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1. Introduction.

Let $X=(\Omega,F,F_t,X_t,\theta_t,P^x)$ be a transient Hunt process with lifetime ζ on a locally compact space $(E,\mathcal{E})$ with countable base. Assume that there is an excessive reference measure $\xi(dx)$ also denoted by dx and a potential kernel $u = u(x,y)$ such that for all nonnegative Borel functions f

$$(1.1) \qquad Uf(x) = E^x\left[\int_0^\infty f(X_t)\,dt\right] = \int u(x,y)\,f(y)\,dy.$$

Let(P_t) be the transition semigroup of X. A Borel measurable function $s \geq 0$ is called excessive if

$$(1.2) \qquad P_t s \leq s \text{ and } \lim_{t\to 0} P_t s = s.$$

An excessive function s is called a natural potential function, if s is finite and

$$(1.3) \qquad \lim_{n\to\infty} P_{T_n} s(x) = 0$$

for every x, whenever $\{T_n\}$ is an increasing sequence of stopping times with limit $T \geq \zeta$ almost surely P^x. Here $P_{T_n}s(x) = E^x[s(X_{T_n}); T_n < \zeta]$.

It is well known that each natural potential function s is generated by a unique integrable natural additive functional A, i.e.,

$$(1.4) \qquad s(x) = E^x(A_\infty).$$

Let us recall the definition of the energy of the natural potential function. Details may be found in [6]. Definitions given there are for almost everywhere finite class (D) potential functions. Since every natural potential function is of class (D), the results are applicable here as well.

The mass functional of an excessive function s is defined as

$$(1.5) \qquad L(s) = \sup\left\{\int s\,d\lambda\,;\ \lambda U \leq \xi\right\}.$$

Let s be a natural potential function and A the corresponding natural additive functional. If $p = E^{\cdot}(A_\infty^2)$ is finite, then p is necessarily a natural potential function. If $L(p) < \infty$, we say that s has finite energy and put

$$(1.6) \qquad \|s\|_e^2 = L(p).$$

If r and s are natural potential functions of finite energy generated by natural additive functionals A and B, respectively, their mutual energy is defined by

$$(1.7) \qquad (r,s)_e = L(E^{\cdot}[A_\infty B_\infty]).$$

Let $\mathcal{R}$ denote the linear space of differences of natural potential functions of finite energy. The above definition extends to $\mathcal{R}$.

Let us now introduce the excessive domination principle.

(**ED**): there exists a positive constant K such that for every $s \in \mathcal{R}$ there exists a natural potential function p satisfying $|s| \leq p$ and $\|p\|_e \leq K\|s\|_e$.

In [**7**] it was proved that if (**ED**) holds, then Hunt's hypothesis (**H**) holds, i.e., every excessive function is regular.

In this note we give a sufficient condition such that (**ED**) holds. We prove our result in the setting of the dual Hunt processes as described in [**1-VI**]. We show that the excessive domination principle is equivalent to the weak sector condition (**S**) defined as follows:

(**S**): for each signed measure ν such that $(U|\nu|, |\nu|) < \infty$ and for each positive measure μ, $(U\nu, \mu)^2 \leq M(U\nu, \nu)(U\mu, \mu)$ where M is a positive constant not depending on μ and ν.

This result gives the potential-theoretic characterization of the weak sector condition.

The main tools in proving this result are capacitary inequalities for energy established by M.Rao in [**7**]. He proves that the energy of a natural potential function is comparable with its capacitary integral. Precisely,

$$\frac{1}{2}\|s\|_e^2 \leq \int_0^\infty tC(s > t)\, dt \leq 2\|s\|_e^2$$

(see Thm.3.1). He also proves that for $s \in \mathcal{R}$ the capacitary integral $\int_0^\infty tC(|s| > t)\, dt$ is finite and that there exists a natural potential function p dominating $|s|$ such that $m\|p\|_e^2 \leq \int_0^\infty tC(|s| > t)\, dt \leq M\|p\|_e^2$ where m and M are positive constants not depending on s (see Theorem 3.2).

The missing link was the estimate of the capacitary integral of $s \in \mathcal{R}$ in terms of the energy of s. In Proposition 2.2 we obtain this estimate for the not necessarily Markov kernel U which satisfies the weak sector condition and both U and $\hat{U}$ satisfy the weak maximum principle. We use the symmetric kernel $V = U + \hat{U}$ and modify the argument from [**5**].

In Section 2. we provide the details of the estimate, while in Section 3. we recall all necessary results and obtain the announced equivalence.

2. The Estimate.

Let E be a locally compact space with countable base. A kernel on E is a nonnegative lower semi-continuous function u defined on the product $E \times E$.

For a positive Radon measure μ we define

$$U\mu(x) = \int u(x,y)\, \mu(dy) \text{ and } \hat{U}\mu(x) = \int u(y,x)\, \mu(dy). \tag{2.1}$$

We also define $U\mu$ and $\hat{U}\mu$ for a signed measure μ. For signed measures μ and ν let us denote

$$(U\mu, \nu) = \int U\mu(x)\nu(dx) \text{ whenever } \int U|\mu|(x)|\nu|(dx) < \infty. \tag{2.2}$$

Then

$$(U\mu, \mu) = \iint u(x,y)\mu(dy)\mu(dx) = (\hat{U}\mu, \mu).$$

Let $v(x,y) = u(x,y) + u(y,x)$ and $V\mu(x) = \int v(x,y)\mu(dy)$. Hence $V = U + \hat{U}$ and v is the symmetric kernel.

We define the capacities with respect to U and V as follows: for $A \subset E$,

$$C_U(A) = \sup\{\mu(E); \mu \text{ measure with compact support } S(\mu) \subset A \text{ and } U\mu \leq 1 \text{ on } S(\mu)\}, \tag{2.3}$$

$$(2.4)\quad C_V(A) = \sup\{\mu(E); \mu \text{ measure with compact support } S(\mu) \subset A \text{ and } V\mu \leq 1 \text{ on } S(\mu)\}.$$

It is well known that such capacities are inner regular (e.g., [2] p.153)

$$C_U(A) = \sup\{C_U(K)\ ;\ K \subset A,\ K \text{ compact}\} \tag{2.5}$$

and similarly for V.

From now on we assume that U satisfies the weak sector condition (**S**) as defined in Section 1.

The weak sector condition immediately implies the positivity of U: if ν is a signed measure such that $(U|\nu|, |\nu|) < \infty$, then $(U\nu, \nu) \geq 0$. Thus both U and $\hat{U}$ are positive. Therefore V is also positive and by using symmetry we have

$$(V\mu, \nu) \leq (V\mu, \mu)^{\frac{1}{2}}(V\nu, \nu)^{\frac{1}{2}} \tag{2.6}$$

for μ, ν signed measures.

For each compact set K there exists a positive measure λ with the support in K such that

$$(V\lambda, \lambda) = C_V(K) = \lambda(K),\ V\lambda \leq 1 \text{ in } S(\lambda) \text{ and } V\lambda \geq 1\ C_V - \text{a.e. on } K \tag{2.7}$$

(e.g., [2] p.159). λ is called the equilibrium measure for K.

It is necessary to compare the capacities with respect to U and V. The following lemma shows that they are comparable.

LEMMA 2.1. *For every set A*

$$C_V(A) \leq C_U(A) \leq 2C_V(A). \tag{2.8}$$

PROOF: By the inner regularity of C_U and C_V it is enough to show the inequalities for compact sets.

Let K be compact, μ a measure such that $S(\mu) \subseteq K$ and $V\mu \leq 1$ on $S(\mu)$. Then $U\mu \leq 1$ on $S(\mu)$, so trivially $C_V(K) \leq C_U(K)$. Therefore, if $C_U(K) = 0$, then $C_V(K) = 0$.

Assume that $C_U(K) > 0$; then there is a measure μ on K such that $U\mu \leq 1$ on $S(\mu)$ and $\mu(E) = \mu(K) > 0$. For the symmetric kernel V,

$$C_V(K) = [\inf(V\nu, \nu)]^{-1}$$

where ν ranges over all measures concentrated on K ([2]). Let $\lambda = \mu/\mu(E)$; then $(V\nu, \nu) \leq (V\lambda, \lambda) = 2(U\lambda, \lambda) = 2/\mu(E) < \infty$ which implies $C_V(K) > 0$.

Hence, the sets of C_V-capacity zero are precisely these which are of C_U-capacity zero.

Now we show the second inequality in (2.8). We may assume that $C_U(K) > 0$. Let μ be a measure on K such that $U\mu \leq 1$ on $S(\mu)$ and let ν be a V-equilibrium measure of K. Since $(U\mu, \mu) \leq (1, \mu) < \infty$, μ does not charge sets of C_U-capacity zero, and hence C_V-capacity zero. Therefore

$$\mu(E) = (1, \mu) \leq (V\nu, \mu) \leq (V\nu, \nu)^{\frac{1}{2}}(V\mu, \mu)^{\frac{1}{2}} = C_V(K)^{\frac{1}{2}}[2(U\mu, \mu)]^{\frac{1}{2}} \leq \sqrt{2}C_V(K)^{\frac{1}{2}}\mu(E)^{\frac{1}{2}}$$

so $\mu(E) \leq 2C_V(K)$. Hence $C_U(K) \leq 2C_V(K)$. ∎

For the following result we need an additional assumption on U. We assume that both U and $\hat{U}$ satisfy the weak maximum principle:

($\mathbf{M_W}$): there exists a positive constant A such that for every positive measure μ with compact support, $U\mu \leq 1$ on $S(\mu)$ implies $U\mu \leq A$ everywhere.

$(\hat{\mathbf{M}}_{\mathbf{W}})$ is defined similarly. Then V also satisfies the weak maximum principle (with constant $2A$).

Now we prove the main estimate.

PROPOSITION 2.2. *Assume that* $(\mathbf{S})$,$(\mathbf{M}_{\mathbf{W}})$, $(\hat{\mathbf{M}}_{\mathbf{W}})$ hold for U. Let ν be a signed measure such that $U\nu \geq 0$ and

$$\int_0^\infty tC_U(U\nu > t)\,dt < \infty. \tag{2.9}$$

Then

$$\int_0^\infty tC_U(U\nu > t)\,dt \leq 24MA^2(U\nu, \nu). \tag{2.10}$$

PROOF: For each integer n let $B_n = \{U\nu > 2^n\}$. Then

$$\begin{aligned}\int_0^\infty tC_U(U\nu > t)\,dt &= \sum_n \int_{2^{n-1}}^{2^n} tC_U(U\nu > t)\,dt \\ &\geq \sum_n \int_{2^{n-1}}^{2^n} tC_U(U\nu > 2^n)\,dt = \frac{3}{8}\sum_n 2^{2n}C_U(B_n)\end{aligned} \tag{2.11}$$

Similarly

$$\int_0^\infty tC_U(U\nu > t)\,dt \leq \frac{3}{2}\sum_n 2^{2n}C_U(B_n). \tag{2.12}$$

Let $\epsilon > 0$. For each integer n let K_n be a compact subset of B_n such that

$$C_V(B_n) \leq C_V(K_n) + \epsilon_n, \tag{2.13}$$

where $\sum_n 2^{2n}\epsilon_n < \epsilon$.

Let μ_n be a V-equilibrium measure of K_n, i.e. μ_n is a positive measure on K_n, $\mu_n(E) = C_V(K_n)$ and $V\mu_n \leq 1$ on $S(\mu_n)$. By the weak maximum principle $V\mu_n \leq 2A$ everywhere. Define the measure μ as

$$\mu = \sum_n 2^n\mu_n. \tag{2.14}$$

Then

$$\begin{aligned}(V\mu, \mu) &= \sum_n\sum_m 2^{n+m}(V\mu_n, \mu_m) \\ &\leq 2\sum_n\sum_{m\leq n} 2^{n+m}(V\mu_m, \mu_n) \leq 2\sum_n\sum_{m\leq n} 2^{n+m}(2A, \mu_n) \\ &= 4A\sum_n C_V(K_n)\sum_{m\leq n} 2^{n+m} = 4A\sum_n 2^{2n}C_V(K_n).\end{aligned} \tag{2.15}$$

Using (2.11), (2.9) and Lemma 2.1, we get $(V\mu, \mu) < \infty$.

On K_n we have $U\nu > 2^n$. Hence

$$\sum_n 2^{2n} C_V(K_n) = \sum_n 2^{2n}(1, \mu_n) = \sum_n 2^n(2^n, \mu_n)$$
$$\leq \sum_n 2^n(U\nu, \mu_n) = (U\nu, \mu) \leq \sqrt{M}(U\nu, \nu)^{\frac{1}{2}}(U\mu, \mu)^{\frac{1}{2}}$$
$$= \frac{\sqrt{M}}{\sqrt{2}}(U\nu, \nu)^{\frac{1}{2}}(V\mu, \mu)^{\frac{1}{2}} \leq 2\sqrt{2}\sqrt{M}A(U\nu, \nu)^{\frac{1}{2}}\left[\sum_n 2^{2n}C_V(K_n)\right]^{\frac{1}{2}}.$$

Therefore

$$\sum_n 2^{2n} C_V(K_n) \leq 8MA^2(U\nu, \nu). \tag{2.16}$$

Further, by (2.13),

$$\sum_n 2^{2n} C_V(K_n) \geq \sum_n 2^{2n} C_V(B_n) - \epsilon$$

Using the above, (2.16), (2.12) and Lemma 2.1 we get

$$\int_0^\infty tC_U(U\nu > t)\, dt \leq 3\sum_n 2^{2n} C_V(B_n)$$
$$\leq 3\sum_n 2^{2n} C_V(K_n) + 3\epsilon \leq 24MA^2(U\nu, \nu) + 3\epsilon. \tag{2.17}$$

Thus (2.10) holds. ∎

Now we show that the excessive domination principle for this situation implies the weak sector condition.

PROPOSITION 2.3. *Assume that for each signed measure ν such that $(U|\nu|, |\nu|) < \infty$, there exists a positive measure λ satisfying $|U\mu| \leq U\lambda$ and $(U\lambda, \lambda) \leq M(U\nu, \nu)$, where M is a positive constant not depending on ν.*

Then the weak sector condition holds.

PROOF: Let ν be a signed measure and μ a positive measure. Then

$$(U\nu, \mu)^2 \leq (|U\nu|, \mu)^2 \leq (U\lambda, \mu)^2 \leq (U\lambda, \lambda)(U\mu, \mu) \leq M(U\nu, \nu)(U\mu, \mu). \quad ∎$$

3. Proof of the equivalence.

In this section we assume that X and $\hat{X}$ are transient Hunt processes on the LCCB space $(E, \mathcal{E})$ with respect to a σ-finite excessive measure $\xi(dx)$ as described in [**1-VI**]. Let $u(x, y)$ be the potential density of the potential operator U which is excessive in the first variable and coexcessive in the second variable. We assume that U and $\hat{U}$ satisfy conditions (2.1), (2.2), (4.1) and (4.2) from [**1-VI**]. Then u is a lower semi-continuous function. By Proposition 2.10 in [**1-VI**] every natural potential function s is a potential of measure.

We assume that U satisfies the weak sector condition (**S**). Then U and $\hat{U}$ are positive kernels. By Theorem 3.2 in [**3**], both U and $\hat{U}$ satisfy the maximum principle (in particular, hypothesis (**H**) holds). Therefore, we may apply the results from Section 2.

In this setting the energy of a natural potential function $s = U\mu$ is simply $2(U\mu, \mu)$ (see [6]).

In [1-VI] the capacity $C(B)$ of a relatively compact Borel set B is defined as

(3.1)
$$C(B) = \sup\{\mu(E);\ \mu \text{ positive measure with support } B \text{ and } U\mu \leq 1 \text{ everywhere }\}.$$

By the maximum principle, $C(B) = C_U(B)$ where C_U is defined in Section 2.

The following two theorems are Theorem 2.4 and Theorem 2.5 from [7].

THEOREM 3.1. *Let s be a natural potential function of finite energy. Then*

$$\frac{1}{2}\|s\|_e^2 \leq \int_0^\infty tC(s > t)\,dt \leq 2\|s\|_e^2. \tag{3.2}$$

THEOREM 3.2. *Let F be a Borel measurable function. Put $g(x) = E^x[F^*]$, where $F^* = \sup_{t>0} |F(X_t)|$. Then g is excessive, $g \geq |F|$ except for a semipolar set and*

$$\|g\|_e^2 \leq 16 \int_0^\infty tC(|F| > t)\,dt. \tag{3.3}$$

Further, if F is finely lower semi-continuous, then $g \geq |F|$ everywhere and

$$\int_0^\infty tC(|F| > t)\,dt \leq 2\|g\|_e^2. \tag{3.4}$$

By Theorem 3.1 we get that for $s = s_1 - s_2 \in \mathcal{R}$

$$\int_0^\infty tC(|s| > t)\,dt < \infty. \tag{3.5}$$

Indeed, $|s| \leq s_1 + s_2$ and $s_1 + s_2$ is the natural potential function of finite energy. Hence

$$\int_0^\infty tC(|s| > t)\,dt \leq \int_0^\infty tC(s_1 + s_2 > t)\,dt \leq 2\|s_1 + s_2\|_e^2 < \infty.$$

We are now ready to prove

PROPOSITION 3.3. *Let $s \in \mathcal{R}$ and assume $s \geq 0$. Then there is a natural potential function p of finite energy such that*

$$s \leq p \text{ and } \|p\|_e^2 \leq K\|s\|_e^2, \tag{3.6}$$

where K is independent of s.

PROOF: By the remarks above, $s = U\nu$ for a signed measure ν and $\int_0^\infty tC(U\nu > t) < \infty$. Using Proposition 2.2, we get

$$\int_0^\infty tC(U\nu > t)\,dt \leq 24M(U\nu, \nu) = 12M\|U\nu\|_e^2. \tag{3.7}$$

Since $U\nu$ is finely continuous, we may apply the second part of Theorem 3.2. The function $p(x) = E^x(s^*)$ is finite, $p \geq s = U\nu$ and

$$\|p\|_e^2 \leq 16 \int_0^\infty tC(U\nu > t)\,dt. \tag{3.8}$$

p is necessarily a natural potential function, so by combining (3.7) and (3.8) we get (3.6) with $K = 172M$. ∎

We have obtained the excessive domination principle for $s \in \mathcal{R}$ nonnegative. To extend the result to an arbitrary $s \in \mathcal{R}$ we need the following result which is proved in [4] (see Theorem 5).

PROPOSITION 3.4. *Let $s \in \mathcal{R}$. Then $u = |s| \in \mathcal{R}$ and $\|u\|_e \leq \|s\|_e$.*

Using the last two propositions, for each $s \in \mathcal{R}$ there is a natural potential function p such that $|s| \leq p$ and $\|p\|_e^2 \leq K\|s\|_e^2$. Together with Proposition 2.3 this gives

THEOREM 3.5. *Excessive domination principle* (**ED**) is equivalent to the weak sector condition (**S**).

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