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1. Introduction. Results.

Let E be a separable Banach space and let E' its topological dual and E_1 the closed unit ball of E'. Our purpose in this paper will be to state a "'majorizing measure" type sufficient condition for checking the law of the iterated logarithm in Banach space. Let $X, X_1, X_2, \ldots$ be a sequence of independent identically distributed random variables with values in E. We denote, as usual, $S_n(X) = X_1 + \cdots + X_n$, $n \geq 1$ and $a(n) = \sqrt{2n \operatorname{llog} n}$, $n \geq 3$. We recall that the random variable X satisfies the bounded law of the iterated logarithm in E, $(BLIL)$, (resp. compact law of the iterated logarithm in E, $(CLIL)$), when the sequence $\{S_n(X)/a(n), n \geq 3\}$ is bounded in E almost surely, (resp. relatively compact in E almost surely). By way of preliminary, we recall the reduction theorem of Ledoux-Talagrand, ([3], theorem 1.1).

THEOREM 1.1.

a) *(BLIL) X satisfies the bounded LIL if, and only if, the following three conditions hold*

(1.1) $\mathsf{E}(\|X\|^2 \operatorname{loglog}\|X\|) < \infty$,

(1.2) *for each* $f \in E', \mathsf{E}(< x, f >^2) < \infty$,

(1.3) *the sequence* $\{S_n(X)/a(n), n \geq 3\}$ *is bounded in* E *in probability.*

b) *(CLIL) X satisfies the compact LIL if, and only if, the following three conditions hold*

(1.1) $E(\|X\|^2 / \log\log \|X\|) < \infty$,

(1.4) $\{\langle X, f\rangle^2 , f \in E_1\}$ *is uniformly integrable,*

(1.5) $S_n(X)/a(n) \to 0$ *as* $n \to \infty$, *in probability.*

This result, which reduces the problem from one of the almost sure behavior to one of the in-probability behavior, let in doubt the question of a possible condition (regarding X and E, instead of $S_n(X)$ and E) ensuring (1.3) or (1.5). Our goal here will be precisely of giving a such kind of condition. For, we introduce some useful notations :

Let $\phi_2(x) = e^{x^2} - 1$, and we consider the usual Orlicz norm associated to ϕ : given a probability $(\Omega, \mathcal{F}, \mu)$, we set for any element f of $L^{\phi_2}(\mu), \|f\|_{\phi_{2,\mu}} = \inf\{c > 0 : \int_\Omega \phi_2(f(x).c^{-1}) d\mu(x) \leq 1\}$.

We refer the reader to [2] for basic results on Orlicz spaces. Throughout this paper, we denote by $(\Omega_X, \mathcal{A}_X, P_X)$ the probability space of the sequence $X, X_1, X_2, \ldots$; we set also for any integer $p \geq 1$, $a_p = a(2^p)$. We introduce the following homogeneous pseudo metrics :

$$(1.6) \qquad \forall p \geq 1, \forall f, g \in E', d_p(f,g) = d_p(0, f-g) = \|\langle X^{(p)}, f-g\rangle\|_{\phi_2, P_X}.$$

Where $X^{(p)} = X.I(\|X\| \leq a_p)$.

We set afterwards for any integer $p \geq 1$,

$$\begin{aligned} &B_p = \{f \in E' : d_p(0,f) \leq 1\} \\ (1.7) \qquad &\mu_p = \inf_{\mu \in \mathbf{M}_1^+(B_p)} \sup_{f \in B_p} \int_0^1 \left(\frac{1}{\mu(B_{d_p}(f,u))}\right) du, \\ &\qquad \text{where } B_{d_p}(f,u) = f + \{g : d_p(0,g) \leq u\} \\ &\Delta_p = \sup\{d_p(0,f)\,,\, f \in E'_1\}. \end{aligned}$$

Our main result can be stated as follows.

THEOREM 1.2.
a) *(BLIL) In order that X satisfies the bounded LIL in E it is enough that conditions* (1.1), (1.2) *and*

$$(1.8) \qquad \limsup_{p \to \infty} \Delta_p \mu_p^2 / \sqrt{\log p} < \infty,$$

are fulfilled.
b) *(CLIL) In order that X satisfies the compact LIL in E, it is enough that conditions* (1.1), (1.4) *and*

$$(1.9) \qquad \lim_{p \to \infty} \Delta_p \mu_p^2 / \sqrt{\log p} = 0,$$

are fulfilled.

2. Preliminaries.

For proving theorem 1.2, we will use the following slight improvement of the well known result of [1]. Its proof is very similar to those of theorem 1.5 in [5].

THEOREM 2.1. — *Let $X = \{X_t, t \in T\}$ be a centered stochastic process, with basic probability space $(\Omega, \mathcal{A}, P)$. We assume that*

$$(2.1) \qquad \forall s,t \in T\ ,\ \|X_s - X_t\|_{\phi_2,P} \leq d(s,t),$$

where d is a pseudo-metric on T. Then for any Borel probability measure on T (i.e. $\mu \in \mathbf{M}_1^+(T)$).

$$(2.2) \quad \text{p.s.} \sup_{(s,t) \in T \times T} |X_s - X_t| \leq C\|X\|_{\phi_2,\mu\otimes\mu} \sup_{t \in T} \int_0^{\frac{\operatorname{diam}(T,d)}{2}} \phi_2^{-1}\left(\frac{1}{\mu(B_d(t,u))}\right) du$$

$$(2.3)\quad \text{p.s. } \forall s,t\in T \quad |X_s - X_t| \le C\|X\|_{\phi_2,\mu\otimes\mu} \sup_{t\in T}\int_0^{\frac{\operatorname{diam}(T,d)}{2}} \phi_2^{-1}\Big(\frac{1}{\mu(B_d(t,u))}\Big)du$$

$$(2.4)\qquad \|\sup_{(s,t)\in T\times T} X_s - X_t\|_{\phi_2,P} \le CI(T,d),$$

where

$$(2.5)\qquad I(T,d) = \inf_{\mu\in \mathbb{M}_1^+(T)} \sup_{t\in T}\int_0^{\frac{\operatorname{diam}(T,d)}{2}} \phi_2^{-1}\Big(\frac{1}{\mu(B_d(t,u))}\Big)du;$$

and $\widetilde{X} = \{(X_s - X_t)/d(s,t), s,t \in T, d(s,t) \neq 0\}$ and $0 < C < \infty$ is a numerical constant.

3. Proof of theorem 1.2.

By a classical symmetrization argument, it is enough to prove theorem 1.2 for symmetric random variables X. In that case, the sequence $X, X_1, X_2, \ldots$ has same law than the sequence $\varepsilon X, \varepsilon_1 X_1, \varepsilon_2 X_2, \ldots$ where $\varepsilon, \varepsilon_1, \varepsilon_1, \varepsilon_2, \ldots$ is a Rademacher sequence defined on another probability space $(\Omega_\varepsilon, \mathcal{A}_\varepsilon, P_\varepsilon)$.

Let p be fixed, we denote again $X^{(p)} = \{\langle X^{(p)}, f\rangle, f \in E'\}$. Then,

$$(3.1)\qquad \sup_{(f,g)\in E_1'}\left|\frac{X^{(p)}(f) - X^{(p)}(g)}{d_p(f,g)}\right| = \sup_{(f,g)\in E_1'}\left|\frac{X^{(p)}(f-g)}{d_p(f,g)}\right| \le \sup_{d_p(0,h)\le 1} |\langle X^{(p)}, h\rangle|.$$

But, $\|X^{(p)}(h) - X^{(p)}(h')\|_{\phi_2,P_X} = \|X^{(p)}(h-h')\|_{\phi_2,P_X} = d_p(h,h')$.

By virtue of theorem 2.1,

$$(3.2)\qquad \|\sup_{d_p(0,h)\le 1} \langle X^{(p)}, h\rangle\|_{\phi_2,P_X} \le C\mu p,$$

where $0 < C < \infty$ is a numerical constant, which may change from line to line. Set now for any integer $n \in [2^p, 2^{p+1}[$,

$$(3.3)\qquad \forall f \in E_1',\ \mathrm{U}_n(f) = \Big(\frac{1}{n}\sum_{j=1}^{n}\langle X_i^{(p)}, f\rangle^2\Big)^{1/2}.$$

Next we use the following elementary fact : if $\phi_1(x) = e^{|x|} - |x| - 1$, then,

(3.4) there exists a number $0 < C < \infty$ such that $\|f^2\|_{\phi_1,P_X} \le \|f\|_{\phi_2,P_X} \le C\|f^2\|_{\phi_1,P_X}$.

Consequently, we get

$$\| \sup_{d_p(0,h)\le 1} \mathsf{U}_n(h) \|_{\phi_2, P_X} \le C \cdot \mu_p. \tag{3.5}$$

Using then the triangular inequality for the l_2-norms, we also have,

$$\begin{aligned} \sup_{(f,g)\in E_1'} \frac{|\mathsf{U}_n(f) - \mathsf{U}_n(g)|}{d_p(f,g)} &\le \sup_{(f,g)\in E_1'} \left| \frac{\mathsf{U}_n(f-g)}{d_p(f,g)} \right| \\ &\le \sup_{d_p(0,h)\le 1} |\mathsf{U}_p(h)|, \end{aligned} \tag{3.6}$$

hence, finally,

$$\| \sup_{(f,g)\in E_1'} \left| \frac{\mathsf{U}_p(f-g)}{d_p(f,g)} \right| \|_{\phi_2, P_X} \le C\mu_p. \tag{3.7}$$

Let $M > 0$, and we set

$$A(M) = \left\{ \sup_{(f,g)\in E_1'} \left| \frac{\mathsf{U}_n(f-g)}{d_p(f,g)} \right| \le M\mu_p \right\}.$$

We have, from (3.7), $P_X\{A^c(M)\} \le \overline{e}^{CM^2}$, and on $A(M)$, denoting

$$\forall n \in [2^{p-1}, 2^p[, \forall f \in E_1', G_n(f) = \frac{1}{\sqrt{n}} \sum_{i=1}^{n} < X_i^{(p)}, f > \varepsilon_i,$$

and using a generalized version of the classical Kahane-Khintchine inequalities (see [4], p. 277) for Rademacher averages,

$$\|G_n(f) - G_n(g)\|_{\phi_2, P_\varepsilon} \le |U_n(f-g)| \le M\mu_p d_p(f,g). \tag{3.8}$$

Hence, in virtue of theorem 2.1, on $A(M)$, we have

$$\| \sup_{f\in E_1'} G_n(f) \|_{\phi_2, P_\varepsilon} \le MC(\mu_p)^2 \Delta_p. \tag{3.9}$$

Since, $\sup_{f\in E_1'} G_n(f) = \left\| \frac{S_n(X^{(p)}\varepsilon)}{\sqrt{n}} \right\|$; we deduce for any p and integer $n \in [2^{p-1}, 2^p[$,

$$\begin{aligned} P\left\{ \frac{\|S_n(X)\|}{\sqrt{2^n \, \mathrm{llog\,n}}} > M^2 \right\} &\le P\{\exists i \le 2^{p+1} : X_i \neq X_i^{(p)} \\ &\quad + \int P_X\{A^c(M)\} dP_\varepsilon + \int_{A(M)} P_\varepsilon \left\{ \frac{\|S_n(X^{(p)}\varepsilon)\|}{\sqrt{n}} > M^2 C \sqrt{\log p} \right\} dP_X \\ &\le 2^p P\{\|X\| > a_p\} + \exp(-CM^2) + \exp\left(-\frac{MC\sqrt{\log p}}{\Delta_p(\mu_p)^2} \right) 2. \end{aligned}$$

Taking into account assumptions (1.1) and (1.8), we thus see, for any $\varepsilon > 0$, that it is possible to find a real $M(\varepsilon) < \infty$ and integer $N(\varepsilon) < \infty$ such that for any $n \geq N(\varepsilon)$

$$P\left\{\frac{\|S_n(X)\|}{\sqrt{2n \operatorname{llog} n}} > M(\varepsilon)\right\} \leq \varepsilon.$$

Hence the bounded LIL is established. We deduce the compact LIL by means of theorem 1.1, and using a quite similar argumentation.

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