

# SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

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*Séminaire de probabilités (Strasbourg)*, tome 25 (1991), p. 121-137

[http://www.numdam.org/item?id=SPS\\_1991\\_\\_25\\_\\_121\\_0](http://www.numdam.org/item?id=SPS_1991__25__121_0)

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# On Newton's method for stochastic differential equations

SHIGETOKU KAWABATA AND TOSHIO YAMADA

## 1. Introduction.

The aim of this paper is to propose a formulation of Newton-Kantorovich's method for Ito-type stochastic differential equations. This note has three sources;

- (1) Newton's method on Banach space by L.V. Kantorovich[6],
- (2) Chaplygin-Vidossich's method for ordinary differential equations [3] [7],
- (3) Newton's method for random operators by A.Bharucha-Reid and R. Kannan [2].

As is well known, S.A. Chaplygin[3] introduced a process for the approximation of solutions for non-linear Cauchy problems for ordinary differential equations;

$$(1.1) \quad x' = f(t, x), \quad x(t_0) = x_0$$

consisting of the iterative solution of a sequence of linear Cauchy problems;

$$(1.2) \quad \begin{aligned} u'_{n+1} &= f(t, u_n(t)) + f_x(t, u_n(t))(u_{n+1}(t) - u_n(t)) \\ u_{n+1}(t_0) &= x_0. \end{aligned}$$

At the end of seventies, G.Vidossich [7] has shown that the Chaplygin sequence is exactly the Newton sequence for the operator;

$$(1.3) \quad F(x)(t) = x(t) - x_0 - \int_{t_0}^t f(s, x(s)) ds$$

For stochastic initial value problems;

$$(1.4) \quad \begin{aligned} dX(t) &= \sigma(t, X(t))dB(t) + b(t, X(t))dt, \quad 0 \leq t \leq T \\ X(0) &= \xi, \end{aligned}$$

one may propose heuristically an analogue of Chaplygin's method in the following iterative scheme;

$$(1.5) \quad \begin{aligned} X_0(t) &= \xi, \\ X_{n+1}(t) &= X(0) + \int_0^t \sigma(s, X_n(s)) dB(s) + \int_0^t b(s, X_n(s)) ds \\ &\quad + \int_0^t \sigma_x(s, X_n(s))(X_{n+1}(s) - X_n(s)) dB(s) \\ &\quad + \int_0^t b_x(s, X_n(s))(X_{n+1}(s) - X_n(s)) ds \end{aligned}$$

We shall show in this paper that the above sequence is the Newton sequence for the stochastic operator;

$$(1.6) \quad \begin{aligned} F(Z)(t) &= Z(t) - Z(0) - \int_0^t \sigma(s, Z(s)) dB(s) \\ &\quad - \int_0^t b(s, Z(s)) ds \end{aligned}$$

We will also discuss the local as well as the global convergence of the sequence to the solution of the equation ( 1.4 ). Our investigation is motivated by the paper by Bharucha-Reid and Kannan [2], where they have developed a probabilistic analogue of Newton-Kantorovich's method for solutions of random operator equations. Applications of their theory are being considered by their school [1], although no explicit application to solutions of Ito-type stochastic differential equations seems exist. To avoid complicated notations, we deal in the present paper with one dimensional case only, but one may generalize the results obtained in this paper to multi-dimensional case without any difficulty.

## 2. Preliminaries.

Let  $\sigma(t, x)$  and  $b(t, x)$  be defined on  $[0, \infty) \times R^1$  and Borel measurable. We consider following Ito-type stochastic differential equation;

$$(2.1) \quad X(t) = X(0) + \int_0^t \sigma(s, X(s)) dB(s) + \int_0^t b(s, X(s)) ds .$$

By a probability family space with an increasing family of  $\sigma$ -fields which is denoted as  $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ , we mean a probability space  $(\Omega, \mathcal{F}, P)$  with right continuous increasing system  $\mathcal{F}_t$  of sub- $\sigma$  fields of  $\mathcal{F}$ , each containing all  $P$ -null sets.

**Definition ( 2.1 )** By a solution of the equation ( 2.1 ), we mean a probability space with an increasing family of  $\sigma$ -fields  $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$  and a family of stochastic processes  $\{X(t), B(t)\}$  defined on it such that

- (1) with probability one,  $X(t)$  and  $B(t)$  are continuous in  $t$  and  $B(0) = 0$ ,
- (2)  $X(t)$  and  $B(t)$  are  $\mathcal{F}_t$ -measurable,
- (3)  $B(t)$  is a  $\mathcal{F}_t$ -martingale such that

$$(2.2) \quad E[(B(t) - B(s))^2 / \mathcal{F}_s] = t - s, \quad t \geq s,$$

- (4)  $X(t)$  and  $B(t)$  satisfy

$$(2:3) \quad X(t) = X(0) + \int_0^t \sigma(s, X(s)) dB(s) + \int_0^t b(s, X(s)) ds,$$

where the integral by  $dB(s)$  is understood in the sense of Ito integral.

**Condition A** We say that  $\sigma(t, x)$  and  $b(t, x)$  satisfy the Condition A, if

- (1)  $\sigma(t, x)$  and  $b(t, x)$  are continuous in  $(t, x)$  and differentiable with respect to  $x$ , moreover  $D_x \sigma(t, x) = \sigma_x(t, x)$  and  $D_x b(t, x) = b_x(t, x)$  are continuous with respect to  $x$ .
- (2) there exist positive constants  $K$  and  $M$  such that,

$$(2.4) \quad |\sigma(t, x)|^2 \leq K(1 + x^2),$$

$$(2.5) \quad |b(t, x)|^2 \leq K(1 + x^2),$$

$$(2.6) \quad |\sigma_x(t, x)| \leq M,$$

and

$$(2.7) \quad |b_x(t, x)| \leq M.$$

**Remark 2.1.** Since the inequalities ( 2.6 ) and ( 2.7 ) imply the global Lipschitz condition for  $\sigma(t, x)$  and  $b(t, x)$ , then with the conditions ( 2.4 ) and ( 2.5 ), there exists a solution  $X(t)$  of the equation ( 2.1 ) defined on  $[0, T]$ , such that

$$(2.8) \quad \sup_{t \in [0, T]} E[|X(t)|^2] < +\infty,$$

where  $T < +\infty$  is an arbitrarily given positive number. Furthermore a solution with the property ( 2.8 ) is pathwise unique. ( see for e.g., [4] and [5] ). In the following in this paper, we assume  $E[|X(0)|^2] < +\infty$  .

### 3. The Gâteaux derivative.

Let  $\mathcal{L}_T$  be the set of  $\varphi : [0, \infty) \times \Omega \rightarrow R$ , such that (i)  $\varphi$  is  $\mathcal{F}_t$ -adapted and continuous with respect to  $t$ , (ii)  $E[\sup_{0 \leq s \leq T} |\varphi(s, \omega)|^2] < \infty$ . Then  $\mathcal{L}_T$  is a Banach space with the norm

$$\|\varphi\|^2 = E[\sup_{0 \leq s \leq T} |\varphi(s, \omega)|^2].$$

Consider the following operator  $F$  defined on  $\mathcal{L}_T$ ;

$$(3.1) \quad \begin{aligned} F(Z) = F(Z)(t) = & Z(t, \omega) - Z(0, \omega) - \int_0^t \sigma(s, Z(s, \omega)) dB(s) \\ & - \int_0^t b(s, Z(s, \omega)) ds \quad 0 \leq t \leq T, \quad Z \in \mathcal{L}_T \end{aligned}$$

**LEMMA(3.1).** Under the condition A the operator  $F$  maps the space  $\mathcal{L}_T$  into itself.

**PROOF:** Let a process  $Z$  belong to  $\mathcal{L}_T$ . It is obvious by the definition of  $F$  that  $F(Z)(t)$   $0 \leq t \leq T$ , is  $\mathcal{F}_t$ -adapted and continuous in  $t$ . To prove that

$$E[\sup_{0 \leq t \leq T} |F(Z)(t)|^2] < +\infty \quad \text{holds,}$$

we first observe that

$$(3.2) \quad \begin{aligned} E[\sup_{0 \leq t \leq T} |F(Z)(t)|^2] \leq & 3E[\sup_{0 \leq t \leq T} |Z(t, \omega) - Z(0, \omega)|^2] \\ & + 3E[\sup_{0 \leq t \leq T} |\int_0^t \sigma(s, Z(s, \omega)) dB(s)|^2] \\ & + 3E[\sup_{0 \leq t \leq T} |\int_0^t b(s, Z(s, \omega)) ds|^2] \quad \text{holds.} \end{aligned}$$

By Doob's martingale inequality and Schwarz's inequality, we get from the above (3.2) that

$$(3.3) \quad \begin{aligned} E[\sup_{0 \leq t \leq T} |F(Z)(t)|^2] \leq & 6E[\sup_{0 \leq t \leq T} |Z(t, \omega)|^2] + 6E[|Z(0, \omega)|^2] \\ & + 12E[|\int_0^T \sigma(s, Z(s, \omega)) dB(s)|^2] \\ & + 3TE[\int_0^T |b(s, Z(s, \omega))|^2 ds] \end{aligned}$$

holds.

Noting that

$$E[|\int_0^T \sigma(s, Z(s, \omega)) dB(s)|^2] = E[\int_0^T \sigma^2(s, Z(s, \omega)) ds],$$

we can conclude from (3.3) with (2.4) and (2.5) in the condition A that

$$(3.4) \quad \begin{aligned} E[\sup_{0 \leq t \leq T} |F(Z)(t)|^2] &\leq 6E[\sup_{0 \leq t \leq T} |Z(t, \omega)|^2] \\ &\quad + 6E[|Z(0, \omega)|^2] + 12KE[\int_0^T [1 + |Z(t, \omega)|^2] dt] \\ &\quad + 3TK E[\int_0^T [1 + |Z(t, \omega)|^2] dt] \\ &\leq 6E[\sup_{0 \leq t \leq T} |Z(t, \omega)|^2] + 6E[|Z(0, \omega)|^2] \\ &\quad + 12KT[1 + E[\sup_{0 \leq t \leq T} |Z(t, \omega)|^2]] \\ &\quad + 3KT^2[1 + E[\sup_{0 \leq t \leq T} |Z(t, \omega)|^2]] < +\infty. \end{aligned}$$

q.e.d.

Now we are in a position to introduce the Gâteaux derivative of the operator  $F$ .

**Definition 3.1** Let  $Z$  belong to  $\mathcal{L}_T$ . If for any  $h \in \mathcal{L}_T$ ,

$$\lim_{u \downarrow 0} \frac{1}{u} [F(Z + uh) - F(Z)]$$

exists in norm convergence sense in the space  $\mathcal{L}_T$ , we call the limit the Gâteaux derivative of the operator  $F$  at  $Z$ . This limit element in  $\mathcal{L}_T$  will be denoted by

$$dF(Z; h) = dF(Z; h)(t), \quad 0 \leq t \leq T.$$

**LEMMA 3.2.** For any  $Z \in \mathcal{L}_T$ , there exists the Gâteaux derivative of the operator  $F$  at  $Z$  and it satisfies

$$(3.5) \quad \begin{aligned} dF(Z; h) &= dF(Z; h)(t) \\ &= h(t, \omega) - h(0, \omega) - \int_0^t \sigma_x(s, Z(s, \omega)) h(s, \omega) dB(s) \\ &\quad - \int_0^t b_x(s, Z(s, \omega)) h(s, \omega) ds. \end{aligned}$$

**PROOF:** By the definition of the operator  $F$ , we observe that

$$\begin{aligned} &\frac{1}{u} [F(Z + uh)(t) - F(Z)(t)] \\ &= \frac{1}{u} [uh(t, \omega) - \int_0^t [\sigma(s, Z(s, \omega) + uh(s, \omega)) - \sigma(s, Z(s, \omega))] dB(s) \\ &\quad - \int_0^t [b(s, Z(s, \omega) + uh(s, \omega)) - b(s, Z(s, \omega))] ds] \\ &= h(t, \omega) - \int_0^t \sigma_x(s, Z(s, \omega)) h(s, \omega) dB(s) \\ &\quad - \int_0^t b_x(s, Z(s, \omega)) h(s, \omega) ds + R(t, \omega), \text{ say.} \end{aligned}$$

Note by the condition A that the functions  $\sigma_x(t, x)$  and  $b_x(t, x)$  are continuous with respect to  $x$ . Then we have

$$(3.6) \quad \begin{aligned} R(t, \omega) &= -\frac{1}{u} \left[ \int_0^t [\sigma_x(s, Z(s, \omega) + \theta uh(s, \omega)) - \sigma_x(s, Z(s, \omega))] uh(s, \omega) dB(s) \right. \\ &\quad \left. + \int_0^t [b_x(s, Z(s, \omega) + \theta uh(s, \omega)) - b_x(s, Z(s, \omega))] uh(s, \omega) ds \right] \end{aligned}$$

where  $\theta, 0 < \theta < 1$ , depends on  $(s, \omega, h)$ .

To complete the proof it suffices to show that

$$(3.7) \quad \lim_{u \downarrow 0} E \left[ \sup_{0 \leq t \leq T} |R(t, \omega)|^2 \right] = 0, \quad \text{holds.}$$

By a similar way as in the proof of Lemma (3.1), we observe that

$$(3.8) \quad \begin{aligned} &E \left[ \sup_{0 \leq t \leq T} |R(t, \omega)|^2 \right] \\ &\leq 2E \left[ \sup_{0 \leq t \leq T} \left| \int_0^t (\sigma_x(s, Z(s, \omega) + \theta uh(s, \omega)) - \sigma_x(s, Z(s, \omega))) \right. \right. \\ &\quad \left. \left. h(s, \omega) dB(s) \right|^2 \right] \\ &\quad + 2E \left[ \sup_{0 \leq t \leq T} \left| \int_0^t (b_x(s, Z(s, \omega) + \theta uh(s, \omega)) - b_x(s, Z(s, \omega))) h(s, \omega) ds \right|^2 \right] \\ &\leq 8E \left[ \int_0^T |\sigma_x(s, Z(s, \omega) + \theta uh(s, \omega)) - \sigma_x(s, Z(s, \omega))|^2 h^2(s, \omega) ds \right] \\ &\quad + 2E \left[ \left( \int_0^T |b_x(s, Z(s, \omega) + \theta uh(s, \omega)) - b_x(s, Z(s, \omega))|^2 ds \right) \right. \\ &\quad \left. \left( \int_0^T h^2(s, \omega) ds \right) \right] \\ &= J_1 + J_2, \text{ say.} \end{aligned}$$

Since the function  $\sigma_x$  and the function  $b_x$  both are continuous with respect to  $x$ , it follows that

$$\lim_{u \downarrow 0} [\sigma_x(s, Z(s, \omega) + \theta uh(s, \omega)) - \sigma_x(s, Z(s, \omega))] = 0$$

and also

$$\lim_{u \downarrow 0} [b_x(s, Z(s, \omega) + \theta uh(s, \omega)) - b_x(s, Z(s, \omega))] = 0$$

hold. Furthermore, we know by the condition A that

$$|\sigma_x| \leq M \text{ and } |b_x| \leq M \text{ hold.}$$

Hence, Lebesgue's convergence theorem implies

$$\lim_{u \downarrow 0} [J_1 + J_2] = 0.$$

Thus by (3.8)

$$\lim_{u \downarrow 0} E \left[ \sup_{0 \leq t \leq T} |R(t, \omega)|^2 \right] = 0.$$

The lemma is proved.

q.e.d.

#### 4. Stochastic analogue of Newton's method for stochastic differential equations.

First of all, we will discuss the existence of the inverse the Gâteaux derivative of  $F$  at  $Z$  which will be denoted by  $dF^{-1}(Z)$ .

LEMMA 4.1. *Let  $Z$  be a given element in  $\mathcal{L}_T$ . Let  $\varphi$  belong to  $\mathcal{L}_T$  such that  $\varphi(0, \omega) = 0$ . Then, there exists one and only one element  $h$  in  $\mathcal{L}_T$  such that,*

$$(4.1) \quad \varphi(t, \omega) = dF(Z; h)(t);$$

i.e.,

$$(4.2) \quad \begin{aligned} \varphi(t, \omega) = h(t, \omega) - \int_0^t \sigma_x(s, Z(s, \omega))h(s, \omega) dB(s) \\ - \int_0^t b_x(s, Z(s, \omega))h(s, \omega) ds. \end{aligned}$$

PROOF: Since the linear stochastic differential equation (4.2) satisfies the global Lipschitz condition for its diffusion coefficient as well as for its drift coefficient, then the existence and the pathwise uniqueness hold for the equation (4.2). From this fact, the lemma follows immediately. q.e.d.

LEMMA 4.2. *Let  $\varphi$  belong to  $\mathcal{L}_T$ , such that  $\varphi(0, \omega) = 0$ . Then, there exists a positive constant  $L < +\infty$ , which is independent of  $Z$  and also of  $t \in [0, T]$ , such that*

$$(4.3) \quad \|dF^{-1}(Z)(\varphi)\|_t^2 \leq 3\|\varphi\|_t^2 e^{Lt}, \quad 0 \leq t \leq T$$

where  $\|\varphi\|_t^2$  stands for  $E[\sup_{0 \leq s \leq t} |\varphi(s, \omega)|^2]$ .

PROOF: Let  $h(t, \omega)$  be

$$h(t, \omega) = dF^{-1}(Z)(\varphi)(t, \omega), \quad 0 \leq t \leq T.$$

Then by (4.2), we get

$$(4.4) \quad \begin{aligned} \|h\|_t^2 &= E[\sup_{0 \leq s \leq t} |h(s, \omega)|^2] \leq 3E[\sup_{0 \leq s \leq t} |\varphi(s, \omega)|^2] \\ &+ 3E[\sup_{0 \leq s \leq t} |\int_0^s \sigma_x(u, Z(u, \omega))h(u, \omega) dB(u)|^2] \\ &+ 3E[\sup_{0 \leq s \leq t} |\int_0^s b_x(u, Z(u, \omega))h(u, \omega) du|^2] \end{aligned}$$

It follows from (4.4) that

$$(4.5) \quad \begin{aligned} \|h\|_t^2 &\leq 3\|\varphi\|_t^2 + 12E[\int_0^t \sigma_x^2(s, Z(s, \omega))h^2(s, \omega) ds] \\ &+ 3E[(\int_0^t b_x^2(s, Z(s, \omega)) ds) \int_0^t h^2(s, \omega) ds] \\ &\leq 3\|\varphi\|_t^2 + 12M^2 E[\int_0^t h^2(s, \omega) ds] + 3M^2 T E[\int_0^t h^2(s, \omega) ds] \\ &\leq 3\|\varphi\|_t^2 + L \int_0^t \|h\|_s^2 ds, \end{aligned}$$

where we have used (2.6) and (2.7) in the condition A, and  $L$  stands for  $12M^2 + 3M^2T$ .

By Gronwall's inequality, it follows from (4.5) that

$$\|h\|_t^2 = \|dF^{-1}(Z)(\varphi)\|_t^2 \leq 3\|\varphi\|_t^2 e^{Lt}, \quad 0 \leq t \leq T.$$

*q.e.d.*

We are now in a position to introduce the Newton sequence for the operator  $F$ .

Let

$$(4.6) \quad \begin{aligned} X_0(t) &= X(0, \omega), \\ X_{n+1}(t) &= X_n(t) - dF^{-1}(X_n)(F(X_n))(t) \quad n = 1, 2, \dots \end{aligned}$$

We call  $X_n(t)$ ,  $n = 1, 2, \dots$ , the Newton sequence for the operator  $F$ .

It follows from (4.6) that the sequence satisfies that

$$(4.7) \quad \begin{aligned} X_{n+1}(t) &= X_0(t) + \int_0^t \sigma(s, X_n(s)) dB(s) \\ &+ \int_0^t b(s, X_n(s)) ds + \int_0^t \sigma_x(s, X_n(s))(X_{n+1}(s) - X_n(s)) dB(s) \\ &+ \int_0^t b_x(s, X_n(s))(X_{n+1}(s) - X_n(s)) ds \end{aligned}$$

Thus the Newton sequence introduced in the above (4.6) is exactly the same sequence as the stochastic analogue of Chaplygin sequence (1.5) discussed in the introduction.

The following theorem concerns the convergence in local sense of the Newton sequence to the solution of the stochastic differential equation (2.1).

**THEOREM 4.1.** *Let  $X(t)$  be the solution of the equation (2.1). Choose a positive number  $\delta$  such that,*

$$(4.8) \quad 120\delta M^2 e^{L\delta} = \alpha < 1$$

*holds. Then,*

$$(4.9) \quad \lim_{n \rightarrow \infty} E\left[\sup_{0 \leq t \leq \delta} |X_n(t) - X(t)|^2\right] = 0$$

*holds with error bound,*

$$(4.10) \quad \begin{aligned} &[E[\sup_{0 \leq t \leq \delta} |X_n(t) - X(t)|^2]]^{1/2} \\ &\leq \frac{\beta^n}{1 - \beta} [E[\sup_{0 \leq t \leq \delta} |X_1(t) - X_0(t)|^2]]^{1/2}, \end{aligned}$$

where  $\beta = \sqrt{\alpha}$ .

**PROOF:** we will divide the proof in two steps. Without loss of generality, we can suppose that  $\delta < 1$  holds.

*First step.* In this step we will show that

$$(4.11) \quad \|X_{n+1} - X_n\|_\delta^2 \leq \alpha \|X_n - X_{n-1}\|_\delta^2$$



holds.

By the definition of the Newton sequence (4.6), we know

$$(4.12) \quad X_{n+1}(t) - X_n(t) = -dF^{-1}(X_n)(F(X_n))(t) .$$

Hence, Lemma 4.2 implies that

$$(4.13) \quad \|X_{n+1} - X_n\|_\delta^2 \leq 3e^{L\delta} \|F(X_n)\|_\delta^2$$

holds.

For  $n \geq 1$ , we observe by (4.6)

$$\begin{aligned} F(X_n)(t) &= F(X_n)(t) - F(X_{n-1})(t) + F(X_{n-1})(t) \\ &= F(X_n)(t) - F(X_{n-1})(t) - dF(X_{n-1}; X_n - X_{n-1})(t) . \end{aligned}$$

Hence, we have

$$\begin{aligned} F(X_n)(t) &= \int_0^t \sigma(s, X_{n-1}(s)) dB(s) - \int_0^t \sigma(s, X_n(s)) dB(s) \\ &\quad + \int_0^t b(s, X_{n-1}(s)) ds - \int_0^t b(s, X_n(s)) ds \\ &\quad + \int_0^t \sigma_x(s, X_{n-1}(s))(X_n(s) - X_{n-1}(s)) dB(s) \\ &\quad + \int_0^t b_x(s, X_{n-1}(s))(X_n(s) - X_{n-1}(s)) ds \\ &= \int_0^t \sigma_x(s, X_{n-1}(s))(X_n(s) - X_{n-1}(s)) dB(s) \\ &\quad + \int_0^t b_x(s, X_{n-1}(s))(X_n(s) - X_{n-1}(s)) ds \\ &\quad - \int_0^t \sigma_x(s, X_{n-1}(s) + \theta(X_n(s) - X_{n-1}(s)))(X_n(s) - X_{n-1}(s)) dB(s) \\ &\quad - \int_0^t b_x(s, X_{n-1}(s) + \theta'(X_n(s) - X_{n-1}(s)))(X_n(s) - X_{n-1}(s)) ds \end{aligned}$$

where  $0 < \theta, \theta' < 1$ .

Thus, we have

$$\begin{aligned}
& \|F(X_n)\|_\delta^2 \\
& \leq 2E\left[\sup_{0 \leq t \leq \delta} \left| \int_0^t (\sigma_x(s, X_{n-1}(s)) - \sigma_x(s, X_{n-1}(s) + \theta(X_n(s) - X_{n-1}(s)))) \right. \right. \\
& \quad \left. \left. (X_n(s) - X_{n-1}(s)) dB(s) \right|^2\right] \\
& + 2E\left[\sup_{0 \leq t \leq \delta} \left| \int_0^t (b_x(s, X_{n-1}(s)) - b_x(s, X_{n-1}(s) + \theta(X_{n-1}(s) - X_n(s)))) \right. \right. \\
& \quad \left. \left. (X_n(s) - X_{n-1}(s)) ds \right|^2\right] \\
& \leq 8E\left[\int_0^t (\sigma_x(s, X_{n-1}(s)) - \sigma_x(s, X_{n-1}(s) + \theta(X_n(s) - X_{n-1}(s))))^2 \right. \\
& \quad \left. (X_n(s) - X_{n-1}(s))^2 ds \right] \\
& + 2E\left[\int_0^t (b_x(s, X_{n-1}(s)) - b_x(s, X_{n-1}(s) + \theta(X_n(s) - X_{n-1}(s))))^2 ds \right. \\
& \quad \left. \int_0^t (X_n(s) - X_{n-1}(s))^2 ds \right]
\end{aligned}$$

Hence, by (2.6) and (2.7) in the condition A, we observe that

$$\begin{aligned}
(4.14) \quad & \|F(X_n)\|_\delta^2 \leq 32M^2\delta \|X_n - X_{n-1}\|_\delta^2 + 8M^2\delta^2 \|X_n - X_{n-1}\|_\delta^2 \\
& \leq 40M^2\delta \|X_n - X_{n-1}\|_\delta^2, \quad (0 < \delta < 1).
\end{aligned}$$

Combine (4.13) with (4.14). Then, we can conclude that (4.11) holds.

*Second step.* Put  $\beta = \sqrt{\alpha}$ . The inequality (4.11) implies

$$(4.15) \quad \|X_{n+1} - X_n\|_\delta \leq \beta \|X_n - X_{n-1}\|_\delta$$

From this it follows immediately

$$(4.16) \quad \|X_{n+1} - X_n\|_\delta \leq \beta^n \|X_1 - X_0\|_\delta$$

Since  $\|\cdot\|_\delta$  is the norm of the Banach space  $\mathcal{L}_\delta$ , we get from (4.16) that

$$\begin{aligned}
(4.17) \quad & \|X_{n+p} - X_n\|_\delta \leq (\beta^{n+p-1} + \dots + \beta^n) \|X_1 - X_0\|_\delta \\
& \leq \frac{\beta^n}{1 - \beta} \|X_1 - X_0\|_\delta.
\end{aligned}$$

Hence, the sequence  $X_n$   $n = 1, 2, \dots$ , is a Cauchy sequence in the Banach space  $\mathcal{L}_\delta$ . Put  $\tilde{X}(t)$   $0 \leq t \leq \delta$  the limit of the sequence  $X_n$   $n = 1, 2, \dots$ . Since the process  $X_n(t)$  satisfies

$$\begin{aligned}
X_n(t) &= X(0) + \int_0^t \sigma(s, X_{n-1}(s)) dB(s) + \int_0^t b(s, X_{n-1}(s)) ds \\
& + \int_0^t \sigma_x(s, X_{n-1}(s))(X_n(s) - X_{n-1}(s)) dB(s) \\
& + \int_0^t b_x(s, X_{n-1}(s))(X_n(s) - X_{n-1}(s)) ds \quad 0 \leq t \leq \delta,
\end{aligned}$$

then, the limit process  $\tilde{X}(t)$  satisfies the equation (2.1);

$$\tilde{X}(t) = X(0) + \int_0^t \sigma(s, \tilde{X}(s)) dB(s) + \int_0^t b(s, \tilde{X}(s)) ds, \quad 0 \leq t \leq \delta.$$

Since the Pathwise uniqueness holds for the equation (2.1), we observe that

$$(4.18) \quad \tilde{X}(t) = X(t) \text{ holds.}$$

Hence we get (4.9). (4.10) follows from (4.17) and (4.18).

*q.e.d.*

## 5. The convergence in the large of the Newton sequence.

In this section we assume for the coefficients the following condition B.

**Condition B :** We say that the coefficients  $\sigma(t, x)$  and  $b(t, x)$  satisfy the Condition B, if they satisfy the Condition A and moreover there exists a positive constant  $N < +\infty$ , such that

$$(5.1) \quad |\sigma(t, x)| \leq N \quad \text{and} \quad |b(t, x)| \leq N$$

hold for all  $t$  and  $x$ .

Under the condition B, we have the following theorem which concerns the convergence in the large.

**THEOREM 5.1.** *Let  $T$  be a fixed positive number. Then the Newton sequence  $X_n(t)$   $n = 1, 2, \dots$  defined by (4.6) converges in the large to the solution  $X(t)$  of the equation (2.1) in the following sense;*

$$(5.2) \quad \lim_{n \rightarrow \infty} E[ \sup_{0 \leq t \leq T} |X_n(t) - X(t)|^2 ] = 0,$$

*if and only if*

$$(5.3) \quad \sup_n E[ \sup_{0 \leq t \leq T} |X_n(t)|^2 ] < +\infty$$

*holds.*

**PROOF:** The necessity is obvious. To prove the sufficiency, we will divide the proof in several steps. In the proof  $K_2 < \infty$  stands for  $\sup_n E[ \sup_{0 \leq t \leq T} |X_n(t)|^2 ]$

*First step :* Let  $T_1$  be defined by

$$(5.4) \quad T_1 = \sup\{t; t \in [0, T] \text{ and } \lim_{n \rightarrow \infty} E[ \sup_{0 \leq s \leq t} |X_n(s) - X(s)|^2 ] = 0\}.$$

Then Theorem 4.1 implies

$$(5.5) \quad 0 < \delta \leq T_1 \leq T.$$

*Second step :* In the present step, we will show that

$$(5.6) \quad \lim_{n \rightarrow \infty} E[ \sup_{0 \leq t \leq T_1} |X_n(t) - X(t)|^2 ] = 0$$

holds.

Let  $\epsilon > 0$  be an arbitrary positive number. Choose  $S_0$  such that

$$(5.7) \quad \begin{aligned} 0 &< S_0 < \min(T_1, 1) \\ (80M^2K_2 + 20N^2)S_0 &< \frac{\epsilon}{10}. \end{aligned}$$

By the definition of  $T_1$ , we get

$$\lim_{n \rightarrow \infty} E\left[ \sup_{0 \leq t \leq T_1 - S_0} |X_n(t) - X(t)|^2 \right] = 0.$$

Hence, for sufficiently large  $N_1$ , we observe that

$$(5.8) \quad E\left[ \sup_{0 \leq t \leq T_1 - S_0} |X_n(t) - X(t)|^2 \right] \leq \frac{\epsilon}{10}, \quad n \geq N_1$$

holds.

On the other hand, we have

$$(5.9) \quad E\left[ \sup_{T_1 - S_0 \leq t \leq T_1} |X_n(t) - X(t)|^2 \right] \leq 3I_1 + 3I_2 + 3I_3,$$

where

$$\begin{aligned} I_1 &= E\left[ \sup_{T_1 - S_0 \leq t \leq T_1} |X_n(t) - X_n(T_1 - S_0)|^2 \right], \\ I_2 &= E\left[ |X_n(T_1 - S_0) - X(T_1 - S_0)|^2 \right], \\ I_3 &= E\left[ \sup_{T_1 - S_0 \leq t \leq T_1} |X(t) - X(T_1 - S_0)|^2 \right]. \end{aligned}$$

First, we will deal with  $I_1$ . We have

$$(5.10) \quad \begin{aligned} X_n(t) - X_n(T_1 - S_0) &= \int_{T_1 - S_0}^t \sigma(s, X_{n-1}(s)) dB(s) \\ &+ \int_{T_1 - S_0}^t b(s, X_{n-1}(s)) ds + \int_{T_1 - S_0}^t \sigma_x(s, X_{n-1}(s)) Y_n(s) dB(s) \\ &+ \int_{T_1 - S_0}^t b_x(s, X_{n-1}(s)) Y_n(s) ds, \end{aligned}$$

where  $Y_n(t) = X_n(t) - X_{n-1}(t)$ .

By Doob's martingale inequality with Schwarz's inequality, it follows from (5.10) that

$$\begin{aligned} &E\left[ \sup_{T_1 - S_0 \leq t \leq T_1} |X_n(t) - X_n(T_1 - S_0)|^2 \right] \\ &\leq 16E\left[ \int_{T_1 - S_0}^{T_1} |\sigma(s, X_{n-1}(s))|^2 ds \right] + 4S_0 E\left[ \int_{T_1 - S_0}^{T_1} |b(s, X_{n-1}(s))|^2 ds \right] \\ &+ 16E\left[ \int_{T_1 - S_0}^{T_1} |\sigma_x(s, X_{n-1}(s)) Y_n(s)|^2 ds \right] \\ &+ 4S_0 E\left[ \int_{T_1 - S_0}^{T_1} |b_x(s, X_{n-1}(s)) Y_n(s)|^2 ds \right]. \end{aligned}$$

Hence, by the condition B, we observe that

$$\begin{aligned}
 (5.11) \quad & E\left[\sup_{T_1-S_0 \leq t \leq T_1} |X_n(t) - X_n(T_1 - S_0)|^2\right] \\
 & \leq 16N^2S_0 + 4N^2S_0 + 16M^2 \left(\int_{T_1-S_0}^{T_1} E\left[\sup_{0 \leq t \leq T} |Y_n(t)|^2\right] ds\right) \\
 & \quad + 4M^2S_0 \int_{T_1-S_0}^{T_1} E\left[\sup_{0 \leq t \leq T} |Y_n(t)|^2\right] ds
 \end{aligned}$$

Here, note that by the condition (5.3) that

$$\begin{aligned}
 (5.12) \quad & E\left[\sup_{0 \leq t \leq T} |Y_n(t)|^2\right] \leq 2E\left[\sup_{0 \leq t \leq T} |X_n(t)|^2\right] \\
 & \quad + 2E\left[\sup_{0 \leq t \leq T} |X_{n-1}(t)|^2\right] \leq 4K_2 < +\infty, \quad n = 1, 2, \dots
 \end{aligned}$$

Then the inequalities (5.11) and (5.12) imply that

$$I_1 \leq 80M^2K_2S_0 + 20N^2S_0.$$

Hence, by (5.7),

$$(5.13) \quad I_1 \leq \frac{\epsilon}{10}$$

holds.

Second, for  $I_2$ , we can choose a number  $N_2$  such that

$$(5.14) \quad I_2 = E[|X_n(T_1 - S_0) - X(T_1 - S_0)|^2] \leq \frac{\epsilon}{10}, \quad n \geq N_2$$

holds.

Finally for  $I_3$ , it is easily seen that

$$\begin{aligned}
 I_3 & \leq 8E\left[\int_{T_1-S_0}^{T_1} \sigma^2(s, X(s)) ds\right] + 2S_0E\left[\int_{T_1-S_0}^{T_1} b^2(s, X(s)) ds\right] \\
 & \leq 8N^2S_0 + 2N^2S_0.
 \end{aligned}$$

Hence by (5.7) we get

$$(5.15) \quad I_3 \leq \frac{\epsilon}{10}.$$

From the inequalities (5.8),(5.9),(5.13),(5.14) and (5.15), we can conclude that

$$(5.6) \quad \lim_{n \rightarrow \infty} E\left[\sup_{0 \leq t \leq T_1} |X_n(t) - X(t)|^2\right] = 0$$

holds.

*Third step*: In this step, we shall show that  $T_1 = T$ , using the method of reduction to absurdity.

Assume  $T_1 \neq T$  and let us find a contradiction.

By what has been proved in the second step, we can choose a sequence of positive numbers  $a_n$ ,  $n = 1, 2, \dots$  such that

$$(5.16) \quad \begin{aligned} a_n &\downarrow 0 \quad (n \rightarrow \infty) \\ E[|X_n(T_1) - X(T_1)|^2] &\leq a_n. \end{aligned}$$

We will divide the step in two substeps.

(i): First, we will find a positive number  $h > 0$  such that

$$(5.17) \quad \begin{aligned} T_1 + h &\leq T, \\ \lim_{n \rightarrow \infty} E\left[\sup_{T_1 \leq t \leq T_1 + h} |Y_n(t)|^2\right] &= 0, \quad \text{where } Y_n(t) = X_n(t) - X_{n-1}(t), \end{aligned}$$

holds.

By the definition of  $Y_n(t)$ , we have for  $T_1 \leq t \leq T$ ,

$$\begin{aligned} Y_n(t) &= X_n(T_1) - X_{n-1}(T_1) \\ &+ \int_{T_1}^t (\sigma(s, X_{n-1}(s)) - \sigma(s, X_{n-2}(s))) dB(s) \\ &+ \int_{T_1}^t (b(s, X_{n-1}(s)) - b(s, X_{n-2}(s))) ds \\ &+ \int_{T_1}^t \sigma_x(s, X_{n-1}(s)) Y_n(s) dB(s) - \int_{T_1}^t \sigma_x(s, X_{n-2}(s)) Y_{n-1}(s) dB(s) \\ &+ \int_{T_1}^t b_x(s, X_{n-1}(s)) Y_n(s) ds - \int_{T_1}^t b_x(s, X_{n-2}(s)) Y_{n-1}(s) ds \end{aligned}$$

Thus, we get from the above that

$$\begin{aligned} &E\left[\sup_{T_1 \leq s \leq t} |Y_n(s)|^2\right] \\ &\leq 7E[|X_n(T_1) - X_{n-1}(T_1)|^2] \\ &+ 7E\left[\sup_{T_1 \leq s \leq t} \left|\int_{T_1}^s (\sigma(u, X_{n-1}(u)) - \sigma(u, X_{n-2}(u))) dB(u)\right|^2\right] \\ &+ 7E\left[\sup_{T_1 \leq s \leq t} \left|\int_{T_1}^s (b(u, X_{n-1}(u)) - b(u, X_{n-2}(u))) du\right|^2\right] \\ &+ 7E\left[\sup_{T_1 \leq s \leq t} \left|\int_{T_1}^s (b_x(u, X_{n-1}(u)) Y_n(u)) du\right|^2\right] \\ &+ 7E\left[\sup_{T_1 \leq s \leq t} \left|\int_{T_1}^s (b_x(u, X_{n-2}(u)) Y_{n-1}(u)) du\right|^2\right] \\ &+ 7E\left[\sup_{T_1 \leq s \leq t} \left|\int_{T_1}^s (\sigma_x(u, X_{n-1}(u)) Y_n(u)) dB(u)\right|^2\right] \\ &+ 7E\left[\sup_{T_1 \leq s \leq t} \left|\int_{T_1}^s (\sigma_x(u, X_{n-2}(u)) Y_{n-1}(u)) dB(u)\right|^2\right]. \end{aligned}$$

Hence, by Doob's martingale inequality with Schwarz's inequality, we observe that

$$\begin{aligned}
(5.18) \quad & E\left[\sup_{T_1 \leq s \leq t} |Y_n(s)|^2\right] \leq 7E[|X_n(T_1) - X_{n-1}(T_1)|^2] \\
& + 28E\left[\int_{T_1}^t |\sigma(u, X_{n-1}(u)) - \sigma(u, X_{n-2}(u))|^2 du\right] \\
& + 7(t - T_1)E\left[\int_{T_1}^t |b(u, X_{n-1}(u)) - b(u, X_{n-2}(u))|^2 du\right] \\
& + 7(t - T_1)M^2E\left[\int_{T_1}^t |Y_n(u)|^2 du\right] + 7(t - T_1)M^2E\left[\int_{T_1}^t |Y_{n-1}(u)|^2 du\right] \\
& + 28M^2E\left[\int_{T_1}^t |Y_n(u)|^2 du\right] + 28M^2E\left[\int_{T_1}^t |Y_{n-1}(u)|^2 du\right]
\end{aligned}$$

holds, where we have used the inequalities (2.6) and (2.7) in the condition A.

Note, by the condition A again, that

$$(5.19) \quad |b(u, X_{n-1}(u)) - b(u, X_{n-2}(u))| \leq M|X_{n-1}(u) - X_{n-2}(u)|$$

and

$$(5.20) \quad |\sigma(u, X_{n-1}(u)) - \sigma(u, X_{n-2}(u))| \leq M|X_{n-1}(u) - X_{n-2}(u)|$$

holds.

Then, the inequalities (5.18), (5.19) and (5.20) imply that

$$\begin{aligned}
(5.21) \quad & E\left[\sup_{T_1 \leq s \leq t} |Y_n(s)|^2\right] \\
& \leq 7E[|X_n(T_1) - X(T_1)|^2] + (56M^2 + 14(t - T_1)M^2)E\left[\int_{T_1}^t |Y_{n-1}(u)|^2 du\right] \\
& + (7(t - T_1)M^2 + 28M^2)E\left[\int_{T_1}^t |Y_n(u)|^2 du\right]
\end{aligned}$$

Choose  $h > 0$ , such that

$$(5.22) \quad \eta = (56M^2h + 14M^2h^2)e^{7M^2h^2 + 28M^2h} < 1$$

Note that

$$(5.23) \quad E[|X_n(T_1) - X_{n-1}(T_1)|^2] \leq 2a_n + 2a_{n-1} \leq 4a_{n-1}$$

Then the (5.21) implies for  $T_1 \leq t \leq T_1 + h$ ,

$$\begin{aligned}
(5.24) \quad & E\left[\sup_{T_1 \leq s \leq t} |Y_n(s)|^2\right] \\
& \leq 28a_{n-1} + (56M^2 + 14M^2h^2) \|\|Y_{n-1}\|\| \\
& + (28M^2 + 7M^2h) \int_{T_1}^t E\left[\sup_{T_1 \leq u \leq s} |Y_n(u)|^2\right] ds
\end{aligned}$$

where  $\|\|\varphi\|\|$  stands for  $E\left[\sup_{T_1 \leq t \leq T_1+h} |\varphi(t)|^2\right]$ .

Hence, by Gronwall's inequality, we observe that

$$(5.25) \quad \begin{aligned} & E\left[\sup_{T_1 \leq s \leq t} |Y_n(s)|^2\right] \\ & \leq \{28a_{n-1} + (56M^2h + 14M^2h^2)\|Y_{n-1}\|\} e^{(7M^2h + 28M^2)(t-T_1)} \end{aligned}$$

Put

$$(5.26) \quad \gamma_n = 28a_n e^{7M^2h^2 + 28M^2h}.$$

Then we get from the above inequality (5.25) that

$$(5.27) \quad \|Y_n\| = E\left[\sup_{T_1 \leq s \leq T_1+h} |Y_n(s)|^2\right] \leq (\gamma_{n-1} + \eta \|Y_{n-1}\|)$$

holds. Now we are in a position to prove

$$(5.17) \quad \lim_{n \rightarrow \infty} E\left[\sup_{T_1 \leq t \leq T_1+h} |Y_n(t)|^2\right] = \lim_{n \rightarrow \infty} \|Y_n\| = 0.$$

Let  $\epsilon > 0$  be an arbitrary positive number. Choose an positive integer  $N_1$  such that,

$$(5.28) \quad \gamma_{n-1} \leq \frac{\epsilon}{2}(1 - \eta), \quad n \geq N_1,$$

holds.

We have by (5.27) that

$$(5.29) \quad \begin{aligned} \|Y_{N_1+m}\| & \leq \gamma_{N_1+m-1} + \eta \|Y_{N_1+m-1}\| \\ & \leq \gamma_{N_1+m-1} + \eta \gamma_{N_1+m-2} + \eta^2 \|Y_{N_1+m-2}\| \\ & \leq \gamma_{N_1-1}(1 + \eta + \eta^2 + \cdots + \eta^m) + \eta^{m+1} \|Y_{N_1-1}\| \\ & \leq \frac{\gamma_{N_1-1}}{1 - \eta} + 4K_2 \eta^{m+1}. \end{aligned}$$

Choose a positive integer  $N_2$  such that

$$(5.30) \quad 4K_2 \eta^{m+1} < \frac{\epsilon}{2}, \quad m \geq N_2,$$

holds. Hence we can conclude that

$$\|Y_n\| < \epsilon, \quad n \geq N_1 + N_2,$$

holds.

(ii) : Here we will show that

$$(5.31) \quad \lim_{n \rightarrow \infty} E\left[\sup_{T_1 \leq t \leq T_1+h} |X_n(t) - X(t)|^2\right] = 0.$$

holds.



Note that (5.16) and (5.17) hold. Then we can choose a sequence of positive numbers  $\delta_n, n = 1, 2, \dots$  such that

$$(5.32) \quad \begin{aligned} & \delta_n \downarrow 0 (n \rightarrow \infty) \\ & 5E[|X_n(T_1) - X(T_1)|^2] \\ & + (60M^2h + 15M^2h^2)E\left[\sup_{T_1 \leq t \leq T_1+h} |Y_n(t)|^2\right] \leq \delta_n \end{aligned}$$

holds.

By the definition of the processes  $X_n$  and  $X$ , we have

$$(5.33) \quad \begin{aligned} & E\left[\sup_{T_1 \leq s \leq t} |X_n(s) - X(s)|^2\right] \\ & \leq 5E[|X_n(T_1) - X(T_1)|^2] \\ & + 5E\left[\sup_{T_1 \leq s \leq t} \left|\int_{T_1}^s (\sigma(u, X_{n-1}(u)) - \sigma(u, X(u))) dB(u)\right|^2\right] \\ & + 5E\left[\sup_{T_1 \leq s \leq t} \left|\int_{T_1}^s (b(u, X_{n-1}(u)) - b(u, X(u))) du\right|^2\right] \\ & + 5E\left[\sup_{T_1 \leq s \leq t} \left|\int_{T_1}^s (\sigma_x(u, X_{n-1}(u))Y_n(u) dB(u))\right|^2\right] \\ & + 5E\left[\sup_{T_1 \leq s \leq t} \left|\int_{T_1}^s (b_x(u, X_{n-1}(u))Y_n(u) du)\right|^2\right], \quad T_1 \leq t \leq T_1 + h. \end{aligned}$$

By (2.6) and (2.7) in the condition A, it follows from the above (5.33) that

$$(5.34) \quad \begin{aligned} & E\left[\sup_{T_1 \leq s \leq t} |X_n(s) - X(s)|^2\right] \\ & \leq 5E[|X_n(T_1) - X(T_1)|^2] \\ & + 20M^2 \int_{T_1}^t E\left[\sup_{T_1 \leq u \leq s} |X_{n-1}(u) - X(u)|^2\right] ds \\ & + 5hM^2 \int_{T_1}^t E\left[\sup_{T_1 \leq u \leq s} |X_{n-1}(u) - X(u)|^2\right] ds \\ & + 20hM^2 E\left[\sup_{T_1 \leq u \leq T_1+h} |Y_n(t)|^2\right] \\ & + 5M^2h^2 E\left[\sup_{T_1 \leq u \leq T_1+h} |Y_n(t)|^2\right], \quad T_1 \leq t \leq T_1 + h. \end{aligned}$$

Note that

$$\begin{aligned} & E\left[\sup_{T_1 \leq u \leq s} |X_{n-1}(u) - X(u)|^2\right] \\ & \leq 2E\left[\sup_{T_1 \leq u \leq T_1+h} |Y_n(u)|^2\right] + 2E\left[\sup_{T_1 \leq u \leq s} |X_n(u) - X(u)|^2\right], \quad T_1 \leq s \leq T_1 + h. \end{aligned}$$

Then, we observe from (5.34) that

$$\begin{aligned} & E\left[\sup_{T_1 \leq s \leq t} |X_n(s) - X(s)|^2\right] \\ & \leq 5E[|X_n(T_1) - X(T_1)|^2] + 40M^2 \int_{T_1}^t E\left[\sup_{T_1 \leq u \leq s} |X_n(u) - X(u)|^2\right] ds \\ & + 10M^2h \int_{T_1}^t E\left[\sup_{T_1 \leq u \leq s} |X_n(u) - X(u)|^2\right] ds \end{aligned}$$

$$\begin{aligned}
& + (40M^2h + 10M^2h^2 + 20M^2h + 5M^2h^2)E\left[\sup_{T_1 \leq t \leq T_1+h} |Y_n(t)|^2\right] \\
& \leq \delta_n + (40M^2 + 10M^2h) \int_{T_1}^t E\left[\sup_{T_1 \leq u \leq s} |X_n(u) - X(u)|^2\right] ds, \quad T_1 \leq t \leq T_1 + h.
\end{aligned}$$

Hence, by Gronwall's inequality, we have

$$E\left[\sup_{T_1 \leq s \leq t} |X_n(s) - X(s)|^2\right] \leq \delta_n e^{(40M^2h + 10M^2h^2)}, \quad T_1 \leq t \leq T_1 + h.$$

Thus we can conclude that

$$\lim_{n \rightarrow \infty} E\left[\sup_{T_1 \leq t \leq T_1+h} |X_n(t) - X(t)|^2\right] = 0.$$

But this contradicts the definition of  $T_1$ .

q.e.d.

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