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The Azéma Martingales as Components of Quantum Independent Increment Processes

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Inspired by the work of J. Azéma [3], M. Emery and P.A. Meyer, K.R. Parthasarathy investigated the quantum stochastic differential equation

$$\mathrm{d}X = (c-1)X\mathrm{d}\Lambda + \mathrm{d}A^{\dagger} + \mathrm{d}A$$

for a real number c; see [7]. The solution of such an equation is called an Azéma martingale. We demonstrate how an Azéma martingale can be regarded as a component of a quantum independent stationary increment process in the sense of [2].

A classical stochastic process (X_{st}) taking values in a semi-group G and indexed by pairs $(s,t) \in \mathbb{R}_{+}^{2}$, $s \leq t$, is an increment process if

$$X_{rs}X_{st} = X_{rt}, \ r \leq s \leq t,$$
$$X_{tt} = e, \ e \text{ the unit element of } G.$$

To give sense to increments in the non-commutative case, we replace the group by a *-bialgebra. This object is defined as follows. A coalgebra $\mathcal{C}$ is a (complex) vector space on which two linear mappings

$$\Delta : \mathcal{C} \to \mathcal{C} \otimes \mathcal{C} \qquad \text{(comultiplication)}$$
$$\delta : \mathcal{C} \to \mathbb{C} \qquad \text{(counit)}$$

are given such that

$$(\Delta \otimes \mathrm{id}) \circ \Delta = (\mathrm{id} \otimes \Delta) \circ \Delta \qquad \text{(coassociativity law)}$$
$$(\delta \otimes \mathrm{id}) \circ \Delta = \mathrm{id} = (\mathrm{id} \otimes \delta) \circ \Delta \qquad \text{(counit property).}$$

A *-bialgebra is a *-algebra which is also a coalgebra in such a way that Δ and δ are *-algebra homomorphisms.

The vector space $L(\mathcal{C}, \mathcal{A})$ formed by the linear mappings from a coalgebra $\mathcal{C}$ to a (complex, unital) algebra $\mathcal{A}$ is an algebra with the multiplication

$$R \star S = M \circ (R \otimes S) \circ \Delta$$

where $M : \mathcal{A} \otimes \mathcal{A} \to \mathcal{A}$ denotes multiplication in $\mathcal{A}$. The unit of $L(\mathcal{C}, \mathcal{A})$ is given by $b \mapsto \delta(b)\mathbf{1}$. Especially, the algebraic dual space $\mathcal{C}^* = L(\mathcal{C}, \mathbb{C})$ of a coalgebra $\mathcal{C}$ is an algebra (with unit δ).

If the *-bialgebra $\mathcal{B}$ has an antipode, that is a linear operator S on $\mathcal{B}$ such that $S \star \mathrm{id} = \mathrm{id} \star S = \delta\mathbf{1}$ (i.e. S is the inverse of the identity with respect to $\star$), then we call $\mathcal{B}$ a *-Hopf algebra.

EXAMPLES:

1) Let G be a semi-group. The semi-group algebra $\mathbb{C}G$ is a *-bialgebra if we define * by antilinear extension of $x^* = x^{-1}$ and Δ and δ by linear extension of $\Delta x = x \otimes x$, $\delta x = 1$, $x \in G$. If G is a group $\mathbb{C}G$ is a *-Hopf algebra with $S(x) = x^{-1}$.

2) Let G be a sub-semi-group of the semi-group $M_{\mathbb{C},d}$ of complex $d \times d$-matrices. Then we denote by $G[d]$ the *-algebra of complex-valued functions on G generated by the functions ξ_{kl}, $k, l = 1, \ldots, d$, which map an element $(\alpha_{mn})_{m,n=1,\ldots,d}$ of G to α_{kl}. If we set

$$\Delta\xi_{kl} = \sum_{n=1}^{d} \xi_{kn} \otimes \xi_{nl}$$
$$\delta(\xi_{kl}) = \delta_{kl}$$

we can extend Δ and δ to *-algebra homomorphisms in a unique way. $G[d]$ becomes a *-bialgebra. We call $G[d]$ the coefficient algebra of G.

3) Denote by $M_{\mathbb{C}}\langle d\rangle$ the free algebra generated by indeterminates x_{kl} and x_{kl}^*, $k, l = 1, \ldots, d$. The mappings *, Δ and δ are given by extending

$$(x_{kl})^* = x_{kl}^*$$
$$\Delta x_{kl} = \sum_{n=1}^{d} x_{kn} \otimes x_{nl} \tag{1}$$
$$\delta x_{kl} = \delta_{kl} \tag{2}$$

in the unique way which makes * an involution and Δ and δ *-algebra homorphisms. Similarly, $M_{\mathbb{R}}\langle d\rangle$ is defined as the free algebra generated by x_{kl}, $k, l = 1, \ldots, d$, with the involution given by $(x_{kl})^* = x_{kl}$ and Δ and δ again defined by (1) and (2). $M_{\mathbb{R}}\langle d\rangle$ is a quotient (i.e. a homomorphic image) of $M_{\mathbb{C}}\langle d\rangle$ (it has the additional relations $x_{kl}^* = x_{kl}$). If we make $M_{\mathbb{K}}\langle d\rangle$ commutative we obtain the coefficient algebra $M_{\mathbb{K}}[d]$ of $M_{\mathbb{K},d}$, $\mathbb{K} = \mathbb{C}$ or $\mathbb{R}$. Any $G[d]$ of Example 2 is a quotient of $M_{\mathbb{R}}[d]$ or at least of $M_{\mathbb{C}}[d]$.

4) Denote by $\mathbb{C}\langle x_1, \ldots, x_d\rangle = \mathbb{C}\langle d\rangle$ the free algebra generated by indeterminates $x_1, \ldots, x_d$. We extend the mappings *, Δ and δ with

$$(x_l)^* = x_l$$
$$\Delta x_l = x_l \otimes \mathbf{1} + \mathbf{1} \otimes x_l$$
$$\delta x_l = 0$$

to obtain a *-bialgebra which is a quotient of $M_{\mathbb{R}}\langle 2d\rangle$. The *-bialgebra $\mathbb{C}\langle d\rangle$ is a *-Hopf algebra with antipode $S(x_{l_1} \ldots x_{l_n}) = (-1)^n x_{l_n} \ldots x_{l_1}$.

5) Divide $M_{\mathbb{C}}\langle d\rangle$ by the ideal J_U generated by the elements

$$\sum_{n=1}^{d} x_{kn} x_{ln}^* - \delta_{kl}\mathbf{1},$$
$$\sum_{n=1}^{d} x_{nk}^* x_{nl} - \delta_{kl}\mathbf{1}.$$

Then J_U is a *-biideal. We denote the *-bialgebra $M_{\mathbb{C}}\langle d\rangle/J_U$ by $U\langle d\rangle$. It can be shown that $U\langle d\rangle$ has no antipode.
6) By making $U\langle d\rangle$ commutative one obtains the coeficient algebra $U[d]$ of the group U_d of unitary $d\times d$-matrices; see [5] where $U\langle d\rangle$ was called the non-commutative analogue of the coefficient algebra of U_d and where a structure theorem for $U\langle d\rangle$ was proved. $U[d]$ is a *-Hopf algebra with the *-algebra homomorphism $S(x_{kl})=x^*_{lk}$ as the antipode.
7) Consider in $M_{\mathbb{R}}\langle 2\rangle$ the ideal generated by the elements $x_{11}-\mathbf{1}$ and x_{21}. This is a *-biideal. We denote the quotient *-bialgebra by $H_0\langle 2\rangle$. It is equal to the free algebra $\mathbb{C}\langle x,y\rangle$ generated by two indeterminates x and y with the involution $x^*=x$, $y^*=y$, and Δ and δ given by

$$\Delta x = x\otimes y+\mathbf{1}\otimes x,\ \ \delta x=0$$
$$\Delta y = y\otimes y,\ \ \delta y=1.$$

8) By making $H_0\langle 2\rangle$ commutative one obtains the coefficient algebra $H_0[2]$ of the semigroup

$$H_0=\{\begin{pmatrix}1 & \alpha\\ 0 & \beta\end{pmatrix} : \alpha,\beta\in\mathbb{R}\}.$$

The set of complex-valued *-algebra homomorphisms on $H_0[2]$ equipped with $\star$ as the multiplication is isomorphic to H_0.
9) A *-Hopf algebra $H\langle 2\rangle$ containing $H_0\langle 2\rangle$ as a sub-*-bialgebra is obtained if we divide the *-bialgebra $\mathbb{C}\langle x,y,y^{-1}\rangle$ with

$$\begin{aligned}\Delta x &= x\otimes y+\mathbf{1}\otimes x,\ \ \delta x=0\\ \Delta y &= y\otimes y,\ \ \delta y=1\\ \Delta y^{-1} &= y^{-1}\otimes y^{-1},\ \ \delta y^{-1}=1\\ x^* &= x,\ y^*=y,\ (y^{-1})^*=y^{-1}\end{aligned}$$

by the *-biideal generated by the elements $yy^{-1}-\mathbf{1}$ and $y^{-1}y-\mathbf{1}$. An antipode is given by extending $S(x)=xy^{-1}$, $S(y)=y^{-1}$, $S(y^{-1})=y$, to a linear anti-homomorphism; see [12].
10) We can make $H\langle 2\rangle$ commutative to obtain the *-Hopf algebra H_2. The set of complex-valued *-algebra homomorphisms on H_2 is isomorphic to the group

$$H=\{\begin{pmatrix}1 & \alpha\\ 0 & \beta\end{pmatrix} : \alpha,\beta\in\mathbb{R},\beta\neq 0\},$$

but H_2 is not equal to $H[2]=H_0[2]$.

GENERAL THEORY:

Let (j_{st}) be a quantum stochastic process in the sense of Accardi, Frigerio and Lewis [1], indexed by pairs $(s,t)\in\mathbb{R}^2_+$, $s\leq t$. The j_{st} are *-algebra homomorphisms from a *-algebra $\mathcal{B}$ to a *-algebra $\mathcal{A}$ where there is also given a state Φ on $\mathcal{A}$. Let $\mathcal{B}$ be a *-bialgebra. We call (j_{st}) a quantum independent stationary increment process if the following conditions are fulfilled (see [2])

(a) $j_{rs}\star j_{st}=j_{rt}$, $r\leq s\leq t$; $j_{tt}=\delta\mathbf{1}$

(b1) The algebras $j_{st}(\mathcal{B})$ and $j_{s't'}(\mathcal{B})$ commute for disjoint intervals (s,t) and (s',t').
(b2) The state Φ factorizes on the sub-algebras $j_{t_1t_2}(\mathcal{B}),\ldots,j_{t_nt_{n+1}}(\mathcal{B})$ of $\mathcal{A}$ for $n\in\mathbb{N}$, $t_1<\cdots<t_{n+1}$.
(c) The states $\Phi\circ j_{st}$ only depend on the difference $t-s$, i.e. $\Phi\circ j_{st}=\varphi_{t-s}$.
(d) $\lim_{t\downarrow 0}\varphi_t(b)=\delta(b)$ for all $b\in\mathcal{B}$.

Two independent stationary increment processes are called equivalent if the numbers $\Phi(j_{s_1t_1}(b_1)\ldots j_{s_nt_n}(b_n))$ are the same for both processes.

Let $\mathcal{B}$ be a *-Hopf algebra and let $(j_t)_{t\in\mathbb{R}_+}$ be a quantum stochastic process over $\mathcal{B}$ in the sense of Accardi, Frigerio and Lewis. Then $j_{st}=(j_s\circ S)\star j_t$ satisfies (a), and (j_t) is called a process with independent and stationary increments if (j_{st}) is an independent stationary increment process.

An independent stationary increment process (j_{st}) is, up to equivalence, determined by its (infinitesimal) generator ψ which is the linear functional on $\mathcal{B}$ given by

$$\psi(b)=\frac{\mathrm{d}}{\mathrm{d}t}\varphi_t(b)|_{t=0}.$$

The set of generators coincides with the elements in $\mathcal{B}$ satisfying

$$\psi(\mathbf{1})=0$$
$$\psi\lceil\text{Kern }\delta\text{ is positive}$$
$$\psi(b^*)=\overline{\psi(b)}.$$

Given ψ satisfying these properties, one can make the following construction (see [9], cf. [8, 6]). Divide $\mathcal{B}$ by the null space of the positive semi-definite sesquilinear form

$$(b,c)=\psi((b-\delta(b)\mathbf{1})^*(c-\delta(c)\mathbf{1}))$$

on $\mathcal{B}$ to obtain the pre-Hilbert space D. Denote by $\eta:\mathcal{B}\to D$ the canonical mapping and define the *-representation ρ of $\mathcal{B}$ on D by

$$\rho(b)\eta(c)=\eta(bc)-\eta(b)\delta(c).$$

We can write down the quantum stochastic integral equations

$$j_{st}(b)=\delta(b)+\int_s^t(j_{s\tau}\star\mathrm{d}I_\tau^\psi)(b) \tag{3}$$

on the Bose Fockspace $\mathcal{F}$ over $L^2(\mathbb{R}_+,H)$, H the completion of D, where $b\in\mathcal{B}$, $s\le t$, and

$$I_t^\psi(b)=A_t^\dagger(\eta(b))+\Lambda_t(\rho(b)-\delta(b)\mathbf{1})+A_t(\eta(b^*))+\psi(b)t.$$

In short-hand differential notation

$$\mathrm{d}j_{st}=j_{st}\star\mathrm{d}I_t^\psi,\ j_{tt}=\delta\mathbf{1}.$$

The operators $j_{st}(b)$ are defined on a dense linear sub-space of $\mathcal{F}$ which is the span of certain exponential vectors; see [4]. In a formal algebraic sense, the j_{st} constitute a version of an independent stationary increment process with generator ψ. We believe that this statement can be made rigorous for an arbitrary *-bialgebra by showing that the linear span of

$$\{j_{s_1t_1}(b_1)\dots j_{s_nt_n}(b_n)\Omega : n \in \mathbb{N}, (s_l,t_l) \in \mathbb{R}_+^2, s_l \leq t_l, b_1,\dots,b_n \in \mathcal{B}\}$$

is in the domain of the closure of the operator $j_{st}(b)$. Only the restriction of j_{st} to this linear subspace of the Fock space can be the independent stationary increment process in question, so that the representation (3) is an embedding theorem. For $\mathbb{C}G$, $\mathbb{C}\langle d\rangle$ and $U\langle d\rangle$ a rigorous treatment of equation (3) can be found in [4, 10], [11] and [9]. For $\mathbb{C}G$, G a group, the operators $j_{st}(x)$, $x \in G$, are unitary and are representations of G of type S (cf. [6]). For $\mathbb{C}\langle d\rangle$ the operators $j_{st}(x_l)$ are sums of creation, preservation, annihilation and scalar processes [11]. For $U\langle d\rangle$ the operators $(j_{st}(x_{kl}))_{k,l=1,\dots,d}$ are increments $(U_s)^\dagger U_t$ of a solution U_t of a linear quantum stochastic differential equation on $\mathbb{C}^d \otimes \mathcal{F}$ with constant coefficients [9].

APPLICATION TO $H\langle 2\rangle$:

We concentrate on Example 7. A generator ψ on $H_0\langle 2\rangle$ can always be constructed by the following procedure. Assume that we are given a pre-Hilbert space D, two hermitian operators ρ_x and ρ_y on D, two vectors η_x and η_y in D and two real numbers ψ_x and ψ_y. We then define the *-representation ρ of $H_0\langle 2\rangle$ by extending $\rho(x) = \rho_x$, $\rho(y) = \rho_y$. Next we define the linear mapping $\eta : H_0\langle 2\rangle \to D$ by the equations

$$\begin{aligned}\eta(x) &= \eta_x\\ \eta(y) &= \eta_y\\ \eta(bc) &= \rho(b)\eta(c) + \eta(b)\delta(c).\end{aligned}$$

Finally, we define $\psi \in H_0\langle 2\rangle^*$ by

$$\begin{aligned}\psi(x) &= \psi_x\\ \psi(y) &= \psi_y\\ \psi(bc) &= \psi(b)\delta(c) + \delta(b)\psi(c) + \langle \eta(a^*), \eta(b)\rangle.\end{aligned}$$

Then ψ is a generator, and the associated equations (3) for $b = x$ and $b = y$ are

$$\begin{aligned}\mathrm{d}X_{st} &= X_{st}(\mathrm{d}A_t^\dagger(\eta_y) + \mathrm{d}\Lambda_t(\rho_y - 1) + \mathrm{d}A_t(\eta_y) + \psi_y \mathrm{d}t)\\ &\qquad + \mathrm{d}A_t^\dagger(\eta_x) + \mathrm{d}\Lambda_t(\rho_x) + \mathrm{d}A_t(\eta_x) + \psi_x \mathrm{d}t\\ X_{ss} &= 0,\end{aligned} \tag{4}$$

and

$$\begin{aligned}\mathrm{d}Y_{st} &= Y_{st}(\mathrm{d}A_t^\dagger(\eta_y) + \mathrm{d}\Lambda_t(\rho_y - 1) + \mathrm{d}A_t(\eta_y) + \psi_y \mathrm{d}t)\\ Y_{ss} &= 1,\end{aligned} \tag{5}$$

where we set $X_{st} = j_{st}(x)$ and $Y_{st} = j_{st}(y)$. By property (a) of an independent stationary increment process we obtain for $r \leq s \leq t$

$$X_{rt} = (j_{rs} \star j_{st})(x) = X_{rs}Y_{st} + X_{st}$$

and

$$Y_{rt} = Y_{rs}Y_{st}.$$

Using this and property (b1) we have for $s \leq t$

$$\begin{aligned} X_{0s}X_{0t} &= X_{0s}(X_{os}Y_{st} + X_{st}) \\ &= X_{0s}Y_{st}X_{0s} + X_{st}X_{0s} \\ &= X_{0t}X_{0s} \end{aligned}$$

and

$$\begin{aligned} Y_{0s}Y_{0t} &= Y_{0s}Y_{0s}Y_{st} \\ &= Y_{0s}Y_{st}Y_{0s} \\ &= Y_{0t}Y_{0s} \end{aligned}$$

showing that both $X_t = X_{0t}$ and $Y_t = Y_{0t}$ are commutative processes.

The equations for the Azéma martingales arise as the following special cases. Choose $D = \mathbb{C}$, $\rho_x = 0$, $\rho_y = c \in \mathbb{R}$, $\eta_x = 1$, $\eta_y = 0$ and $\psi_x = \psi_y = 0$. This determines a generator $\psi^{(c)}$ on $H_0\langle 2\rangle$. Equation (4) and (5) become

$$\mathrm{d}X_{st} = (c-1)X_{st}\mathrm{d}\Lambda_t + \mathrm{d}Q_t,\ X_{ss} = 0 \tag{6}$$

(where we put $Q_t = A_t^\dagger + A_t$) and

$$\mathrm{d}Y_{st} = (c-1)Y_{st}\mathrm{d}\Lambda_t,\ Y_{ss} = \mathbf{1}. \tag{7}$$

Equation (7) is the one for the second quantization operator

$$Y_{st} = \Gamma(\chi_{[0,s]} + c\chi_{[s,t]} + \chi_{[t,\infty)}),$$

equation (6) is solved by $X_{st} = X_t - X_sY_{st}$ and X_t satisfies the Azéma martingale equation

$$\mathrm{d}X_t = (c-1)X_t\mathrm{d}\Lambda_t + \mathrm{d}Q_t,\ X_0 = 0.$$

We have

$$\begin{aligned} \psi^{(c)}(xyx) &= \overline{\eta(x)}\eta(yx) \\ &= \overline{\eta(x)}(\rho(y)\eta(x) + \eta(y)\delta(x)) \\ &= c. \end{aligned}$$

But

$$\begin{aligned} \psi^{(c)}(x^2y) &= \overline{\eta(x)}\eta(xy) \\ &= \overline{\eta(x)}(\rho(x)\eta(y) + \eta(x)\delta(y)) \\ &= 1, \end{aligned}$$

which shows that for $c \neq 1$ the process (X_{st}, Y_{st}) cannot be reduced to an independent stationary increment process over $H_0[2]$.

In the case $c \neq 0$ we can extend the generator $\psi^{(c)}$ to a generator on $H\langle 2\rangle$ in the only possible way by setting $\rho(y^{-1}) = c^{-1}$, $\eta(y^{-1}) = 0$, and $\psi^{(c)}(y^{-1}) = 0$. Then $(X_t, Y_t, (Y_t)^{-1})$ is a process with independent stationary increments over $H\langle 2\rangle$.

REMARK: Nothing has been said about the domains of our processes. However, for $-1 \leq c \leq 1$ the Y_{st} are bounded and for $-1 \leq c < 1$ this is also true for X_{st} (see [7]). For $c = 1$ we have $X_{st} = Q_{st}$ and this is actually the case of Brownian motion and the *-bialgebra $\mathbb{C}\langle 1 \rangle$. Also from [7] we know that for $-1 \leq c \leq 1$ the process X_t has the chaos completeness property which means that the embedding of j_{st} into (X_{st}, Y_{st}) is an isomorphism.

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