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MULTIPLICATIVE DECOMPOSITION OF NONSINGULAR MATRIX VALUED SEMIMARTINGALES

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1. INTRODUCTION

It was shown in Karandikar (1982) that a continuous semimartingale Z with values in the space of d×d nonsingular matrices admits a multiplicative decomposition

$$Z = NB$$

where N is a continuous local martingale and B is a continuous process of locally bounded variation. We extend this result to general semimartingales Z. In general, the decomposition is not unique. Conditions are given under which a decomposition with B predictable exists, and it is shwon that under these conditions, the decomposition is unique. An example is given of a bounded semimartingale Z which does not admit a decomposition with B predictable.

We obtain a formula for inverse of a multiplicative integral and also integration by parts formula for multiplicative stochastic integration, which like in Karandikar (1982) is the main tool of this paper. For multiplicative decomposition of real valued semimartingales, see Ito-Watanabe (1965), Meyer (1967), Jacod (1979).

2. PRELIMINARIES

Let $(\Omega,\mathcal{F},P)$ be a fixed complete probability space, and $\mathcal{F} = (\mathcal{F}_t)$ be a filtration satisfying usual hypothesis. All processes we consider are $(\mathcal{F}_t)$-adapted . For an integer d, L(d) will denote the set of all d×d matrices and $L_o(d)$ will denote the set of all invertible elements of L(d). For A L(d), $|A|$ will denote the Hillbert-Schmidt norm of A.

An L(d) valued process $X = (X_{ij})$ is said to be a semimartingale if each of its components X_{ij} is a semimartingale. For a rcll (cadlag) process X, X_- denotes the process : $X_-(t) = X(t-)$ for $t > 0$ and $X_-(0) = 0$. Here, X(t-) denotes the left limit at t.

For an L(d) valued locally bounded predictable process f and an L(d) valued semimartingale X, the stochastic integral $\int f dX$, which we denote by $f.X$ is defined by

$$(f.X)_{ij} = \sum_k \int f_{ik} dX_{kj}.$$

Also the integral $X:f$ is defined by $X:f = (f'.X')'$, where A' denotes transpose of A. Clearly

$$(X : f)_{ij} = \sum_k \int f_{kj} dX_{ik}.$$

For L(d) - valued semimartingales X,Y, let

$$[X,Y]_{ij} = \sum_k [X_{ik}, Y_{kj}]$$

and for continuous semimartingales X,Y (or L^2-local martingales X,Y), let

$$\langle X,Y\rangle = \sum_k \langle X_{ik}, Y_{kj}\rangle.$$

For a semimartingale $X = (X_{ij})$, X^c denotes its continuous martingale part defined componentwise by

$$(X^c)_{ij} = (X_{ij})^c.$$

For a process X, S(X) will denote the process

$$S(X) = \begin{cases} \sum_{0\le s\le t} X(s) & \text{if } \sum_{0\le s\le t} |X(s)| < \infty \\ 0 & \text{otherwise.} \end{cases}$$

For a rcll process X, the process ΔX is defined by

$$\Delta X = X - X_- .$$

It is well known that if X,Y are semimartingales, (real valued) then

$$\sum_{s\le t} |\Delta X(s)|\ |\Delta Y(s)| < \infty \quad \text{a.s.,} \tag{1}$$

and the same can be seen to hold for L(d) valued semimartingales. It follows as in real valued case that for L(d) valued semimartingales X,Y,

$$[X,Y] = \langle X^c, Y^c\rangle + S(\Delta X \Delta Y). \tag{2}$$

From now on, all processes we consider one L(d) valued. For semimartingales X,Y and locally bounded predictable processes f,g, the following identities are easy to verify

$$(f.X) : g = f.(X : g) \tag{3}$$

$$[f.X,Y:g] = f.[X,Y]:g \tag{4}$$

$$[X:f,Y] = [X,f.Y]. \tag{5}$$

In view of (2), f.X : g is unambigiously defined. The integration by parts formula takes the form

$$XY = X_-.Y + X : Y_- + [X,Y]. \tag{6}$$

It is well known that for a semimartingle X, the equation

$$Z = I + X(0) + Z_-.X \tag{7}$$

admits a unique solution. (See e.g. Emery (1978)). The solution Z to (7) is called multiplicative stochastic integral $\Pi(I + dX)$, which we denote by $\varepsilon(X)$. For more information on multiplicative stochastic integral, see Emery (1978), Karandikar (1981, 1982).

For a process Z, define Z^- by, $Z^-(t) = Z(t-)$ for $t > 0$ and $Z^-(0) = I$. The equation (7) can be rewritten as

$$Z = I + Z^-.X \tag{8}$$

From this it follows that $Z = Z^-(I+\Delta X)$ and hence a necessary condition for Z to be $L_o(d)$ valued is that $(I+\Delta X)$ be invertible. We show in the next result that the condition is sufficient and obtain a formula for the inverse Z^{-1}.

For a process Y and a subset E of L(d), we say $Y \notin E$ if

$$P(w : Y(t,w) \in E \text{ for some } t) = 0.$$

THEOREM 1: Let X be a semimartingale such that $(I + \Delta X) \in L_o(d)$ and let $Z = \varepsilon(X)$. Then $Z, Z^- \in L_o(d)$ and

$$[\varepsilon(X)]^{-1} = [\varepsilon(Y')]' \tag{9}$$

where

$$Y = -X + \langle X^c,X^c\rangle + S((I + \Delta X)^{-1} - I + \Delta X) \tag{10}$$

Proof: Let $W = [\varepsilon(Y')]'$. It follows that

$$W = I + Y : W^- = I + Y(0) + Y : W_-$$

Note that $\langle X^c,Y^c\rangle = -\langle X^c,X^c\rangle$ and that $\Delta Y = (I + \Delta X)^{-1} - I$. From this it is easy to check that

$$X + Y + [X,Y] = 0 \tag{11}$$

and that $(I + \Delta X(0))(I + \Delta Y(0)) = I$. Now by (6),

$$\begin{aligned} ZW &= Z_-.W+Z : W_-+[Z,W] \\ &= Z_-.Y : W_-+Z_-.X : W_-+(I+\Delta X(0))(I+\Delta Y(0))+[Z_-.X,Y : W_-] \\ &= Z_-.(Y+X+[X,Y]):W_- + I \\ &= I \end{aligned}$$

using (11). This proves $Z, Z_- \in L_o(d)$ and that $Z^{-1} = W$.

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REMARK: It has been pointed out to the author that this result has appeared in an article by R.Leandre (Sem. Prob. XLX, p. 271). This result can be restated as : if Z is a solution to

$$Z = I + Z^-.X,$$

then Z^{-1} is the unique solution to

$$W = I + Y : W^-$$

where Y is given by (10).

Let M_{loc} denote the class of local martingales (L(d) valued) and V denote the class of processes with bounded variation paths on bounded intervals.

For a semimartingale Z such that $Z, Z^- \in L_o(d)$, let $\mathcal{L}(Z)$ be defined by

$$\mathcal{L}(Z) = (Z^-)^{-1}. (Z-I). \tag{12}$$

Then clearly $\mathcal{L}(Z)$ is also a semimartingale. The following result follows easily from the definitions and well known properties of stochastic integrals.

THEOREM 2: Let Z be a semimartingale such that $Z, Z^- \in L_o(d)$ and let X be a semimartingale such that $I + \Delta X \in L_o(d)$. Then

(i) $\varepsilon(\mathcal{L}(Z)) = Z$

(ii) $\mathcal{L}(\varepsilon(X)) = X$

(iii) $Z \in M_{loc} \iff \mathcal{L}(Z) \in M_{loc}$

(iv) $Z \in V \iff \mathcal{L}(Z) \in V$

(v) $X \in M_{loc} \iff \varepsilon(X) \in M_{loc}$

(vi) $X \in V \iff \varepsilon(X) \in V$.

3. MULTIPLICATIVE DECOMPOSITION

In Karandikar (1982), it was proved that an $L_o(d)$ valued continuous semimartingale Z admits a multiplicative decomposition Z = NB, into a local martingale N and $B \in V$. This result for real valued case is proved in Ito-Watanabe (1965) and Meyer (1967). For a complete discussion of the real valued case, see Jacod (1979).

Here we will show that a semimartingale Z with $Z, Z^- \in L_o(d)$ admits a decomposition Z = NB with $N \in M_{loc}$, $B \in V$. Of course in general the decomposition is not unique, just as additive decomposition is not unique. It will be proved that if such a decomposition exists with B predictable, then such a decomposition is unique.

We will give a counter example to show that even if Z is a special semimartingale, it may not admit a decomposition Z = NB with B predictable (and $N \in M_{loc}$, $B \in V$).

The main tool in Karandikar (1982) was integration by parts formula for the multiplicative integral. We need its analogue for rcll semimartingales, which we obtain next.

THEOREM 3: Let X,Y be semimartingales such that

$$(I + \Delta X) \in L_o(d) \text{ and } (I + \Delta Y) \in L_o(d).$$

Then

$$\varepsilon(X+Y+[X,Y]) = \varepsilon(W^-.X : (W^-)^{-1})\varepsilon(Y) \tag{13}$$

where $W = \varepsilon(Y)$.

Proof: Let $Z = \varepsilon(W^-.X : (W^-)^{-1})$. Then by (6)

$$\begin{aligned} ZW &= Z_-.W+Z : W_-+[Z,W] \\ &= (Z_-W_-).Y + (Z_-W^-).X : ((W^-)^{-1}W_-)+(I+X(0))(I+Y(0)) \\ &\qquad +((Z_-W^-).[X : (W^-)^{-1},W_-.Y] \\ &= (Z_-W_-).Y+(Z_-W_-).X:I.1_{(0,\infty)}+(I+X(0))(I+Y(0))+Z_-W_-.[X,I\,1_{(0,\infty)}.Y]. \end{aligned}$$

Using the fact that if $U(0) = 0$, $U : 1_{(0,\infty)} = U = 1_{(0,\infty)}.U$, it follows that

$$\begin{aligned} ZW &= (Z_-W_-).Y+(Z_-W_-).X+(Z_-W_-)1_{(0,\infty)}.[X,Y]+(I+X(0))(I+Y(0)) \\ &= (Z_-W_-).(X+Y+[X,Y])+(I+X(0))(I+Y(0)) \\ &= (Z_-W_-).(X+Y+[X,Y])+I+X(0)+Y(0)+[X,Y](0). \end{aligned}$$

Hence

$$ZW = \varepsilon(X+Y+[X,Y]).$$

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The idea of obtaining multiplicative decomposition is as follows. Let Z be a semimartingale with $Z,Z^- \in L_o(d)$ and let $X = \mathcal{L}(Z)$. If X can be written as

$$X = M + A + [M,A] \tag{14}$$

with

$$M \in \mathcal{M}_{loc},\ A \in \mathcal{V} \tag{15}$$

$$(I + \Delta A) \in L_o(d) \tag{16}$$

then $(I+\Delta X) = (I+\Delta M)(I+\Delta A)$ and hence $(I+\Delta M) \in L_o(d)$. Then by integration by part formula, $Z = NB$, where $B = \varepsilon(A)$, $N = \varepsilon(B^-.M : (B^-)^{-1})$ and by Theorem 3, $B \in \mathcal{V}$, $N \in \mathcal{M}_{loc}$. Thus the whole thing reduces to obtaining a decomposition (14) satisfying (15) and (16). We show that this can be done in the next result.

THEOREM 4: Let Z be a semimartingale such that $Z, Z^- \in L_o(d)$. Then Z admits a decomposition

$$Z = NB \tag{17}$$

where $N \in \mathcal{M}_{loc}$, $B \in \mathcal{V}$.

Proof: Let $X = \mathcal{L}(Z)$. For $0 < a \leq \frac{1}{2}$ (fixed), let

$$Y = X - S(\Delta X\, 1_{\{|\Delta X| \geq a\}}). \tag{18}$$

Then $|\Delta Y| \leq a$ and hence Y is a special semimartingale. Let $Y = U + V$ be the canonical decomposition of Y, with $U(0) = 0$, $U \in \mathcal{M}_{loc}$, $V \in \mathcal{V}$

Since $U \in \mathcal{M}_{loc}$ and $U(0) = 0$ one has ${}^P(\Delta U) = 0$, (hence ${}^PD$ denotes predictable projection of a process D) and hence ${}^P\Delta Y = \Delta V$, since ΔV is predictable. Thus

$$|\Delta V| = |{}^P\Delta Y| \leq {}^P|\Delta Y| \leq a. \tag{19}$$

This implies $(I + \Delta V) \in L_o(d)$. Let

$$M = U : (I + \Delta V)^{-1}.$$

This is well defined as $(I + \Delta V)^{-1}$ is a bounded predictable process and as a consequence $M \in \mathcal{M}_{loc}$, $M(0) = 0$. Then

$$U = M : (I + \Delta V)$$
$$= M + S(\Delta M \Delta V)$$
$$= M + [M,V].$$

Thus $Y = U + V = U + M + [M,V]$. As a consequence

$$(I + \Delta Y) = (I + \Delta M)(I + \Delta V)$$

and hence $(I + \Delta M) \in L_o(d)$. Now

$$X = M + V + [M,V] + S(\Delta X\, 1_{(|\Delta X| \geq a)})$$
$$= M + W$$

where $W \in \mathcal{V}$. Define $A \in \mathcal{V}$ by

$$A = W + S(\{(I + \Delta M)^{-1} - I\}\, \Delta W).$$

Then it is easy to check that $[M,A] = W-A$ and hence that

$$M + A + [M,A] = M + W = X.$$

Here $(I + \Delta M) \in L_o(d)$ and since $(I + \Delta X) \in L_o(d)$, $(I + \Delta A) \in L_o(d)$ as well. Then if

$$B = \varepsilon(A)$$

and

$$N = \varepsilon(B^{-}.M : (B^{-})^{-1})$$

then by integration by points formula, $NB = \varepsilon(M + A + [M,A]) = \varepsilon(X)$ and hence $Z = NB$.

□

It can be easily seen that the decomposition $Z = NB$ need not be unique unless we require that B be predictable.

The questions then are (i) when does a decomposition $Z = NB$ exist with B predictable? (ii) If such a decomposition exists, is it unique?

These questions are answered in the next result.

It suffices to consider the case $Z(0) = I$ for a given Z can be written as

$$Z(t) = \tilde{Z}(t).\, Z(0)$$

where $\tilde{Z}(t) = Z(t).[Z(0)]^{-1}$.

For a process W, let $|W|^*$ denote the real valued process

$$|W|_t^* = \sup\{|Z_s| : 0 \leq s \leq t\}.$$

THEOREM 5: Let Z be a semimartingale such that $Z(0) = I$; $Z^{-} \in L_o(d)$. Then Z admits a decomposition

$$Z = NB$$

with

$$N \in \mathcal{M}_{loc}, N(0) = I,\ B \in \mathcal{V} \text{ and } B \text{ predictable} \tag{20}$$

if and only if

$$|Z|^* \text{ is locally integrable} \tag{21}$$

and

$$^{P}Z \in L_o(d). \tag{22}$$

Finally, if (20), (21) hold then the decomposition (18) satisfying (19) is unique.

Proof: Suppose the decomposition (18) exists with N, B satisfying (19). Then $|N|^*$ is locally integrable and $|B|^*$ is locally bounded since B is predictable. Thus $|Z|^*$ is locally integrable. Also, (18) implies that $N, N^-, B, B^- \in L_o(d)$. So let $U = \mathcal{L}(N)$, $V = \mathcal{L}(B)$ and $M = (B^-)^{-1}. U : B^-$ so that

$$N = \varepsilon(U) = \varepsilon(B^-. M : (B^-)^{-1}).$$

Then by integration by parts formula,

$$Z = \varepsilon(M + V + [M,V])$$

and thus

$$(Z^-)^{-1}Z = (I + \Delta M + \Delta V + \Delta M \Delta V)$$
$$= (I + \Delta M)\ (I + \Delta V).$$

Now (19) implies that $M \in \mathcal{M}_{loc}$, $M(0) = 0$, $V \in \mathcal{V}$ and V is predictable. Taking predictable projection on both sides in (22) and using that $Z^-, \Delta V$ are predictable and ${}^P\Delta M = 0$ one gets

$$(Z^-)^{-1}({}^PZ) = (I + \Delta V)$$

which yields

$${}^PZ = (Z^-)(I+\Delta V).$$

Since $Z^- \in L_o(d)$ and $(I + \Delta V) = (B^-)^{-1}B \in L_o(d)$ as observed above, this yield (21).

Conversely suppose (20), (21) hold. Then (20) implies that Z is a special semimartingale and this $X = \mathcal{L}(Z)$ is also a special semimartingle. Let $X = U + V$ be the canonical decomposition of X with $U \in \mathcal{M}_{loc}$, $U(0) = 0$, $V \in \mathcal{V}$, V predictable. Then

$$(Z^-)^{-1}Z = (I + \Delta X)$$
$$= (I + \Delta U + \Delta V)$$

and thus taking predictable projection, one gets

$$(Z^-)^{-1}\ {}^P(Z) = (I + \Delta V)$$

and hence that $(I + \Delta V) \in L_o(d)$ in view of (21). Now defining $M = U : (I + \Delta V)^{-1}$, one has $M \in \mathcal{M}_{loc}$ and that

$$M + V + [M,V] = U + V = X$$

as in the proof of Theorem 4. This yields

$$Z = \varepsilon(X) = \varepsilon(M + V + [M,V])$$
$$= NB$$

with $B = \varepsilon(V)$ and $N = \varepsilon(B^-.M : (B^-)^{-1})$. Clearly N,B satisfy (19).

Remains to prove that if (20), (21) hold, then the decomposition is unique subject to (19). Let $Z = N_1B_1 = N_2B_2$ where N_i, B_i satisfy (19). (Here the suffix 1,2 does not represent component). Let $U_i = \mathcal{L}(N_i)$, $V_i = \mathcal{L}(B_i)$ and $M_i = (B_i^-)^{-1}$, $U_i. (B_i^-)$. Then it follows that for i = 1,2,

$$Z = \varepsilon(M_i + V_i + [M_i,V_i])$$

for i = 1,2. Thus

$$X = \mathcal{E}(Z) = M_i + V_i + [M_i,V_i]$$

with $V_i \in \mathcal{V}$, V_i predictable, $M_i \in \mathcal{M}_{loc}$, $M_i(0) = 0$. It follows that $[M_i,V_i] \in \mathcal{M}_{loc}$ and $[M_i,V_i](0) = 0$. Thus defining $W_i = M_i + [M_i,V_i]$, one gets

$$X = W_i+V_i \tag{23}$$

with $W_i \in \mathcal{M}_{loc}$, $W_i(0) = 0$. The uniqueness of additive decomposition (23) yields $V_1 = V_2$, which in turn implies $B_1 = B_2$ as $B_i = \varepsilon(V_i)$. Since $B_i \in L_o(d)$, this clearly implies $N_1 = N_2$ as well completing the proof. □

We will now give an example to show that (20) does not imply (21). This shows that (20) is not sufficient to guarantee the decomposition (18) - (19). Indeed, this example is for the case d = 1.

Let $\Omega = \{1, -1\}$, $A = \mathcal{P}(\Omega)$, $P(\{1\}) = P(\{-1\}) = \frac{1}{2}$, and $\eta(w)$ for $w \in \Omega$. Let $F_t = \{\phi,\Omega\}$ for $t < 1$ and $F_t = P(\Omega)$ for $t \geq 1$. Let

$$Z(t) = 1_{[0,1)}(t) + \eta 1_{[1,\infty)}(t). \tag{24}$$

Clearly Z is a bounded semimartingale and $Z(t) \neq 0$ for all t. However, $({}^PZ)(t) = 1_{[0,1)}(t) + \eta 1_{(1,\infty)}(t)$ and thus $({}^PZ)(1) = 0$.

The above example shows that a special semimartingale need not admit a decomposition (18) with B-predicable.

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