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*Séminaire de probabilités (Strasbourg)*, tome 25 (1991), p. 354-373
<http://www.numdam.org/item?id=SPS_1991__25__354_0>

# An extension of Krein's inverse spectral theorem to strings with nonreflecting left boundaries

by

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Abstract: Krein's inverse spectral theorem describes the spectral measures $\tau$ of the differential operators $D_m D_x$ with boundary condition $f_-'(0) = 0$, if $m$ runs through all nondecreasing functions on $[0,\infty)$. This result will be extended to boundary conditions of the type $af_-'(0) - f(0) = 0$ $(a \in [0,\infty))$.
Other conditions as in Krein's theorem appear.

Key words: gap-diffusions, quasidiffusions, generalized second order differential operator, spectral measures, local times, Krein's inverse spectral theorem, Krein's correspondence

60J35, 60J60, 34B20

## 1. Introduction

It is well-known that every nondecreasing function $m$ on $[0,\infty)$ performed with appropriate boundary conditions at zero and at $l := \sup \operatorname{supp} m$ (a so-called string) generates a strong Markov process $(X_t)$ on supp $m$, where supp $m$ denotes the set of points where $m$ increases. This process has as its (selfadjoint) infinitesimal generator in $L_2(m)$ the generalized second order differential operator $D_m D_x$ together with the mentioned boundary conditions. $(X_t)$ is called a quasi- (or gap-) diffusion with speed measure $m$. Examples are diffusions and birth- and death-processes. Several probabilistic quantities of $(X_t)$ as e.g. transition densities, first hitting time densities, Lévy-measures of the inverse local time at zero, can be expressed in terms of spectral measures $\tau^{(m)}$ of $D_m D_x$ under different boundary conditions, see e.g. Ito, McKean [2], Küchler [7], [8], Küchler, Salminen [9].

An essential result concerning these spectral measures is M.G. Krein's inverse spectral theorem, in a more extended form known as Krein's correspondence theorem, see Kac, Krein [3], Kotani, Watanabe [6]. Roughly speaking it states that the mapping $m \longrightarrow \tau^{(m)}$ is a one-to-one and onto correspondence between the strings $m$ with the "reflecting" boundary condition $f^-(0) := f'(0-) = 0$ and the set of all measures $\tau$ on $[0,\infty)$ that integrate $(1+\mu)^{-1}$ thereon, see Theorem 2.2 below. What we are going to do is to study the situation for the boundary conditions

$$af^-(0) - f(0) = 0 ,$$

where $a \in [0,\infty)$ is fixed. (The case above corresponds to $a = \infty$.)
If $a \in (0,\infty)$ ("elastic killing boundary"), then there is still a one-to-one and into correspondence (Theorem 2.4). If $a = 0$, then $m \longrightarrow \tau^{(m)}$ maps the strings $m$ with the "killing" boundary condition $f(0) = 0$ onto the set of measures on $(0,\infty)$ that integrate $[\mu(1+\mu)]^{-1}$, but not one-to-one. In Theorem 3.2 we shall describe the preimages for every $\tau$ which form one-parametric families.
As an application we get the description of all measures $\nu$ that can appear as the Lévy-measure of the inverse local times at zero for quasidiffusions (see Remark 3.6). This result was proved by other (probabilistic) means in Knight [5]. Here we shall present an analytical approach.
Moreover, a generalization of Lemma 1 of Karlin, McGregors paper [4] concerning birth- and death-processes to strings is given (see Corollary 3.7).

## 2. Strings, spectral measures and Krein's theorem

Here we shall summarize some facts from the theory of generalized second order differential operators $D_m D_x$. For details the reader is referred to Kac, Krein [3] or Dym, McKean [1], the latter uses another terminology.
By $R$ and $K$ we denote the real axis and the complex plane, respectively. $R_+$ stands for $[0,\infty)$, $K_-$ for $K \setminus R_+$. Put $\bar{R}_+ := [0,\infty]$ and $\frac{1}{0} := \infty$, $\frac{1}{\infty} := 0$. Let $m$ be a nondecreasing right-continuous extended real-valued function on $R$ with $m(x) \equiv 0$, $x < 0$. Define $E_m$ to be the set of points where $m$ increases and is finite:

$$E_m := \{x \in R_+ \mid \exists\, \varepsilon_0 > 0:\ m(x-\varepsilon) < m(x+\varepsilon) < \infty \quad \forall\, \varepsilon \in (0,\varepsilon_0)\}.$$

We shall assume $E_m \neq \emptyset$ and denote by the same letter $m$ the measure generated by the function $m$. Such a measure $m$ is called a speed measure.

Introduce $c$, $l$ and $r$ by

$$c := \inf E_m = \inf \{x \geq 0 \mid m(x) > 0\},$$

$$l := \sup E_m \leq \infty,$$

$$r := \sup \{x \geq 0 \mid m(x) < \infty\} \leq \infty .$$

We have $0 \leq c \leq l \leq r$ and put $h := r - l$ if $l < \infty$. Otherwise $h$ is irrelevant and for convenience we put $h = 0$ in this case. Note that $h = 0$ if $l < \infty$ and $m(l) = \infty$. If $l + m(l) < \infty$ and $m(\{l\}) > 0$, then $h$ must be greater than zero. The number $r = l + h$ is called the length of the string.

Sometimes we shall write $c_m$, $l_m, \ldots$ to express that these numbers come from $m$.

By $\vartheta$ we denote the set of all real functions $f$ on $R$ having a representation

$$f(x) = \bar{a} + \bar{b} \cdot x + \int_0^x (x-s) g(s) m(ds) \tag{2.1}$$

for some measurable $g$ on $R$ and some reals $\bar{a}$, $\bar{b}$.

Note that every $f \in \vartheta$ is continuous and linear on the open intervals of $R \setminus E_m$.

On $\vartheta$ we define a generalized second order differential operator $D_m D_x$ by $D_m D_x f = g$, details can also be found in Küchler [7], [8].

For every fixed $a \in [0, \infty]$ the restriction $A_a$ of $D_m D_x$ to

$$\Delta_a := \{f \in \vartheta \cap L_2(m) \mid D_m D_x f \in L_2(m),\ af^-(0) - f(0) = 0\} \tag{2.2}$$

(for $a = \infty$ we mean $f^-(0) = 0$) is a nonnegative selfadjoint operator in $L_2(m)$.

(By $f^+$ and $f^-$ we denote the right- and left-hand-side derivative of $f$, respectively.)

Note that $f \in \vartheta \cap L_2(m)$ implies $f(r) = 0$ if $r = l + h < \infty$. Because of the linearity of $f$ on the intervals of $R \setminus E_m$ this can be written as a boundary condition $hf^+(l) + f(l) = 0$ with $f^+(l) = 0$ if $h = \infty$. Otherwise, the boundary condition appearing in (2.2) can also be included in $f \in \vartheta \cap L_2(m)$ if we change $m$ to the left of $-a$ into $m(x) = -\infty$, $x < -a$.

In the following, $m$ will be understood in this way.

This change of $m$ charges $-a$ with infinite mass. The original measure $m$ on $R_+$ remains unchanged by this procedure if $a > 0$. In case $a = 0$, the value of $m(\{0\})$ is not reconstructable. But this does not disturb the corresponding spectral theory, as we will see below. Thus we suppose $m(0) = 0$ if we consider $a = 0$. Now $m$ has infinite mass at $-a$ (and $r$ if $r < \infty$) and thus $f \in \vartheta \cap L_2(m)$ implies also $f(-a) = 0$, i.e. $af^-(0) - f(0) = 0$.

Therefore, the selfadjoint operators $A_a$ are characterized by the (changed) function $m$, or by $(m,a)$. We call the pair $(m,a)$ a <u>string</u> and denote it by $S_a(m)$. If the length $r = l + h$ is infinite, then we say that the string $S_a(m)$ is infinite. Depending on $l + m(l-) < \infty$ or $= \infty$ the string $S_a(m)$ is called regular or singular. The resolvent operator $R_{\lambda,a} := (A_a - \lambda I)^{-1}$ exists for $\lambda \in (-\infty, 0)$, and it can be shown analogously to Dym, McKean [1] that $R_{\lambda,a}$ is given by

$$(R_{\lambda,a}f)(x) = \int_0^l r_{\lambda,a}(x,y)f(y)m(dy) , \qquad f \in L_2(m),$$

where

$$r_{\lambda,a}(x,y) := \frac{\Phi_a^{\uparrow}(x \wedge y, \lambda)\,\Phi^{\downarrow}(x \vee y, \lambda)}{W} .$$

Here $\Phi_a^{\uparrow}$ and $\Phi^{\downarrow}$ denote the solutions $f \in \vartheta$ of

$$D_m D_x f + \lambda f = 0$$

satisfying the boundary conditions

$$\Phi_a^{\uparrow}(0,\lambda) = 1, \quad a \in (0,\infty]; \quad \Phi_0^{\uparrow -}(0,\lambda) = 1, \tag{2.3}$$

$$a\,\Phi_a^{\uparrow -}(0,\lambda) - \Phi_a^{\uparrow}(0,\lambda) = 0, \quad a \in [0,\infty); \quad \Phi_\infty^{\uparrow -}(0,\lambda) = 0, \tag{2.4}$$

and

$$\Phi^{\downarrow -}(0,\lambda) = -1, \qquad \text{and} \tag{2.5}$$

$$h\,\Phi^{\downarrow +}(l,\lambda) + \Phi^{\downarrow}(l,\lambda) = 0 . \tag{2.6}$$

Note that $\Phi_a^{\uparrow}(\cdot,\lambda)$ is increasing and $\Phi^{\downarrow}(\cdot,\lambda)$ is decreasing for fixed $\lambda < 0$.

$W$ denotes the Wronskian:

$$W = W(\lambda) := \Phi_a^{\uparrow -}\Phi^{\downarrow} - \Phi_a^{\uparrow}\Phi^{\downarrow -}$$

Several times we will use that $\Phi_a^{\uparrow}(\cdot,\lambda)$ is the uniquely determined solution of

$$\Phi(x,\lambda) = 1 + \frac{x}{a} - \lambda\int_0^x (x-s)\,\Phi(s,\lambda)m(ds), \quad x\in[0,r) \tag{2.7}$$

for $a\in(0,\infty]$, and of

$$\Phi(x,\lambda) = x - \lambda\int_0^x (x-s)\,\Phi(s,\lambda)m(ds), \quad x\in[0,r) \tag{2.8}$$

for $a = 0$.
Similarly, $\Phi^{\downarrow}(\cdot,\lambda)$ is the unique solution of

$$\Phi(x,\lambda) = \Phi(0,\lambda) - x - \lambda\int_0^x (x-s)\,\Phi(s,\lambda)m(ds), \quad x\in[0,r). \tag{2.9}$$

DEFINITION 2.1: Assume $S_a(m)$ is a string with $a\in[0,\infty]$. Then a measure $\tau$ on $[0,\infty)$ is called a spectral measure of $S_a(m)$, if

$$r_{\lambda,a}(x,y) = \int_0^\infty \frac{\Phi_a^{\uparrow}(x,\mu)\,\Phi_a^{\uparrow}(y,\mu)}{\mu-\lambda}\,d\tau(\mu), \quad \lambda<0;\ x,y\in E_m.$$

The set $\operatorname{supp}\tau$ is called the spectrum of $S_a(m)$.

As for the case of $a=\infty$, treated in Kac, Krein [3] and Dym, McKean [1], one can show that for every string $S_a(m)$ a unique spectral measure $\tau$ exists (on $(0,\infty)$ if $a\neq 0$). It will often be denoted by $\tau_a^{(m)}$. (We shall identify measures $\tau$ on $R_+$ and their generating function $\mu \longrightarrow \tau([0,\mu])$.)
Note that $\Phi_0^{\uparrow}(\cdot,\lambda)$, and therefore $\tau_0^{(m)}$ does not depend on the mass of $m$ at zero. Thus, considering a string $S_0(m)$ we shall always suppose that $m(0) = 0$.
If the string $S_a(m)$ is regular, then $\tau_a^{(m)}$ is given by

$$\tau_a^{(m)}(\mu) = \sum_{k=0}^{\infty} \tau_a^{(m)}(\{\mu_k\})\cdot \mathbb{1}_{[0,\mu_k]}(\mu) \tag{2.10}$$

where $(\mu_k)_{k\geq 0}$ denotes the sequence of solutions of

$$h\,\Phi_a^{\uparrow,+}(1,\mu) + \Phi_a^{\uparrow}(1,\mu) = 0$$

and

$$\tau_a^{(m)}(\{\mu_k\}) = \Big[\int_0^1 [\Phi_a^{\uparrow}(x,\mu_k)]^2 dm\Big]^{-1}.$$

We have

$$0 \le \mu_0 < \mu_1 < \dots < \mu_n < \dots \quad \text{and} \quad \sum_{n\ge 1} \mu_n^{-1} < \infty .$$

The following theorem answers the question which measures may appear as spectral measures for strings $S_\infty(m)$. Its second part is M.G. Krein's inverse spectral theorem, (i) and (ii) together are known as Krein's correspondence (Kotani, Watanabe [6]).

<u>THEOREM 2.2:</u>

(i) For every string $S_\infty(m)$ its spectral measure $\tau = \tau_\infty^{(m)}$ satisfies

$$\int_{0-}^{\infty} \frac{d\tau(\mu)}{1+\mu} < \infty . \tag{2.11}$$

(ii) For every measure $\tau$ on $R_+$ with $\tau(R_+) > 0$ and (2.11) there exists one and only one string $S_\infty(m)$ with $c_m = 0$ having $\tau$ as its spectral measure.

Note that the condition $c_m = 0$ in (ii) ensures the unicity of $m$. Indeed, all "shifted" strings $S_\infty(m(\cdot - c))$ $(c > 0)$ have the same spectral measure, compare Proposition 2.3 below.
For every string $S_\infty(m)$ its characteristic function $\Gamma_m(\cdot)$ is given by (see Chapter 4 below)

$$\Gamma_m(\lambda) := c_m + \int_{0-}^{\infty} \frac{d\tau_\infty^{(m)}(\mu)}{\mu-\lambda} = \lim_{x\uparrow r} \frac{\Phi_0^\uparrow(x,\lambda)}{\Phi_\infty^\uparrow(x,\lambda)}, \quad \lambda \in K_- . \tag{2.12}$$

Because of the definition of the spectral measure we obtain

$$\Phi^\downarrow(0,\lambda) = r_{\lambda,\infty}(0,0) = \Gamma_m(\lambda), \quad \lambda < 0. \tag{2.13}$$

Letting $\lambda \uparrow 0$ in (2.12) we get the formula

$$c_m + \int_{0-}^{\infty} \frac{d\tau_\infty^{(m)}(\mu)}{\mu} = r = l + h \tag{2.14}$$

with the understanding that $h = 0$ if $l + m(l-) = \infty$.
Krein's theorem says that $\Gamma_m(\cdot)$ determines $S_\infty(m)$ uniquely.
In Chapter 4 below we shall see that it holds

$$-\frac{1}{\Gamma_m(\lambda)} = \lambda\, m(\{0\}) - r_m^{-1} - \int_0^\infty \left(\frac{1}{\mu} - \frac{1}{\mu-\lambda}\right)\tau_0^{(m)}(d\mu), \quad \lambda \in K_- . \tag{2.15}$$

For every string $S_\infty(m)$ with $c_m = 0$ we have

$$\tau_\infty^{(m)}(R_+) = [m(\{0\})]^{-1} . \qquad (2.16)$$

Indeed, consider $\lambda \Gamma_m(\lambda)$ for $\lambda \downarrow -\infty$ and compare (2.12) and (2.15), then (2.16) is obvious.
Finally, note that $\tau_\infty^{(m)}(\{0\}) > 0$ if and only if the constant function $\Phi_\infty^\uparrow(\cdot,0) \equiv 1$ is an eigenfunction of $D_m D_x$. This holds if and only if $r = l + h = \infty$ (or $l = \infty$) and $m(l-) < \infty$. Moreover, in this case we have

$$\tau_\infty^{(m)}(\{0\}) = [m(l)]^{-1} . \qquad (2.17)$$

The next proposition shows how the spectral measure changes if $m$ suffers certain transformations.

<u>PROPOSITION 2.3:</u> Let $S_a(m)$ be a string with $a \in [0,\infty]$ and assume $u,v \in (0,\infty)$, $w \in [0,\frac{a}{u}]$ and $w < \infty$. Define

$$\tilde{m}(x) := v\cdot m(u(x-w)) , \qquad x \in R,$$

$$\tilde{a} := \frac{a}{u} - w .$$

Then, for the spectral measures $\tilde{\tau}_{\tilde{a}} := \tau_{\tilde{a}}^{(\tilde{m})}$ and $\tau_a := \tau_a^{(m)}$ of $S_{\tilde{a}}(\tilde{m})$ and $S_a(m)$, respectively, we have

(i) If $a = \infty$, then for all $w \in [0,\infty)$ we have $\tilde{a} = \infty$ and

$$\tilde{\tau}_\infty(\mu) = v^{-1}\tau_\infty(\tfrac{v}{u}\cdot\mu) , \qquad \mu \geq 0.$$

(ii) If $a \in (0,\infty)$, $0 < w < \frac{a}{u}$, then $\tilde{a} \in (0,\infty)$ and

$$\tilde{\tau}_{\tilde{a}}(\mu) = v^{-1}(1 - \tfrac{uw}{a})^2\tau_a(\tfrac{v}{u}\cdot\mu) , \qquad \mu \geq 0.$$

(iii) If $a \in (0,\infty)$, $w = \frac{a}{u}$, then $\tilde{a} = 0$ and

$$\tilde{\tau}_0(\mu) = v^{-1}(\tfrac{u}{a})^2\tau_a(\tfrac{v}{u}\cdot\mu) , \qquad \mu \geq 0.$$

(iv) If $a = w = 0$, then

$$\tilde{\tau}_0(\mu) = v^{-1}\cdot u^2\tau_0(\tfrac{v}{u}\cdot\mu) , \qquad \mu \geq 0.$$

To prove this proposition one calculates the relevant $\tilde{\Phi}_{\tilde{a}}^\uparrow$ and $\tilde{\Phi}^\downarrow$ in terms of $\Phi_a^\uparrow$ and $\Phi^\downarrow$, respectively, using (2.7 - 2.9). This gives the relation between the resolvent kernels $\tilde{r}_{\tilde{\lambda},\tilde{a}}$ and $r_{\lambda,a}$. Definition 2.1 leads to the assertion of Proposition 2.3.

## 3. Results

In this chapter we shall formulate correspondence theorems for strings $S_a(m)$ with $a \neq \infty$ which extend Krein's result. The proofs can be found in Chapter 4.
We shall start with the case of $a = 0$. For this purpose we still need a preparation.
Denote by $\Sigma$ the set of all strings $S_0(m)$ with $m(0) = 0$. We introduce a relation $\sim$ in $\Sigma$ by defining $S_0(m) \sim S_0(n)$ if there exists a real number $t \geq -\frac{1}{r_n}$ such that the transformation

$$x \longrightarrow T_t x := \frac{x}{1 - tx}, \qquad x \in R$$

maps $(0, r_m)$ onto $(0, r_n)$ and such that

$$m(x) = \int_{0+}^{x} (1-ts)^{-2} dn(T_t s), \qquad x \in (0, r_m) \tag{3.1}$$

(indeed, $t = \frac{1}{r_m} - \frac{1}{r_n}$.) It is easy to see that $\sim$ forms an equivalence relation in $\Sigma$. Put $\hat{\Sigma} := \Sigma/\sim$ and for every string $S_0(m) \in \Sigma$ denote by $\hat{S}(m)$ the element of $\hat{\Sigma}$ generated by $S_0(m)$. For every string $S_0(m)$ and every $t \geq -\frac{1}{r_m}$ we define a new string $S_0(m_t)$ by $r_{m_t} := \frac{r_m}{1 + tr_m}$ and

$$\left.\begin{aligned} m_t(x) &:= \int_{0+}^{x} (1-ts)^{-2} dm(T_t s), \qquad && x \in (0, r_{m_t}) \\ m_t(x) &:= \infty, && x \geq r_{m_t}. \end{aligned}\right\} \tag{3.2}$$

Obviously, we have

$$c_{m_t} = \frac{c_m}{1 + tc_m}, \quad l_{m_t} = \frac{l_m}{1 + tl_m}, \qquad t \geq -\frac{1}{r_m}, \tag{3.3}$$

and $S_0(m_t) \sim S_0(m)$ for every $t \geq -\frac{1}{r_m}$.
Otherwise, if $S_0(n) \sim S_0(m)$, then, by definition, there exists a real number $t \geq -\frac{1}{r_n}$ such that $m = n_t$. Observe $r_{m_t} = \infty$ if and only if $t = -\frac{1}{r_m}$.
Thus we have proved the following

LEMMA 3.1:

(i) For every string $S_o(m)$ its equivalence class $\hat{S}(m)$ is equal to $\{S_o(m_t) \mid t \geq -\frac{1}{r_m}\}$.

(ii) Every equivalence class $\hat{S} \in \hat{\Sigma}$ contains one and only one infinite string $S_o(m)$.

Now we are ready to formulate the analogue of Krein's correspondence for strings $S_o(m)$.

THEOREM 3.2:

(i) For every string $S_o(m)$ its spectral measure $\tau = \tau_o^{(m)}$ is supported by $(0,\infty)$ and has the property

$$\int_{0+}^{\infty} \frac{d\tau(\mu)}{\mu(1+\mu)} < \infty \quad . \tag{3.4}$$

Moreover, it holds

$$\int_{0+}^{\infty} \frac{d\tau(\mu)}{\mu} = c_m^{-1} - r_m^{-1} \quad . \tag{3.5}$$

(ii) If two strings $S_o(m)$ and $S_o(n)$ are equivalent (with respect to $\sim$), then $\tau_o^{(m)} = \tau_o^{(n)}$.

(iii) For every measure $\tau$ on $(0,\infty)$ $(\tau((0,\infty))>0)$ with (3.4) and every $r \in (0,\infty]$ there exists one and only one string $S_o(m)$ with length $r = r_m$ having $\tau$ as its spectral measure. If $S_o(m)$ and $S_o(m')$ are strings with the lengths $r$ and $r'$, respectively, having the same spectral measure, then $S_o(m') = S_o(m_t)$ holds with $t = \frac{1}{r'} - \frac{1}{r}$. ($m_t$ was defined in (3.2).)

This theorem can be reformulated in a shorter way as follows.

COROLLARY 3.3: There is a one-to-one and onto correspondence between the set $\hat{\Sigma}$ of equivalence classes $\hat{S}$ of strings $S_o(m)$ and the set of measures $\tau$ on $(0,\infty)$ satisfying (3.4), where $\tau$ is the spectral measure $\tau_o^{(m)}$ of every string $S_o(m)$ from $\hat{S}$.

Now let us turn to the case of $a \in (0,\infty)$.

THEOREM 3.4: Assume $a \in (0,\infty)$. Then it holds:

(i) For every string $S_a(m)$ with $c_m = 0$ and $m(0) \geq 0$ its spectral measure $\tau = \tau_a^{(m)}$ is supported on $(0,\infty)$ and has the property

$$\int_{0+}^{\infty} \frac{d\tau(\mu)}{\mu} = \left(\frac{1}{r_m} + \frac{1}{a}\right)^{-1} < \infty \tag{3.6}$$

(ii) If $\tau$ is a measure on $(0,\infty)$ with nonzero mass, then there exists a string $S_a(m)$ with $c_m = 0$ having $\tau$ as its spectral measure if and only if

$$\varrho(\tau) := \int_{0+}^{\infty} \frac{d\tau(\mu)}{\mu} \leq a . \tag{3.7}$$

In this case, $S_a(m)$ is uniquely determined.
Moreover, if $S_a(m)$ and $S_{a'}(m')$ with $a, a' \in (0,\infty)$, $c_m = c_{m'} = 0$, have the same spectral measure, then

$$m'(x - a') = m_t(x - a) , \qquad x \in R_+$$

with $t := \frac{1}{a'} - \frac{1}{a}$, where $m_t$ was defined in (3.2).

Consider a speed measure $m$ on $[0,\infty)$ with $c_m = 0$, $m(0) \geq 0$ and form strings $S_\infty(m)$, $S_a(m)$ and $S_0(m)$ for some $a \in (0,\infty)$. (Note that $m(\{0\})$ disappears if we construct $S_0(m)$.)
Then we have

PROPOSITION 3.5: Between the spectral measures $\tau_\infty$, $\tau_a$ and $\tau_0$ of $S_\infty(m)$, $S_a(m)$ with $a \in (0,\infty)$ and $S_0(m)$, respectively, the following equation holds:

$$\left[\lambda m(0) - r_m^{-1} - \int_{0+}^{\infty} \left(\frac{1}{\mu} - \frac{1}{\mu - \lambda}\right) d\tau_0(\mu)\right] \cdot$$

$$\cdot \left[\int_{0+}^{\infty} \frac{d\tau_a(\mu)}{\mu - \lambda}\right] \cdot \left[a + \int_{0-}^{\infty} \frac{d\tau_\infty(\mu)}{\mu - \lambda}\right] = -a , \qquad \lambda \in K_- . \tag{3.8}$$

This generalizes a formula which was used by Knight [5], p. 60. Consider a string $S_\infty(m)$ and add to $m$ some point mass $m_0 > 0$ at zero if necessary, i.e. if $c_m > 0$. As we know, this does not touch the spectral measure $\tau_0^{(m)}$ of $S_0(m)$. Now, let $l(t,0)$, $t \geq 0$, be the local time at zero of the quasidiffusion generated by $S_\infty(m)$. Since $0 \in E_m$, this notion makes sense. Then $(l^{-1}(t,0), t \geq 0)$ is an increasing process with independent stationary increments and it holds

$$E_0 \exp(\lambda l^{-1}(t,0)) = \exp(-\frac{t}{\Gamma_m(\lambda)}), \qquad \lambda<0,\ t\geq 0.$$

(See e.g. Knight [5] or Küchler [8].)
For $\lambda < 0$, (2.15) implies

$$-\frac{1}{\Gamma_m(\lambda)} = \lambda\, m(\{0\}) - \frac{1}{r_m} - \int_0^\infty (1 - e^{\lambda y})\Big[\int_{0+}^\infty e^{-\mu y}\tau_0^{(m)}(d\mu)\Big]dy.$$

Thus, by Theorem 3.2(ii) and Lemma 3.1(i) the Lévy-measure $n$ of $l^{-1}(\cdot,0)$, given by

$$dn(y) := \int_{0+}^\infty e^{-\mu y}\tau_0^{(m)}(d\mu)dy, \qquad y\in R_+, \tag{3.9}$$

is the same for all $S_\infty(m_t)$, $t \geq -\frac{1}{r_m}$.
This means that the inverse local times at zero of the quasidiffusions corresponding to $S_\infty(m_t)$ differ in their killing rate $k = \frac{1}{r_m} + t$ only.
Now Theorem 3.2 implies

COROLLARY 3.6: For every nontrivial measure $\tau$ on $(0,\infty)$ with (3.4), every $m(\{0\})>0$ and every constant $k\geq 0$ there exists a quasidiffusion with speed measure $m$, a reflecting boundary at zero and length $\frac{1}{k}$ of the string $S_\infty(m)$ such that $l^{-1}(\cdot,0)$ has the Lévy-measure (3.9).

This result was proved by other means in Knight [5].
As an example consider a birth- and death-process on the set of nonnegative integers with the intensities $\mu_0\geq 0$, $\lambda_i>0$, $\mu_{i+1}>0$, $i\geq 0$. Then

$$m(x) := \sum_{i=0}^\infty m_i\cdot \mathbb{1}_{[0,x]}(x_i)$$

with
$$x_0 := 0, \qquad x_i := \sum_{j=0}^{i-1}\frac{1}{\lambda_j m_j},$$

$$m_0 := 1, \qquad m_i := \prod_{j=1}^{i}\frac{\lambda_{j-1}}{\mu_j}, \qquad i\geq 1$$

and $a := \mu_0^{-1}$, $h\geq 0$ define a string $S_a(m)$. (Necessarily, $h = 0$ if $m$ is singular.) We have

$$D_m D_x f(x_i) = \left[ \frac{\Delta f(x_i)}{\Delta x_i} - \frac{\Delta f(x_{i-1})}{\Delta x_{i-1}} \right] \cdot m_i^{-1} =$$

$$= \lambda_i \; f(x_{i+1}) - (\lambda_i + \mu_i) f(x_i) + \mu_i f(x_{i-1}), \qquad i \geq 1$$

with $\Delta u(x_j) := u(x_{j+1}) - u(x_j)$.
Moreover,

$$D_m D_x f(x_o) = \frac{\dfrac{\Delta f(x_o)}{x_1} - f^-(x_o)}{m_o} \qquad \text{and}$$

the boundary condition

$$af^-(x_o) - f(x_o) = 0$$

is equivalent to

$$D_m D_x f(x_o) = -(\lambda_o + \mu_o) f(x_o) + \lambda_o f(x_1) \; .$$

Thus, we have

$$\Phi_a^{\uparrow}(x_i, \lambda) = Q_i(\lambda) \; , \qquad i \geq 0, \; \lambda \in R$$

in the terminology of Karlin, McGregor [4].
The spectral measure $\tau_a^{(m)}$ of $S_a(m)$ is a solution of the Stieltjes moment problem connected with the Jacobi-matrix $(a_{ij})$ with

$$a_{ij} := \lambda_i \, 1\!\!1_1(j-i) + \mu_i \, 1\!\!1_1(i-j) - (\lambda_i + \mu_i) \, 1\!\!1_o(i-j) \qquad (i,j \geq 0).$$

Indeed, for $\lambda \longrightarrow -\infty$ we have

$$\| -\lambda R_{\lambda,a} f - f \|_{L_2(m)} \longrightarrow 0 \; , \qquad f \in L_2(m).$$

Consequently,

$$\langle -\lambda R_{\lambda,a} f, g \rangle_{L_2(m)} \longrightarrow \langle f, g \rangle_{L_2(m)} \; , \qquad f, g \in L_2(m).$$

Choosing $f = 1\!\!1_{\{x_i\}}$, $g = 1\!\!1_{\{x_j\}}$ we obtain

$$\lim_{\lambda \to -\infty} -\lambda \, r_{\lambda,a}(x_i, x_j) =$$

$$\int_0^\infty \Phi_a^{\uparrow}(x_i, \mu) \, \Phi_a^{\uparrow}(x_j, \mu) \, d\tau_a^{(m)}(\mu) = \frac{\delta_{ij}}{m_i} \; , \quad i,j \geq 0.$$

Compare this equation with Theorem 1 of Karlin, McGregor [4], p. 494 to get the assertion.

Now, Lemma 1 of Karlin, McGregor [4] can be generalized to strings as follows.

COROLLARY 3.7: Given a string $S_\infty(m)$ with $c_m = 0$ and with the spectral measure $\tau$ and assume $a > 0$. Then there exists a string $S_a(m')$ with $c_{m'} = 0$ having the same spectral measure $\tau$ if and only if

$$r_m = l_m + h_m \leq a . \tag{3.10}$$

Proof: If $\tau(\{0\}) > 0$, then there does not exist such a string $S_a(m')$ because, for $a \neq \infty$, the spectral measure is concentrated on $(0, \infty)$. Otherwise, $r_m = \infty$, see the remarks before (2.17).
Assume $\tau(\{0\}) = 0$. From (2.14) we know $r_m = \int_0^\infty \frac{d\tau(\mu)}{\mu}$. Now apply Theorem 3.4(ii).

## 4. Proofs

At first we shall collect some results of the spectral theory of $D_m D_x$. For details see e.g. Kac, Krein [3]. Let us given a string $S_\infty(m)$. The characteristic function $\Gamma(\cdot)$ of $S_\infty(m)$ is given by the limit (see (2.12))

$$\Gamma(\lambda) = \lim_{x \uparrow r} \frac{\Phi_0^{\uparrow}(x,\lambda)}{\Phi_\infty^{\uparrow}(x,\lambda)} , \qquad \lambda \in K_- . \tag{4.1}$$

In the regular case we have for $h < \infty$

$$\Gamma(\lambda) = \frac{\Phi_0^{\uparrow}(r,\lambda)}{\Phi_\infty^{\uparrow}(r,\lambda)} = \frac{\Phi_0^{\uparrow,+}(1,\lambda)\cdot h + \Phi_0^{\uparrow}(1,\lambda)}{\Phi_\infty^{\uparrow,+}(1,\lambda)\cdot h + \Phi_\infty^{\uparrow}(1,\lambda)} , \tag{4.2}$$

and for $h = \infty$ it holds

$$\Gamma(\lambda) = \frac{\Phi_0^{\uparrow,+}(1,\lambda)}{\Phi_\infty^{\uparrow,+}(1,\lambda)} . \tag{4.3}$$

If $S_\infty(m)$ is singular, then besides of (4.1) it holds

$$\Gamma(\lambda) = \lim_{x \uparrow r} \frac{\Phi_0^{\uparrow,+}(x,\lambda)}{\Phi_\infty^{\uparrow,+}(x,\lambda)} , \qquad \lambda \in K_- . \tag{4.4}$$

Moreover, we have the representation (see (2.12))

$$\Gamma(\lambda) = c_m + \int_{0-}^{\infty} \frac{d\tau_\infty^{(m)}(\mu)}{\mu - \lambda} , \qquad \lambda \in K_- . \tag{4.5}$$

In particular, by Krein's Theorem 2.2 and the remarks after this theorem, the string $S_\infty(m)$ is uniquely determined by $\Gamma$.
Assume $S_a(m)$ is a string ($a = 0$ or $= \infty$). Consider the right-continuous inverse function $m^d$ of $m$. Then, by definition of $S_a(m)$, we have $m^d(x) \equiv 0$, $x < 0$, if $a = 0$, and $m^d(x) \equiv -\infty$, $x < 0$, if $a = \infty$. Therefore, as the dual string $S_0^d(m)$ of $S_0(m)$ ($S_\infty^d(m)$ of $S_\infty(m)$) we define $S_0^d(m) := S_\infty(m^d)$ ($S_\infty^d(m) := S_0(m^d)$, respectively).
All quantities connected with the dual string are superscripted by $^d$. Note that it holds

$$l^d = m(l) \ , \quad h^d = \infty \ , \quad \text{if} \quad m(l-) + l < \infty, \ h \in [0,\infty), \tag{4.6}$$

$$l^d = m(l-), \quad h^d = m(\{l\}) < \infty \quad \text{if} \quad m(l-) + l < \infty \ , \ h = \infty, \tag{4.7}$$

$$l^d = m(l-), \quad \text{if} \quad m(l-) + l = \infty \ . \tag{4.8}$$

Moreover, we have

$$(S_0^d(m))^d = S_\infty^d(m^d) = S_0(m) \quad \text{and}$$

$$(S_\infty^d(m))^d = S_0^d(m^d) = S_\infty(m) \ .$$

<u>LEMMA 4.1:</u> For all $x \in [0,l)$ and all $\lambda \in K_-$ it holds with the notation $x_+ := \inf(E_m \cap (x,\infty))$

$$\Phi_0^{\uparrow,d}(m(x),\lambda) = -\lambda^{-1}\Phi_\infty^{\uparrow,+}(x,\lambda) = -\lambda^{-1}\Phi_\infty^{\uparrow,-}(x_+,\lambda),$$

$$\Phi_0^{\uparrow,d,+}(m(x),\lambda) = \Phi_\infty^{\uparrow}(x,\lambda) + (x_+-x)\Phi_\infty^{\uparrow,+}(x,\lambda) = \Phi_\infty^{\uparrow}(x_+,\lambda),$$

$$\Phi_\infty^{\uparrow,d}(m(x),\lambda) = \Phi_0^{\uparrow,+}(x,\lambda) = \Phi_0^{\uparrow,-}(x_+,\lambda)$$

$$\Phi_\infty^{\uparrow,d,+}(m(x),\lambda) = -\lambda\Phi_0^{\uparrow}(x,\lambda) - \lambda(x_+-x)\Phi_0^{\uparrow,+}(x,\lambda) = -\lambda\cdot\Phi_0^{\uparrow}(x_+,\lambda)$$

The equations remain valid for $x = l$ with $l_+ := l + h$ in the case $l + m(l-) < \infty$, $h \in [0,\infty)$.

The proof is similar to those of Proposition 2.3. Indeed we have to show that the right-hand side of the first und third equation under consideration satisfy the equations (2.8), (2.7) for $\Phi_0^{\uparrow,d}(m(x),\lambda)$ and $\Phi_\infty^{\uparrow,d}(m(x),\lambda)$, respectively.
The corresponding equations for the derivatives $\Phi_a^{\uparrow,d,+}(m(x),\lambda)$, $a = 0,\infty$ follow from (2.7), (2.8) by differentiation (the details are given in Neumann [10]).

<u>COROLLARY 4.2:</u> For every string $S_\infty(m)$ the characteristic functions $\Gamma(\lambda)$ and $\Gamma^d(\lambda)$ of $S_\infty(m)$ and $S_\infty(m^d)$, respectively, are connected by

$$\Gamma^d(\lambda) = \frac{-1}{\lambda\Gamma(\lambda)} \qquad \lambda \in K_- . \tag{4.9}$$

<u>Proof:</u> If $S_\infty(m)$ is regular and $h \in [0,\infty)$, then $l^d < \infty$ and $h^d = \infty$. Thus

$$\Gamma^d(\lambda) = \frac{\Phi_0^{\uparrow,d,+}(l^d,\lambda)}{\Phi_\infty^{\uparrow,d,+}(l^d,\lambda)} = -\frac{\Phi_\infty^{\uparrow}(l+h,\lambda)}{\lambda\,\Phi_0^{\uparrow}(l+h,\lambda)} = -\frac{1}{\lambda\Gamma(\lambda)}$$

If $h = \infty$, then $l^d + h^d < \infty$ and

$$\Gamma^d(\lambda) = \frac{\Phi_0^{\uparrow,d}(l^d+h^d,\lambda)}{\Phi_\infty^{\uparrow,d}(l^d+h^d,\lambda)} = -\frac{\Phi_\infty^{\uparrow,+}(l,\lambda)}{\lambda\,\Phi_0^{\uparrow,+}(l,\lambda)} = -\frac{1}{\lambda\Gamma(\lambda)} .$$

In the singular case the proof is obvious by $r = 1$, (4.4) and Lemma 4.1.

(For the singular case, (4.9) is well known from Kac, Krein [3].)

For singular strings $S_\infty(m)$ the following lemma is known (Kac, Krein [3], p. 83):

<u>LEMMA 4.3:</u> For the spectral measures $\tau_0^{(m)}$ and $\tau_\infty^{(m^d)}$ of $S_0(m)$ and $S_\infty(m^d)$, respectively, it holds

$$\tau_0^{(m)}(d\mu) = \mu\cdot\tau_\infty^{(m^d)}(d\mu) \qquad \text{on } R_+ . \tag{4.10}$$

<u>Proof:</u> We sketch the proof for the regular case $l + m(l-) < \infty$ only. Obviously, in this case we have $l^d + m^d(l^d-) < \infty$ also.
The spectrum of $D_m D_x$ with left boundary condition $af^-(0) - f(0) = 0$ consists of the zeros $\{\mu_k : k \geq 0\}$ of

$$\Phi_a^{\uparrow}(l+h,\cdot) = 0 \qquad \text{if} \quad h < \infty \quad \text{and}$$

$$\Phi_a^{\uparrow,+}(l,\cdot) = 0 \qquad \text{if} \quad h = \infty .$$

(See (2.10) above.)
Moreover, we have

$$\tau_a^{(m)}(\{\mu_k\}) = \left[\int_0^l [\,\Phi_a^{\uparrow}(x,\mu_k)]^2 m(dx)\right]^{-1} , \qquad k \geq 0 \tag{4.11}$$

($a = 0$ or $a = \infty$).

Firstly, let us assume $h < \infty$. Then $l^d = m(l)$ and $h^d = \infty$ (see (4.6)) and by Lemma 4.1 it holds

$$\Phi_{\infty}^{\uparrow, d, +}(l^d, \lambda) = -\lambda \Phi_0^{\uparrow}(r, \lambda) . \tag{4.12}$$

If $h = \infty$, then it follows also from (4.7) that $l^d = m(l-)$, $h^d < \infty$ and from Lemma 4.1 we get

$$\Phi_{\infty}^{\uparrow, d}(l^d + h^d, \lambda) = \Phi_0^{\uparrow, +}(l, \lambda). \tag{4.13}$$

Thus we get that the spectra of $S_0(m)$ and $S_{\infty}(m^d)$ outside of zero are the same.
Now, the assertion (4.10) follows from (4.11) and the formula

$$\lambda \int_0^x [\Phi_0^{\uparrow}(y, \lambda)]^2 m(dy) = \int_0^{m(x)} [\Phi_{\infty}^{\uparrow, d}(y, \lambda)]^2 m^d(dy), \qquad \lambda \in K_- . \tag{4.14}$$

(Use Lemma 4.1.)

Now we are ready to prove Theorem 3.2.
The property (3.4) immediately follows from (4.10) and (2.11). We have $c_m = m^d(0)$ and $m^d(0) = [\tau_{\infty}^{(m^d)}([0, \infty))]^{-1}$ (see (2.16)).
It is known that $\tau_{\infty}^{(m^d)}(\{0\}) > 0$ implies $l^d = \infty$ with $m^d(l^d -) < \infty$ or $l^d + m^d(l^d) < \infty$ with $h^d = \infty$. In both cases (2.17) implies

$$\tau_{\infty}^{(m^d)}(\{0\}) = (m^d(l^d))^{-1} = (l+h)^{-1} = r_m^{-1} .$$

(Put $h = 0$ if $m(l-) + l = \infty$.)
Thus we get

$$c_m^{-1} = r_m^{-1} + \int_{0+}^{\infty} \frac{d\tau_0^{(m)}(\mu)}{\mu} ,$$

i.e., (3.5) holds. Therefore (i) is proved.
The crucial point to show (ii) and (iii) is (4.10). Indeed, introduce for $s \geq 0$ measures $\sigma_s$ on $[0, \infty)$ by

$$\sigma_s(d\mu) := s \cdot \varepsilon_0(d\mu) + \tau_{\infty}^{(m^d)}(d\mu)\, \mathbb{1}_{(0,\infty)}(\mu) , \qquad \mu \geq 0,$$

where $\varepsilon_0$ denotes the measure concentrated with unit mass at zero.
Note that $\tau_{\infty}^{(m^d)}(\cdot) = \sigma_{r_m^{-1}}(\cdot)$ and $\tau_{\infty}^{(m^d)}(\{0\}) = r_m^{-1}$. Then by Krein's Theorem 2.2 for every $s \geq 0$ there exists a string $S_{\infty}(n_s)$ with $n_s(x) > 0$ for $x > 0$, i.e. $c_{n_s} = 0$, having $\sigma_s$ as its spectral measure.

From (2.17) it follows for $s \geq 0$ that $n_s(l_{n_s}) = s^{-1}$ with $s^{-1} = \infty$ if $s = 0$.

Put $q_s := n_s^d$, $s \geq 0$. Then the original $m$ is included for $s = r_m^{-1}$ and from (4.10) we get that the spectral measures $\tau_o^{(q_s)}$ do not depend on $s \geq 0$ and are equal to $\tau_o^{(m)}$. If $s > 0$ then

$$s^{-1} = \sigma_s(\{0\})^{-1} = (n_s(l_s)) = r_{q_s} < \infty , \tag{4.15}$$

and if $s = 0$ we get $n_o(l_o-) = \infty$, i.e. $l_{q_o} = \infty$.

Thus, among all $q_s$, $s \geq 0$ we find exactly one infinite string, namely $m_o$. Note that $q_s(0) = c_{n_s} \equiv 0$.

To finish the proof of Theorem 3.2 it suffices to identify the equivalence class $\hat{S}(m)$ introduced in Chapter 3 with $\{q_s \mid s \geq 0\}$.

We remark that the characteristic function $\Gamma_s$ of $q_s$ satisfies (see (4.9), (2.17))

$$\frac{1}{\Gamma_s(\lambda)} = -\lambda \Gamma_{n_s}(\lambda) = -\lambda\left(-\frac{s}{\lambda} + \int_{0-}^{\infty} \frac{d\tau_\infty^{(m^d)}(\mu)}{\mu - \lambda} + \frac{1}{r_m \lambda}\right)$$

$$= \left(s - \frac{1}{r_m}\right) - \lambda \Gamma_{m^d}(\lambda) = \left(s - \frac{1}{r_m}\right) + \frac{1}{\Gamma_m(\lambda)}, \qquad \lambda \in K_- . \tag{4.16}$$

Let us calculate the characteristic function of $S_\infty(m_t)$ with $m_t \in \hat{S}$, where $m_t$ was defined in Lemma 3.1.

<u>LEMMA 4.4:</u> For every $t \geq -\frac{1}{r_m}$ the corresponding to $m_t$ functions $\Phi^\uparrow_{o,t}$, $\Phi^\uparrow_{\infty,t}$ are given by

$$\Phi^\uparrow_{o,t}(x,\lambda) = (1-tx)\Phi^\uparrow_o\left(\frac{x}{1-tx}, \lambda\right) \tag{4.17}$$

$$\Phi^\uparrow_{\infty,t}(x,\lambda) = (1-tx)\Phi^\uparrow_\infty\left(\frac{1}{1-tx}, \lambda\right) + t(1-tx)\Phi^\uparrow_o\left(\frac{1}{1-tx}, \lambda\right) \tag{4.18}$$

<u>Proof:</u> The left hand sides of (4.17) and (4.18) are the unique solutions of (2.7) and (2.8) with $m$ replaced by $m_t$, respectively. After scale transformations and some calculations it is seen that the right-hand sides of (4.17) and (4.18) satisfy these equations. This proves the lemma.

<u>COROLLARY 4.5:</u> We have

$$\frac{1}{\Gamma_{m_t}(\lambda)} = \lim_{x \uparrow r_{m_t}} \frac{\Phi^\uparrow_{\infty,t}(x,\lambda)}{\Phi^\uparrow_{o,t}(x,\lambda)} = \frac{1}{\Gamma_m(\lambda)} + t , \qquad \lambda \in K_- . \tag{4.19}$$

The proof follows immediately from (4.1), (4.17) and (4.18).

Now, compare (4.19) with (4.16). From Krein's inverse spectral theorem we get $m_t = q_s$ for $t = s - r_m^{-1}$.
Thus Theorem 3.2 is proved.

As a consequence of (4.9), (4.10) we get the formula (2.15):

$$- \frac{1}{\Gamma_m(\lambda)} = \lambda \Gamma_{m^d}(\lambda) = \lambda \int_{0-}^{\infty} \frac{d\tau_\infty^{(m^d)}(\mu)}{\mu - \lambda}$$

$$= - \tau_\infty^{(m^d)}(\{0\}) - \int_{0+}^{\infty} \left(\frac{1}{\mu} - \frac{1}{\mu - \lambda}\right) d\tau_o^{(m)}(\mu)$$

$$= - r_m^{-1} - \int_{0+}^{\infty} \left(\frac{1}{\mu} - \frac{1}{\mu - \lambda}\right) d\tau_o^{(m)}(d\mu) , \qquad \lambda \in K_- . \tag{4.20}$$

Note, that we have supposed $m(0) = 0$. If some $m(\{0\}) > 0$ is added to $m$ at zero, the term $\lambda\, m(\{0\})$ is added on the right-hand side of (4.20).

The Corollary 3.3 follows immediately from the Theorem 3.2.

<u>Proof of Theorem 3.4:</u>
Let $S_a(m)$ be a string with $a \in (0,\infty)$ and $c_m = 0$. Put $w := a$ and define $\tilde{m}(x) := m(x-a)$, $x \in R$. Obviously, it holds $c_{\tilde{m}} = a$ and $r_{\tilde{m}} = r_m + a$.
If $\tau_a$ and $\tilde{\tau}_o$ denote the spectral measures of $S_a(m)$ and $S_o(\tilde{m})$, respectively, then we have by Proposition 2.3.(iii)

$$d\tau_a(\mu) = a^2 d\tilde{\tau}_o(\mu) , \qquad \mu > 0$$

From (3.5) it follows

$$\int_{0+}^{\infty} \frac{d\tau_a(\mu)}{\mu} = a^2 \int_{0+}^{\infty} \frac{d\tilde{\tau}_o(\mu)}{\mu} = a^2(a^{-1} - (r_m+a)^{-1}) = a(1 - \frac{a}{a+r_m}) ,$$

i.e. (3.6) and (3.7) hold.
Conversely, if $a \in (0,\infty)$ is fixed and $\tau$ is a measure on $(0,\infty)$ with $\tau((0,\infty)) > 0$ and (3.7) then choose a number $u \in (0,\infty]$ with

$$\int_{0+}^{\infty} \frac{d\tau(\mu)}{\mu} = a(1 - \frac{a}{a+u}) .$$

Put

$$\sigma(d\mu) := a^{-2}\tau(d\mu)\ , \qquad \mu \in (0,\infty),$$

and choose the string $S_o(m)$ with $m(0) = 0$ and $l_m = \infty$ having $\sigma$ as its spectral measure (see Theorem 3.2.(iii)).
By the same theorem, for every $s \in [0,\infty)$ the string $S_o(m_s)$ with

$$m_s(x) := (1 - sx)^2 m(\frac{x}{1 - sx})\ , \qquad x \in [0, s^{-1}],$$
$$= \infty \qquad x > s^{-1}$$

has the same spectral measure $\sigma$ as $S_o(m)$.
It holds by (3.5)

$$c_{m_s}^{-1} = \int_{0+}^{\infty} \frac{d\sigma(\mu)}{\mu} + r_{m_s}^{-1} = \int_{0+}^{\infty} \frac{d\sigma(\mu)}{\mu} + s = a^{-1}(1 - \frac{a}{a+u}) + s\ .$$

Now choose $s$ in such a way that $c_{m_s} = a$ holds, i.e. put $s = \frac{1}{a + u}$.
By shifting $m_s$ to the left

$$\tilde{m}_s(x) := m_s(x + a)$$

we get a string $S_a(\tilde{m}_s)$ with $c_{\tilde{m}_s} = 0$ having $\tau$ as its spectral measure. The uniqueness follows from the uniqueness of $S_o(m)$ with $l_m = \infty$.
For the last part of Theorem 3.4.(ii) note that the strings $S_o(\frac{m'(\cdot - a')}{(a')^2})$ and $S_o(\frac{m(\cdot - a)}{a^2})$ have the common spectral measure $\tau$ (see Proposition 2.3.(iii)).
From Theorem 3.2.(iii) it follows

$$S_o(\frac{m'(\cdot - a')}{(a')^2}) = S_o((\frac{m(\cdot - a)}{a^2})_t) \quad \text{with}$$

$$t = \frac{1}{r' - a'} - \frac{1}{r - a}\ .$$

<u>Proof of Proposition 3.5:</u>
Choose $a' \in (0,\infty]$ and consider a string $S_{a'}(m)$. Then it holds (see the definition of $r_{\lambda,a'}(x,y)$)

$$r_{\lambda,a'}(0,0) = \frac{\Phi^{\downarrow}(0,\lambda)}{\frac{1}{a'}\Phi^{\downarrow}(0,\lambda) + 1} = \frac{1}{\frac{1}{a'} + \frac{1}{\Gamma_m(\lambda)}} \tag{4.21}$$

and, by definition of the spectral measure $\tau_{a'}^{(m)}$,

$$r_{\lambda,a'}(0,0) = \int_0^\infty \frac{d\tau_{a'}^{(m)}(\mu)}{\mu - \lambda} . \qquad (4.22)$$

Now let be $a \in (0,\infty)$. Then (3.8) is a consequence of

$$-\frac{1}{\Gamma_m(\lambda)} \frac{1}{\frac{1}{a} + \frac{1}{\Gamma_m(\lambda)}} (a + \Gamma_m(\lambda)) = -a , \qquad (4.23)$$

(2.15), (4.21), (4.22) for $a' = a$ and $a' = \infty$.
Letting $a \downarrow 0$ in (4.23) divided by $a$ we get Knight's formula.

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