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The Markov Process of Total Spins

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We consider the quantum stochastic process of independent addition of spins. Meyer observed [3], that the total spins form a commuting system of operators and may be interpreted as a classical stochastic process. The law of this process has been calculated by Biane [1] in two special cases. We want to calculate it in general. One obtains a Markov chain homogenous in time.

Our notation is that of [4] and differs a bit from [1]. The spin matrices are

$$\sigma_1 = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \frac{1}{2}\begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.$$

A state ω on the algebra M_2 of complex 2×2 - matrices is given by a density matrix ρ which without loss of generality we assume to be given in the form

$$\rho = \begin{pmatrix} \rho_1 & 0 \\ 0 & \rho_2 \end{pmatrix} = \begin{pmatrix} 1/2 + z & 0 \\ 0 & 1/2 - z \end{pmatrix}, \quad 0 \leq z \leq 1/2.$$

If $A \in M_2$, then

$$\omega(A) = \mathrm{Tr}\, \rho A\,.$$

Consider $(C^2)^{\otimes N}$ and $(M_2)^{\otimes N}$ and on this algebra the state $\omega^{\otimes N}$ given by the density matrix $\rho^{\otimes N}$. We define for $1 \leq n \leq N$

$$\sigma_{i,n} = 1 \otimes \ldots \otimes 1 \otimes \sigma_i \otimes \ldots \otimes 1,$$

where the σ_i stands on the n-th place. Define

$$\sigma_i^{(n)} = \sigma_{i,1} + \ldots + \sigma_{i,n}$$

and

$$\sigma^{(n)2} = (\sigma_1^{(n)})^2 + (\sigma_2^{(n)})^2 + (\sigma_3^{(n)})^2.$$

By a remark of Meyer [3] the $\sigma^{(n)2}$ commute for $1 \leq n \leq N$. Hence together with $\omega^{\otimes N}$ one can define a corresponding classical stochastic process. This process was calculated by Biane [1] for $z = 0$, or $\rho_1 = \rho_2$ (symmetric case) and for $z = 1/2$ or $\rho_1 = 1, \rho_2 = 0$ (empty state). In the symmetric case Biane obtained the random walk on dual hypergroup of SU(2) considered previously by Eymard and Roynette [2]. This is no accidental coincidence, because the random walk on the dual hypergroup of a compact group is a special case of a non-commutative random walk on the group. These topics shall be discussed in a forthcoming paper.

We observe that the $\sigma_i^{(n)}$ have the same commutation relations as the σ_i :

$$[\sigma_i, \sigma_j] = i\sigma_3$$

(and cyclic permutations), so they form a representation of the Lie algebra su(2) and of the group SU(2) of unitary 2×2 - matrices with determinant 1.

Let V be a finite dimensional unitary vector space and let S_1, S_2, S_3 be hermitian operators on V with the same commutation relations as the σ_i. If V is irreducible, then it induces an irreducible representation $\mathcal{D}^\ell$ of su(2) or SU(2), where

$$\ell \in \Lambda = \{0,1/2,1,3/2,2,...\}.$$

The dimension of V is $2\ell + 1$. There exists a basis ψ_m, $m = \ell, -\ell + 1,..., \ell$ such that

$$S_3\psi_m = m\psi_m .$$

The operator $S^2 = S_1^2 + S_2^2 + S_3^2$ has the property

$$S^2\psi = \ell(\ell + 1)\psi,$$

for all $\psi \in V$.

If V is not irreducible it can be split into the orthogonal sum of irreducible vector spaces with representations $\mathcal{D}^{\ell_1}$ and $\mathcal{D}^{\ell_2}$. Then $V_1 \otimes V_2$ splits into the orthogonal sum

$$V_1 \otimes V_2 = \bigoplus_\ell W_\ell ,$$

where $\ell = |\ell_1 - \ell_2|, |\ell_1 - \ell_2| + 1, ..., \ell_1 + \ell_2$. The spaces W_ℓ induce the representation $\mathcal{D}^\ell$ and are determined by V_1 and V_2 and ℓ in a unique way.

The vector space $\mathbf{C}^2$ with $\sigma_1, \sigma_2, \sigma_3$ induces the irreducible representation $\mathcal{D}^{1/2}$. We want to split $(\mathbf{C}^2)^{\otimes N}$ into irreducible subspaces. An *admissible path* of length n is a sequence

$$\Gamma = (\ell_0,...,\ell_n),$$

with $\ell_i \in \Lambda = \{0,1/2,1,...\}$, with $\ell_0 = 0$ and $\ell_n - \ell_{n-1} = \pm 1/2$ for $k = 1,...,n$. In Fig. 1 the admissible paths are drawn and in the points of the diagram the numbers of admissible paths leading to this point are indicated.

Proposition 1. Let $\Gamma = (\ell_0,...,\ell_n)$ be an admissible path. Then there exists exactly one irreducible subspace V_Γ of $(\mathbf{C}^2)^{\otimes N}$. The subspace V_Γ induces the representation $\mathcal{D}^{\ell_n}$. One has

$$(\mathbf{C}^2)^{\otimes N} = \bigoplus_\Gamma V_\Gamma ,$$

where V_Γ runs over all admissible paths of lengths n. The space V_Γ consists of the vectors ψ obeying the equation

$$\sigma^{(n)2}\psi = \ell_n(\ell_n + 1)\psi,$$

for $1 \le k \le n$.

Proof. The proposition is clear for $n = 1$. We prove it by induction from $n - 1$ to n. One has

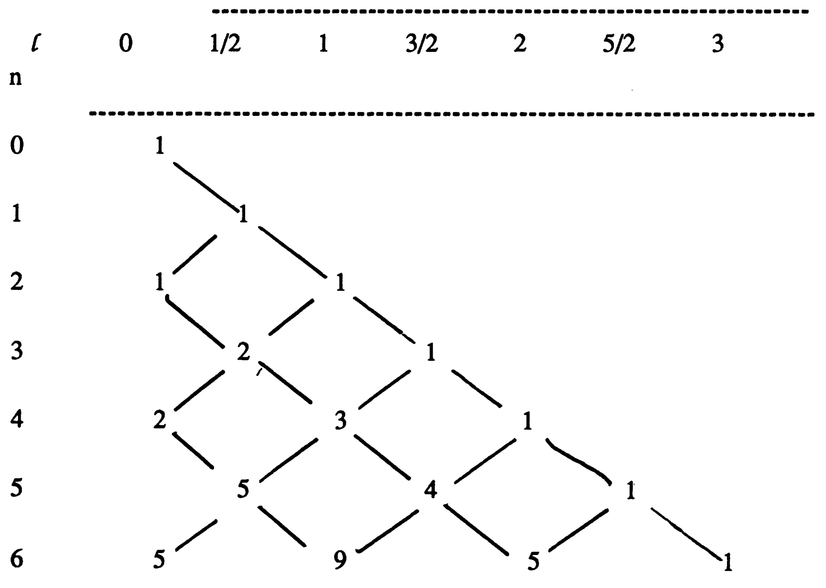
$$(\mathbf{C}^2)^{\otimes(n-1)} = \bigoplus_{\Gamma'} V_\Gamma$$

where the orthogonal sum runs over all admissible paths $\Gamma' = (\ell_0,...,\ell_{n-1})$ of length n-1. Then

$$(\mathbf{C}^2)^{\otimes n} = \bigoplus_{\Gamma'} (V_{\Gamma'} \otimes \mathbf{C}^2).$$

If $\ell_{n-1} = 0$ then $V_{\Gamma'}$ belongs to $\mathcal{D}^0$ and $V_{\Gamma'} \otimes C^2$ is irreducible of type $\mathcal{D}^{1/2}$. If $\ell_{n-1} > 0$, then $V_{\Gamma'} \otimes C^2$ splits into two irreducible subspaces of types $\mathcal{D}^{\ell_{n-1} \pm 1/2}$. Denote them by V_Γ with $\Gamma = (\ell_0, \ldots, \ell_{n-1}, \ell_{n-1} \pm 1/2)$.

Figure 1.



Proposition 2. Let $V \subset (C^2)^{\otimes n}$ be an irreducible representation of type $\mathcal{D}^\ell$ and let $\mathcal{P}_V$ be the orthogonal projection on V. Then

$$\omega^{\otimes N}(\mathcal{P}_V) = w_{n,\ell} = \begin{cases} (2\ell + 1)2^{-n} & \text{for } \rho_1 = \rho_2 = 1/2 \\ (\rho_1\rho_2)^{n/2} \cdot \dfrac{\rho_1^{2\ell+1} - \rho_2^{2\ell+1}}{\rho_2 - \rho_1} & \text{for } \rho_1 \neq \rho_2 \end{cases}$$

Proof. We choose in V a basis ψ_m, $m = -\ell, \ldots, +\ell$ such that

$$\sigma_3^{(n)} \psi_m = m \psi_m .$$

We choose in C^2 the basis

$$\varphi(-\tfrac{1}{2}) = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \quad \varphi(+\tfrac{1}{2}) = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

and in $(C^2)^{\otimes n}$ the basis

$$\varphi(\varepsilon_1, \ldots, \varepsilon_n) = \varphi(\varepsilon_1) \otimes \ldots \otimes \varphi(\varepsilon_n)$$

with $\varepsilon_i = \pm \frac{1}{2}$. Then

$$\sigma_3^{(n)} \varphi(\varepsilon_1, \ldots, \varepsilon_n) = (\varepsilon_1 + \ldots + \varepsilon_n) \varphi(\varepsilon_1, \ldots, \varepsilon_n) = m \varphi(\varepsilon_1, \ldots, \varepsilon_n)$$

and

$$\rho^{\otimes n}\varphi(\varepsilon_1,\ldots,\varepsilon_n) = \rho_1^{\frac{n}{2}-m}\rho_2^{\frac{n}{2}+m}\varphi(\varepsilon_1,\ldots,\varepsilon_n).$$

As ψ_m is a linear combination of $\varphi(\varepsilon_1,\ldots,\varepsilon_n)$ with $\varepsilon_1 + \ldots + \varepsilon_n = m$ one has

$$\rho^{\otimes n}\psi_m = \rho_1^{\frac{n}{2}-m}\rho_2^{\frac{n}{2}+m}\psi_m$$

and

$$w_{n,\ell} = \sum_{m=-\ell}^{\ell} \rho_1^{\frac{n}{2}-m}\rho_2^{\frac{n}{2}+m}.$$

Proposition 3. The number of admissible paths of length n ending in ℓ is

$$d_{n,\ell} = \binom{n}{n/2-\ell} - \binom{n}{n/2-\ell-1}.$$

For the proof see [4], eq.(38).

We define on Λ^N, $\Lambda = \{0,1/2,1,\ldots\}$ a probability measure by putting

$$P\{(\ell_0,\ldots,\ell_N)\} = \begin{cases} w_{N,\ell_N} & \text{if } (\ell_0,\ldots,\ell_n) \text{ is admissible} \\ 0 & \text{if } (\ell_0,\ldots,\ell_n) \text{ is not admissible} \end{cases}.$$

Define a stochastic process $L_0, L_1,\ldots, L_N$ on Λ^N by

$$L_0 = 0$$
$$L_n(\ell_1,\ldots,\ell_N) = \ell_n .$$

Proposition 4. One has

$$P\{L_n = \ell\} = d_{n,\ell}w_{n,\ell}$$

and

$$P\{L_1 = \ell_1,\ldots,L_n = \ell_n\} = \begin{cases} w_{n,\ell_n} & \text{if } (\ell_0,\ldots,\ell_n) \text{ is admissible} \\ 0 & \text{if } (\ell_0,\ldots,\ell_n) \text{ is not admissible} \end{cases}.$$

Proof. If S is the set of admissible paths of length N starting with $\Gamma_0 = (\ell_0,\ldots,\ell_n)$, then

$$P\{L_1 = \ell_1,\ldots,L_n = \ell_n\} = \sum_{\Gamma\in S} P(\Gamma) = \sum_{\Gamma\in S} \omega^{\otimes N}(P_{V_\Gamma}) = \omega^{\otimes N}(P_{V_{\Gamma_0}}\otimes(C^2)^{N-n}) = \omega^{\otimes n}(P_{V_{\Gamma_0}}) = w_{n,\ell_n}.$$

This gives the second assertion of the proposition. The first one is immediate

Proposition 5. The process L_n , $n = 0,1,2,\ldots$ is a homogeneous Markov chain with transition probability

$$P\{L_n = \ell \mid L_{n-1} = \ell'\} = \begin{cases} \dfrac{2\ell+1}{2(2\ell'+1)} & \text{for } \rho_1 = \rho_2 \\ (\rho_1\rho_2)^{\ell'-\ell}\dfrac{\rho_1^{2\ell+1}-\rho_2^{2\ell+1}}{\rho_1^{2\ell'+1}-\rho_2^{2\ell'+1}} & \text{for } \rho_1 \neq \rho_2 \end{cases} \text{ if } \ell-\ell' = \pm\frac{1}{2} \text{ and } 0 \text{ otherwise.}$$

Proof. We calculate

$$P\{L_n = \ell_n \mid L_{n-1} = \ell_{n-1},\ldots,L_1 = \ell_1\} = \frac{P\{L_n = \ell_n,\ldots,L_1 = \ell_1\}}{P\{L_{n-1} = \ell_{n-1},\ldots,L_1 = \ell_1\}} = \frac{w_{n,\ell_n}}{w_{n,\ell_{n-1}}}.$$

On the other hand

$$P\{L_n = \zeta_n , L_{n-1} = \zeta_{n-1}\} = d_{n-1,\zeta_{n-1}} w_{n,\zeta_n} ,$$

as the number of admissible paths of length n ending with ζ_{n-1} , ζ_n is equal to the number of admissible paths of length n - 1 ending in $\zeta_{n-1} S_0$ by proposition 4

$$P\{L_n = \zeta_n \mid L_{n-1} = \zeta_{n-1}\} = \frac{w_{n,\zeta_n}}{w_{n-1,\zeta_{n-1}}} .$$

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