $\ell(x, \omega)$ in $L^2(\mu)$ and $H = 0$. Then (6) becomes

$$\ell(\phi)(x) = \int_\Omega |\ell(x, \omega)|^2 \{\phi(T(\omega)x) - \phi(x)\} dP(\omega) \quad (7)$$

and $\{j_t|_{L^\infty(\mu)}, t \geq 0\}$ is an Evans-Hudson flow describing a classical Markov flow with generator given by (7).

References

[1]. K.R. Parthasarathy : Realisation of a class of Markov processes through unitary evolutions in Fock space, Preprint, Indian Statistical Institute, Delhi (1990).