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Séminaire de probabilités (Strasbourg), tome 25 (1991), p. 31-36
<http://www.numdam.org/item?id=SPS_1991__25__31_0>

REALISATION OF A CLASS OF MARKOV PROCESSES
THROUGH UNITARY EVOLUTIONS IN FOCK SPACE

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1. Introduction: Pursuing the chain of ideas initiated in [1, 2, 3] and further discussed in [4] we modify the notations of quantum stochastic calculus in Fock space and demonstrate how a class of continuous as well as discrete state space Markov processes can be realised through unitary operator evolutions in the tensor product of an initial Hilbert space with a boson Fock space.

2. The basic results of quantum stochastic calculus in a new notation: Let

$$\tilde{H} = h_0 \otimes \Gamma(L^2(\mathbb{R}_+) \otimes k) \tag{2.1}$$

where h_0 and k are complex separable Hilbert spaces and for any Hilbert space H $\Gamma(H)$ denotes the boson Fock space over H. Put

$$h = h_0 \otimes (\mathbb{C}\, e_{-\infty} \oplus k \oplus \mathbb{C}\, e_{\infty}) \tag{2.2}$$

where $e_{\pm\infty}$ are unit vectors and $\oplus$ indicates Hilbert space direct sum. Fix an orthonormal basis $\{e_i | i \in S\}$ in k and put $\tilde{S} = S \cup \{-\infty\} \cup \{\infty\}$. The basic noise processes $\{\Lambda_j^i\}$ of boson stochastic calculus in $\tilde{H}$ can be expressed as

$$\Lambda_i^j = \Lambda_{|e_i\rangle\langle e_j|}, \quad i,j \in S,$$

$$\Lambda_{-\infty}^j = \Lambda_{|e_{-\infty}\rangle\langle e_j|} = A_j, \quad j \in S,$$

$$\Lambda_i^{\infty} = \Lambda_{|e_i\rangle\langle e_\infty|} = A_i^{\dagger}, \quad i \in S,$$

$$\Lambda_{-\infty}^{\infty}(t) = tI, \quad t \geq 0$$

where Λ_i^j, $i,j \in S$ are the conservation (or exchange) processes, A_j, $j \in S$ are the annihilation processes and $A_i^\dagger$, $i \in S$ are the creation processes. We adopt the convention that $\Lambda_i^{-\infty} = \Lambda_\infty^j = 0$.

Inspired by a conversation with V.P.Belavkin in Moscow in 1989 we introduce a subalgebra $\mathcal{I}(h) \subset \mathcal{B}(h)$ with a special involution as follows:

$$\mathcal{I}(h) = \{L \mid L \in \mathcal{B}(h),\ L\, f \otimes e_{-\infty} = L^* f \otimes e_\infty = 0 \text{ for all } f \in h_0\}, \tag{2.3}$$

$$L^b = F L^* F \tag{2.4}$$

where $\mathcal{B}(h)$ is the algebra of all bounded operators on h and F is the unique unitary (flip) operator in h satisfying

$$F\, f \otimes e_{-\infty} = f \otimes e_\infty,\ F\, f \otimes e_\infty = f \otimes e_{-\infty},\ F\, f \otimes u = f \otimes u$$

for all $f \in h_0$, $u \in k$. Then $\mathcal{I}(h)$ is a subalgebra of $\mathcal{B}(h)$ and the correspondence $L \to L^b$ is an involution under which $\mathcal{I}(h)$ is closed. To any $L \in \mathcal{I}(h)$ we associate the family $\{L_j^i \mid i, j \in \tilde{S}\}$ of operators in h_0 by putting

$$\langle f, L_j^i g\rangle = \langle f \otimes e_i, L\, g \otimes e_j\rangle,\ i,j \in \tilde{S},\ f,g \in h_0. \tag{2.5}$$

Then by (2.3)

$$L_j^\infty = L_{-\infty}^i = 0 \text{ for all } i,j \in \tilde{S},$$

$$\sum_{i \in \tilde{S}} \|L_j^i f\|^2 = \|L\, f \otimes e_j\|^2,\quad f \in h_0.$$

Hence by the basic results of quantum stochastic calculus (q.s.c.) there exists a unique adapted process Λ_L in $\tilde{H}$ satisfying

$$\Lambda_L(0) = 0,\ d\Lambda_L = \sum_{i,j \in \tilde{S}} L_j^i\, d\Lambda_i^j,\quad L \in \mathcal{I}(h). \tag{2.6}$$

(See, for example, Proposition 27.1 in [4]). The following two propositions are immediate from the methods of q.s.c. (Ch. III, [4]).

<u>Proposition 2.1</u>. The processes $\{\Lambda_L \mid L \in \mathcal{I}(h)\}$ defined by (2.6) satisfy the followin

(i) $\langle fe(u), \Lambda_L(t) ge(v)\rangle = \int_0^t \langle f\otimes(e_{-\infty}+u(s)), Lg\otimes(v(s)+e_\infty)\rangle ds \langle e(u), e(v)\rangle,$

(ii) If $\Lambda_L^\dagger(t) = \Lambda_{L^b}(t)$ then $\{\Lambda_L, \Lambda_L^\dagger\}$ is an adjoint pair;

(iii) $d\Lambda_L \, d\Lambda_M = d\Lambda_{LM}$.

In particular, Λ_L is independent of the orthonormal basis $\{e_i | i \in S\}$ employed in its definition.

Proposition 2.2. Let $L \in \mathcal{I}(h)$. Then there exists a unique unitary operator valued adapted process U_L satisfying the quantum stochastic differential equation (q.s.d.e.)

$$U_L(0) = 0 \; , \; dU_L = (d\Lambda_L) \, U_L$$

if and only if

$$L+L^b + L^b L = L+L^b + LL^b = 0. \tag{2.7}$$

If h_i, $i = 1,2$ are Hilbert spaces and X is a bounded operator in h_1 we adopt the convention of denoting by the same symbol X, the operator $X \otimes 1$ in $h_1 \otimes h_2$ where 1 dnotes the identify operator in h_2. For any $L \in \mathcal{I}(h)$ and $X \in \mathcal{B}(h_o)$ the operators XL and LX belong to $\mathcal{I}(h)$. Furthermore $X d\Lambda_L = d\Lambda_{XL}$, $(d\Lambda_L) X = d\Lambda_{LX}$.

Proposition 2.3. Let $L \in \mathcal{I}(h)$. Suppose (2.7) holds and U_L is the unitary operator valued process defined by Proposition 2.2. Then

$$d \, U_L^* \, X \, U_L = U_L^* \, d\Lambda_{L^b X + XL + L^b XL} \, U_L \quad \text{for all } X \in \mathcal{B}(h_o).$$

If

$$T_t(X) = \mathbb{E}_o \, U_L^*(t) \, X \, U_L(t)$$

where $\mathbb{E}_o$ denotes the boson vacuum conditional expectation map from $\mathcal{B}(\tilde{H})$ onto $\mathcal{B}(h_o)$ then $\{T_t | t \geq 0\}$ is a uniformly continuous one parameter semigroup of operators on the Banach space $\mathcal{B}(h_o)$ whose infinitesimal generator $\mathcal{L}$ is given by

$$\mathcal{L}(X) = \left. \frac{dT_t(X)}{dt} \right|_{t=0} ,$$

$$\langle f, \mathcal{L}(X) g \rangle = \langle f \otimes e_{-\infty}, (L^b X + XL + L^b XL) g \otimes e_\infty \rangle \quad \text{for all } f, g \in h_o.$$

<u>Proof:</u> Propositions 1-3 are the basic results of q.s.c. and we refer to Chapter III, [4]. □

3. <u>Construction of some classical Markov flows through unitary evolutions</u> :

Let G be a locally compact second countable group acting on a separable σ-finite measure space (χ, F, μ) with G-invariant measure μ. (Obvious generalizations can be worked out when μ is only quasi invariant). Define $h_o = L^2(\mu)$ and $k = L^2(G)$ with respect to a left invariant Haar measure. Express any element $\underline{f} \in h = h_o \otimes (\mathbb{C}\, e_{-\infty} \oplus k \oplus \mathbb{C}\, e_{\infty})$ as a column vector

$$\underline{f} = \begin{pmatrix} f_-(x) \\ f_o(x,g) \\ f_+(x) \end{pmatrix} \qquad x \in \chi \quad , g \in G.$$

Let $\lambda(x,g)$ be any complex valued measurable function on $X \times G$ satisfying

$$\text{ess. sup}_\mu \int_G |\lambda(x,g)|^2 \, dg < \infty \tag{3.1}$$

where dg indicates integration with respect to the left invariant Haar measure.
Define the operator L_λ associated with λ in h by

$$L_\lambda \underline{f} = \begin{pmatrix} -\int_G \{\overline{\lambda(x,g)}\, f_o(x,g) + \frac{1}{2}\, |\lambda(x,g)|^2 f_+(x)\} dg \\ f_o(g^{-1}x,\, g) - f_o(x,g) + \lambda(g^{-1}x,g)\, f_+(g^{-1}x) \\ 0 \end{pmatrix}$$

Then (3.1) implies that $L_\lambda \in \mathcal{B}(h)$. Furthermore the following holds:

(i) $L_\lambda \in \ (h)$;

$$L_\lambda^b \underline{f} = \begin{pmatrix} \int_G \{\overline{\lambda(x,g)}\, f_o(g\,x,g) - \frac{1}{2}\, |\lambda(x,g)|^2 \, f_+(x)\} \, dg \\ f_o(gx,g) - f_o(x,g) - \lambda(x,g)\, f_+(x) \\ 0 \end{pmatrix} ;$$

(iii) $L_\lambda^b L_\lambda + L_\lambda^b + L_\lambda = L_\lambda L_\lambda^b + L_\lambda + L_\lambda^b = 0.$

Using Proposition 2.2 construct the unitary operator valued process $U_\lambda = U_{L_\lambda}$ in

$\tilde{H}$ satisfying

$$U_\lambda(0) = 1,\ dU_\lambda = (d\Lambda_{L_\lambda})U_\lambda.$$

Consider the Evans-Hudson flow $\{j_t | t > 0\}$ induced by U_λ:

$$j_t(X) = U_\lambda(t)^* \, X \, U_\lambda(t),\ X \in \mathcal{B}(h_o).$$

If $\{e_i | i \in S\}$ is any fixed orthonormal basis in $L^2(G)$ then the structure maps $\{\theta^i_j | i,j \in \tilde{S}\}$ of the flow $\{j_t\}$ are given by

$$\theta^i_j(X) = (L^b_\lambda X + XL_\lambda + L^b_\lambda XL_\lambda)^i_j$$

with the convention $\theta^\infty_j = \theta^i_{-\infty} = 0$. Denote by A_o the abelian von Neumann algebra $L^\infty(\mu)$ where any function $\phi \in L^\infty(\mu)$ is interpreted as the operator of multiplication by ϕ in $L^2(\mu) = h_o$ Then a routine computation yields the following: θ^i_j leaves A_o invariant and

$$\theta^i_j(\phi)(x) = \int_G \phi(gx)\bar{e}_i(g)e_j(g)dg - \delta^i_j\,\phi(x),\quad i,j \in S,$$

$$\theta^{-\infty}_j(\phi)(x) = \int_G \overline{\lambda(x,g)}[\phi(gx) - \phi(x)]\ e_j(g)dg,\quad j \in S,$$

$$\theta^i_\infty(\phi)(x) = \int_G \lambda(x,g)\ \overline{e_i(g)}[\phi(gx) - \phi(x)]\ dg,\ i \in S,$$

$$\theta^{-\infty}_\infty(\phi)(x) = \int_G |\lambda(x,g)|^2[\phi(gx) - \phi(x)]dg.$$

It now follows from [2,3] (and also Section 27, 28 in [4]) that

$$[j_s(\phi),\ j_t(\psi)] = 0 \quad \text{for all } s,t \geq 0,\quad \phi,\psi \in A_o.$$

In other words $\{j_t|_{A_o},\ t \geq 0\}$ is a classical Markov flow in the Accardi-Frigerio-Lewis' formalism with infinitesimal generator L given by

$$L(\phi)(x) = \theta^{-\infty}_\infty(\phi)(x) = \int_G |\lambda(x,g)|^2\,[\phi(gx) - \phi(x)]dg.$$

Thus $\lambda(x,g)$ can be interpreted as the rate of change of amplitude density from the state x to the state gx.

When G and X are finite this result reduces to the description in [1, 3]. If G and X are countable we obtain the picture of a Markov flow in [2].

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