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ON CONVERGENCE OF SEMIMARTINGALES

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Let X be a semimartingale. A norm commonly used on the space of semimartingales is the \mathcal{H}^p norm: One defines

$$j_p(M, A) = \| [M, M]_\infty^{1/2} + \int_0^\infty |dA_s| \|_{L^p}$$

for any decomposition $X = M + A$ with M a local martingale and A an adapted, right continuous process with paths of finite variation on compacts. Then

$$\|X\|_{\mathcal{H}^p} = \inf_{X=M+A} j_p(M, A)$$

where the infimum is taken over all such decompositions of X . Then as is well known (see, for example, Emery [2], Meyer [7], or Protter [8], Theorem 2 of Chapter V):

$$\|X^*\|_{L^p} \leq c_p \|X\|_{\mathcal{H}^p} \quad (1 \leq p < \infty)$$

where $X^* = \sup_t |X_t|$, and c_p is a universal constant. An immediate consequence is that if a sequence of semimartingales X^n converges to X in \mathcal{H}^1 , then

$$\lim_{n \rightarrow \infty} E\{(X^n - X)^*\} = 0$$

as well.

In this paper we examine the converse question: if $X^n = M^n + A^n$ is a sequence of semimartingales converging uniformly in L^1 to a process X , what can be said about the convergence of the M^n and A^n processes of the decompositions? Such a question is closely related to recent work on weak convergence of semimartingales: In particular Jacod-Shiryaev [3], Jakubowski-Mémin-Pages [4], and Kurtz-Protter [5].

The examination of two simple examples illustrates the problems that arise and shows that one cannot expect a full converse.

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Let Y be any continuous, adapted process with $Y_0 = 0$ and Y constant on $[1, \infty)$; set

$$X_t^n = n \int_{t-1/n}^t Y_s ds 1_{\{t > 1/n\}}.$$

Then X^n is a differentiable function of t in $[\frac{1}{n}, \infty)$ for each n and in particular each X^n is of finite variation (and hence it is a semimartingale). However the limit Y need not be a semimartingale.

The preceding example indicates that we have to impose some type of uniform bound on the total variation of the A^n processes. But even if we do this we cannot hope always to obtain convergence of the A^n processes in total variation norm. Indeed, let $0 \leq t \leq \frac{\pi}{2}$, and define $A_t^n = \frac{1}{n} \sin nt$. Then $\int_0^{\pi/2} |dA_s^n| = 1$, but $(A^n)^*$ converges to zero.

The following theorem avoids the pathologies of the two preceding examples. Recall that a semimartingale X in \mathcal{H}^1 is special: that is, it always has a unique decomposition $X = X_0 + M + A$, where $M_0 = A_0 = 0$, and the finite variation process A is predictable. Such a decomposition is said to be the canonical decomposition.

Theorem 1. Let X^n be a sequence of semimartingales in \mathcal{H}^1 with canonical decomposition $X^n = X_0^n + M^n + A^n$, satisfying for some constant K ,

$$(1a) \quad E\left\{\int_0^\infty |dA_s^n|\right\} \leq K$$

$$(1b) \quad E\{(M^n)^*\} \leq K.$$

Let X be a process, and suppose that

$$(2) \quad E\{(X^n - X)^*\} \rightarrow 0 \quad \text{as } n \rightarrow \infty.$$

Then X is a semimartingale in \mathcal{H}^1 , and if $X = X_0 + M + A$ is its canonical decomposition we have

$$(3) \quad E\{M^*\} \leq K, \quad E\left\{\int_0^\infty |dA_s|\right\} \leq K$$

and

$$(4a) \quad \lim_{n \rightarrow \infty} \|M^n - M\|_{\mathcal{H}^1} = 0,$$

$$(4b) \quad \lim_{n \rightarrow \infty} E\{(A^n - A)^*\} = 0.$$

Corollary 2. Let (X^n) be a sequence of special semimartingales with canonical decomposition $X^n = X_0^n + M^n + A^n$, where the A^n satisfy (1a). Then if X is a process such

that $\lim_{n \rightarrow \infty} \|(X^n - X)^*\|_{L^1} = 0$, X is a special semimartingale. Further if $X = X_0 + M + A$ is its canonical decomposition, then

$$\lim_{n \rightarrow \infty} \|M^n - M\|_{\mathcal{H}^1} = 0, \quad \lim_{n \rightarrow \infty} E\{(A^n - A)^*\} = 0, \quad E\left\{\int_0^\infty |dA_s|\right\} \leq K.$$

Proof. By deleting a finite number of terms in the sequence (X^n) , we may suppose that $E\{(X^n - X)^*\} \leq K$ for $n \geq 1$. But then

$$\begin{aligned} E\{(M^n - M^1)^*\} &\leq E\{|X_0^n - X_0^1|\} + E\{(X^n - X)^*\} + E\{(A^n - A^1)^*\} \\ &\leq 4K. \end{aligned}$$

So write $\tilde{X}^n = X^n - M^1 = X_0^n + (M^n - M^1) + A^n$, $\tilde{X} = X - M^1$. Then the hypotheses of Theorem 1 hold for \tilde{X}^n , \tilde{X} and the conclusion follows easily. \square

The proof of Theorem 1 uses some ideas from Kurtz and Protter [5], and it also needs the following martingale inequality.

Proposition 3. Let $p \geq 1/2$, M be a martingale in \mathcal{H}^{2p} and K be a predictable process with $K^* \in L^{2p}$. Then

$$\|(K \cdot M)^*\|_{L^p} \leq c_p \|K^*\|_{L^{2p}} \|M^*\|_{L^{2p}}.$$

Proof. Recall the Davis decomposition of M — see Meyer [6, p. 80–81]. Let $\Delta M_s = M_s - M_{s-}$. Let $A_t = \sup_{s \leq t} |\Delta M_s|$; then $M = N + U$, where N is a martingale with $|\Delta N_t| \leq A_{t-}$, and U is a martingale with paths of integrable variation satisfying

$$\left\| \int |dU_s| \right\|_{L^q} \leq c_q \|A_\infty\|_{L^q}, \quad q \geq 1.$$

Further, we have the pointwise inequalities

$$\begin{aligned} A_\infty &\leq 2M^*, \\ [N]_\infty^{1/2} &\leq [M]_\infty^{1/2} + [U]_\infty^{1/2}, \\ [U]_\infty^{1/2} &\leq 4A_\infty. \end{aligned}$$

Now $(K \cdot M)^* \leq (K \cdot N)^* + (K \cdot U)^*$, and $|\Delta(K \cdot N)_t| \leq K_t^* A_t$. Hence, by Meyer [6], Theorem 2 on p. 76,

$$\begin{aligned} \|(K \cdot M)^*\|_{L^p} &\leq c_p (\|[K \cdot N]_\infty + (K^* A_\infty)^2\|_{L^p}^{1/2} + \|(K \cdot U)^*\|_{L^p}) \\ &\leq c_p (\|[K \cdot N]_\infty^{1/2} + K^* A_\infty\|_{L^p} + \|(K \cdot U)^*\|_{L^p}) \\ &\leq c_p (\|K^* [N]_\infty^{1/2}\|_{L^p} + \|K^* M^*\|_{L^p} + \left\| \int |K_s| |dU_s| \right\|_{L^p}) \\ &\leq c_p (\|K^* [M]_\infty^{1/2}\|_{L^p} + \|K^* M^*\|_{L^p} + \|K^* \int |dU_s|\|_{L^p}). \end{aligned}$$

The proof is concluded by applying Holder's inequality, and noting that $\|\int |dU_s|\|_{L^{2p}} \leq c_p \|M^*\|_{L^{2p}}$. (The constant c_p changes from place to place in the preceding.) \square

Remarks. 1. Of course, for $p \geq 1$ this inequality is an immediate consequence of the Burkholder-Davis-Gundy inequalities.

2. This inequality is not true in general for $0 < p < 1/2$.

Proof of Theorem 1. First note that as X is the a.s. uniform limit of a subsequence of the X^n , X is cadlag. Also, as $\|X_0^n - X_0\|_{L^1} \rightarrow 0$, we may take $X_0^n = X_0 = 0$.

Let H be an elementary predictable process, that is a process of the form

$$H_t = \sum_{i=1}^k h_i 1_{(t_i, t_{i+1}]}(t),$$

where $h_i \in \mathcal{F}_{t_i}$, $|h_i| \leq 1$, and $t_1 < t_2 < \dots < t_k$. Then writing $H \cdot X$ for the elementary stochastic integral of H with respect to X , $t_{k+1} = \infty$, we have

$$\begin{aligned} E\{(H \cdot X)_\infty\} &= E\left\{\sum_{i=1}^{k+1} h_i (X_{t_{i+1}} - X_{t_i})\right\} \\ &= \lim_{n \rightarrow \infty} E\left\{\sum_{i=1}^{k+1} h_i (X_{t_{i+1}}^n - X_{t_i}^n)\right\} \\ &= \lim_{n \rightarrow \infty} E\left\{\int_0^\infty H_t dA_t^n\right\} \leq K. \end{aligned}$$

So by the Bichteler-Dellacherie theorem (e.g., Dellacherie-Meyer [1]) X is a quasimartingale, and therefore a special semimartingale. Hence X has a canonical decomposition $X = M + A$, with M a local martingale and A a predictable finite variation process. Choose a sequence (T_k) reducing M . Then, if H is an elementary predictable process, $E\{(H \cdot A)_{T_k}\} = E\{(H \cdot X)_{T_k}\} = \lim_n E\{(H \cdot X^n)_{T_k}\} \leq K$. Thus

$$E\left\{\int_0^{T_k} |dA_s|\right\} \leq K, \quad \text{for each } k \geq 1,$$

and hence $E\{\int_0^\infty |dA_s|\} \leq K$.

Now $M = X - A = (X - X^n) + (M^n + A^n) - A$, and so

$$M^* \leq (X - X^n)^* + (M^n)^* + \int_0^\infty |dA_s^n| + \int_0^\infty |dA_s|.$$

Thus $E\{M^*\} \leq 3K < \infty$, and M is a martingale in \mathcal{H}^1 . Set $Y^n = X^n - X$, $N^n = M^n - M$, $B^n = A^n - A$. We have

$$E\left\{\int_0^\infty |dB_s^n|\right\} \leq 2K, \quad E\{(N^n)^*\} \leq 2K, \quad \lim_n E\{(Y^n)^*\} = 0.$$

To complete the proof it is enough to prove that

$$(5) \quad \lim_{n \rightarrow \infty} E\{[Y^n]_\infty^{1/2}\} = 0.$$

For then, by Dellacherie and Meyer [1], section VII.95, we have $E\{[B^n]^{1/2}\} \leq 2E\{[Y^n]^{1/2}\}$. Hence, as $[N^n]^{1/2} \leq [B^n]^{1/2} + [Y^n]^{1/2}$, $E\{[N^n]_\infty^{1/2}\} \leq 3E\{[Y^n]_\infty^{1/2}\}$, so that $\lim_{n \rightarrow \infty} \|N^n\|_{\mathcal{H}^1} = 0$. This implies that $E\{(M^n - M)^*\} \rightarrow 0$, and hence that $\infty E\{(A^n - A)^*\} \rightarrow 0$. Finally, $E\{M^*\} \leq K$ follows from (4a) and (1b).

To show that $\lim_{n \rightarrow \infty} E\{[Y^n]_\infty^{1/2}\} = 0$, use integration by parts to conclude

$$[Y^n]_\infty = (Y_\infty^n)^2 - 2 \int_0^\infty Y_{s-}^n dN_s^n - 2 \int_0^\infty Y_{s-}^n dA_s^n,$$

and so, writing $U^n = Y_-^n \cdot N^n$,

$$(6) \quad E\{[Y^n]_\infty^{1/2}\} \leq E\{(Y^n)^*\} + 2^{1/2} E\{((U^n)^*)^{1/2}\} + 2^{1/2} E\{(\int_0^\infty |Y_{s-}^n| |dA_s^n|)^{1/2}\}.$$

By Proposition 2

$$\begin{aligned} E\{((U^n)^*)^{1/2}\} &\leq c(E\{(Y^n)^*\})^{1/2} (E\{(N^n)^*\})^{1/2} \\ &\leq cK^{1/2} (E\{(Y^n)^*\})^{1/2}. \end{aligned}$$

Similarly, the third term in (6) is dominated by

$$\begin{aligned} E\{((Y^n)^* \int_0^\infty |dA_s^n|)^{1/2}\} &\leq (E\{(Y^n)^*\})^{1/2} (E\{\int_0^\infty |dA_s^n|\})^{1/2} \\ &\leq K^{1/2} (E\{(Y^n)^*\})^{1/2}. \end{aligned}$$

Thus $\lim_{n \rightarrow \infty} E\{[Y^n]_\infty^{1/2}\} = 0$. □

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