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A Probabilistic Approach to the Boundedness of Singular Integral Operators

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1. Introduction.

Suppose K is a real-valued function and the linear operator T is defined formally by

$$Tf(x) = \int_{\mathbb{R}} K(x-y)f(y)dy.$$

A central area of harmonic analysis has been to find conditions on K so that T is a bounded operator on $L^p(dx), p \in (1,\infty)$. A typical theorem is

Theorem 1.1. *Suppose K is an odd integrable function and suppose that there exist $c_1, c_2 > 0$ and $\delta \in (0,1)$ such that*

$$|K(x)| \le c_1 |x|^{-1}, \quad x \in \mathbb{R} - \{0\}, \tag{1.1}$$

and

$$|K(y) - K(x)| \le c_2 \frac{|y-x|^\delta}{|x|^{1+\delta}}, \quad |y-x| \le \frac{7}{8}|x|. \tag{1.2}$$

Then for all $p \in (1,\infty)$, there exists a constant $c_3(p)$, depending on p, c_1, and c_2, but not *on the L^1 norm of K, such that*

$$\|Tf\|_{L^p(dx)} \le c_3(p)\|f\|_{L^p(dx)}. \tag{1.3}$$

There are two main approaches to proving Theorem 1.1. One involves the Calderón-Zygmund decomposition, establishing a weak $(1,1)$ inequality, and using the Marcinkiewicz interpolation theorem (see Stein [17], Ch. 2). The other involves Littlewood-Paley functions and Fourier multiplier techniques (see [17], Ch. 4).

The purpose of this paper is to give a probabilistic proof of Theorem 1.1 For $\alpha \in (0,\delta)$ and $r > 0$, define $w_r(x)$ by

$$w_r(x) = c_\alpha r^{-1}(1 + \frac{|x|^2}{r^2})^{-((1+\alpha)/2)}, \tag{1.4}$$

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where c_α is chosen so that $\int_{\mathbb{R}} w_r(x)dx = 1$. In Section 2 we use the Burkholder-Davis-Gundy inequalities and another well-known inequality from probability theory to show that to prove Theorem 1.1, it suffices to obtain the L^2 inequality

$$(1.5) \qquad \|Tf(\cdot) - \int Tf(v)w_r(v)dv\|_{L^2(w_r(x)dx)} \le c_4 \|f\|_{L^2(w_r(x)dx)}$$

with c_4 depending on c_1 and c_2 but not on r or the L^1 norm of K. (We also give an analytic proof of this fact.)

In Section 3 we prove (1.5). The tool we use is the elementary Cotlar's lemma (see Theorem 3.2), which reduces the proof of (1.5) to obtaining suitable estimates for certain nonsingular kernels. These estimates are obtained in Section 4.

A side benefit of our method is that with virtually no extra work we obtain the H^1 and BMO boundedness of the operator T. Also, although we do the case $d = 1$ for simplicity, our method extends, with only minor modifications, to the case $K : \mathbb{R}^d \to \mathbb{R}$, $d > 1$.

Ours is by no means the first probabilistic approach to singular integrals. A probabilistic proof of the L^p boundedness of the Hilbert transform has been known for some time (see Durrett [8] or Burkholder [5]). The Riesz transforms have been studied by Meyer [15], Gundy-Varopoulos [12], Gundy-Silverstein [11], Bañuelos [1], and Bennett [2]. The Littlewood-Paley approach has been viewed probabilistically by Meyer [15], Varopoulos [19], McConnell [14], Marias [13], Bouleau-Lamberton [3], and Bourgain [4]. Our approach is quite different from all of these. In particular, we make no use of Littlewood-Paley functions, Fourier multipliers, nor the method of rotation. Rubio de Francia [16] has some results related to our Theorem 2.1.

The letters c and β will denote constants whose value is unimportant and may change from line to line. We will henceforth denote both the operator T and the function K by T. The adjoint of T will be denoted T^*. When we write $f * g$, we mean the convolution of f and g in the usual sense, i.e., with respect to Lebesgue measure.

2. Probability

In this section we show that to prove Theorem 1.1 it suffices to establish (1.5). We prove

Theorem 2.1. *Suppose T is odd and integrable, let $\alpha \in (0,1)$, and suppose there exists a constant c_4 independent of r and the L^1 norm of T such that*

$$(2.1) \qquad \|Tf(\cdot) - \int Tf(v)w_r(v)dv\|_{L^2(w_r(x)dx)} \le c_4 \|f\|_{L^2(w_r(x)dx)}, r > 0,$$

where w_r is defined by (1.4). Then there exists a constant $c_5(p)$ depending only on α, c_4 and p, but not the L^1 norm of T, such that

$$\|Tf\|_{L^p(dx)} \leq c_5(p)\|f\|_{L^p(dx)}. \tag{2.2}$$

We first give a probabilistic proof, then an analytic proof.

In this section we work in the half space $\mathbb{R} \times [0, \infty)$ with points $z = (x, y)$, $x \in \mathbb{R}^d$, $y \in [0, \infty)$. Let X_t be a standard Brownian motion on $\mathbb{R}$ and let Y_t be a Bessel process of index γ on $[0, \infty)$, independent of X_t, where $\gamma = 2 - \alpha$. Thus Y_t is a strong Markov process with continuous paths and infinitesimal generator $\frac{1}{2}f''(y) + \frac{\gamma-1}{2y}f'(y)$. Since $\gamma \in (1, 2), Y_t$ hits 0. Let

$$\tau = \inf\{t : Y_t = 0\}.$$

We will only need to consider Y_t up to time τ, so its boundary behavior at 0 is irrelevant.

Write $Z_t = (X_t, Y_t)$. The infinitesimal generator L of Z_t is given by

$$Lf(z) = \frac{1}{2}\Delta f(z) + \frac{\gamma - 1}{2y}\frac{\partial f}{\partial y}(z), \quad z = (x, y). \tag{2.3}$$

We first compute the $P^{(0,y)}$ distribution of X_τ.

Lemma 2.2. *(cf. Marias [13]).* $P^{(0,r)}(X_\tau \in A) = \int_A w_r(x)dx.$

Proof. Since $\tau < \infty$, a.s., then $P^{(0,r)}(X_\tau \in dx)$ is a probability density. So it suffices to show $P^{(0,r)}(X_\tau \in dx) = cw_r(x)dx$. We do this by calculating the characteristic function of X_τ.

Using the independence of X_t and Y_t, hence of X_t and τ,

$$\begin{aligned} E^{(0,y)} \exp(iuX_\tau) &= \int_0^\infty E^{(0,y)} \exp(iuX_t) P^{(0,y)}(\tau \in dt) \\ &= \int_0^\infty \exp(-u^2 t/2) P^{(0,y)}(\tau \in dt) \\ &= E^y \exp((-\frac{u^2}{2})\tau). \end{aligned} \tag{2.4}$$

By [10, Prop. 5.7 (i)], (2.4) is equal to $c(ux)^{-\nu}K_\nu(ux)$, where $\nu = 1 - \gamma/2$ and K_ν is the usual modified Bessel function. Lemma 2.1 follows by inverting the Fourier transform (see [9]). □

Next we recall an elementary probability inequality (see [7], for example). For completeness and to emphasize its simplicity, we give a proof.

Lemma 2.3. *Suppose A_t and B_t are two increasing continuous processes with $A_0 \equiv 0$. Suppose for some constant $c_6 > 0$,*

$$E(A_\infty - A_t|\mathcal{F}_t) \le c_6 E(B_\infty|\mathcal{F}_t), \quad a.s. \text{ for all } t. \tag{2.5}$$

Then for $p \in [1,\infty)$,

$$EA_\infty^p \le c_7(p) EB_\infty^p,$$

where $c_7(p)$ depends only on p and c_6.

Proof. The case $p = 1$ follows by taking $t = 0$ in (2.5), and then taking expectations, so suppose $p > 1$. Suppose first that A_t is bounded. By integration by parts,

$$\begin{aligned} EA_\infty^p &= pE\int_0^\infty (A_\infty - A_t)dA_t^{p-1} = pE\int_0^\infty E(A_\infty - A_t|\mathcal{F}_t)dA_t^{p-1} \\ &\le c_6 pE\int_0^\infty E(B_\infty|\mathcal{F}_t)dA_t^{p-1} = c_6pE\int_0^\infty B_\infty dA_t^{p-1} \\ &\le c_6 p(EB_\infty^p)^{1/p}(EA_\infty^p)^{\frac{p-1}{p}}. \end{aligned}$$

Dividing through by $(EA_\infty^p)^{\frac{p-1}{p}}$ gives our result with $c_7(p) = (c_6p)^p$.

If A_t is not bounded, note that the process $A_{t\wedge T_N}$ satisfies (2.5), where $T_N = \inf\{t : A_t \ge N\}$. Apply the above argument to $A_{t\wedge T_N}$ to get $EA_{T_N}^p \le c_7(p)EB_\infty^p$, let $N \to \infty$, and use monotone convergence. □

Proposition 2.4. *Under the hypotheses of Theorem 2.1,*

$$E^{(x,y)}[Tf(X_\tau) - E^{(x,y)}Tf(X_\tau)]^2 \le c_4 E^{(x,y)}[f(X_\tau) - E^{(x,y)}f(X_\tau)]^2. \tag{2.6}$$

Proof. Let $f_1(\cdot) = f(\cdot) - \int f(v)w_r(v)dv$. Since T is odd, $T1 \equiv 0$, hence $Tf = Tf_1$. Applying (2.1) to f_1, we get

$$\|Tf(\cdot) - \int Tf(v)w_r(v)dv\|_{L^2(w_r(x)dx)} \le c_4\|f - \int f(v)w_r(v)dv\|_{L^2(w_r(x)dx)}. \tag{2.7}$$

Using Lemma 2.2, (2.7) can be rewritten as

$$E^{(0,r)}[Tf(X_\tau) - E^{(0,r)}Tf(X_\tau)]^2 \le c_4^2 E^{(0,r)}[f(X_\tau) - E^{(0,r)}f(X_\tau)]^2. \tag{2.8}$$

Let $f_2(\cdot) = f(\cdot + x)$. By the translation invariance of X_t and of the operator T, applying (2.8) to f_2 gives (2.6). □

Suppose $f \in C_K^\infty$, that is, C^∞ with compact support. Define

$$u_f(z) = E^{(x,y)} f(X_\tau), \quad u_{Tf}(z) = E^{(x,y)} Tf(X_\tau), \quad z = (x,y),$$

and define

$$M_t^f = u_f(Z_{t\wedge\tau}), \quad M_t^{Tf} = u_{Tf}(Z_{t\wedge\tau}).$$

Since T is in L^1, Tf is also, hence u_{Tf} is finite everywhere. As is well-known, u_f is L-harmonic in $\mathbb{R} \times (0,\infty)$, and by Ito's lemma, M_t^f is a local martingale with

$$\langle M_t^f \rangle = \int_0^{t\wedge\tau} |\nabla u_f(Z_s)|^2 ds, \tag{2.9}$$

with a similar statement holding for M_t^{Tf}. Let

$$A_t = \langle M^{Tf} \rangle_t, \quad B_t = \langle M^f \rangle_t. \tag{2.10}$$

Proof of Theorem 2.1 (Probabilistic). Since T is in L^1,

$$\|Tf\|_{L^p(dx)} = \|T * f\|_{L^p(dx)} \le \|T\|_{L^1(dx)} \|f\|_{L^p(dx)}. \tag{2.11}$$

This is not what we want, since in (2.2) it is important that $c_5(p)$ not depend on the L^1 norm of T. But (2.11) does show that T is a bounded operator on L^p, and so to establish (2.2) for all $f \in L^p$, it suffices to verify (2.2) for $f \in C_K^\infty$. So suppose that $f \in C_K^\infty$.

We do the case $p \ge 2$ first. By the strong Markov property, if $t \le \tau$,

$$\begin{aligned} E^{(0,s)}(A_\infty - A_t | \mathcal{F}_t) &= E^{Z_t} A_\infty = E^{Z_t} < M^{Tf} >_\tau \\ &= E^{Z_t}(M_\tau^{Tf} - M_0^{Tf})^2 = E^{Z_t}[Tf(X_\tau) - E^{Z_t} Tf(X_\tau)]^2, \end{aligned} \tag{2.12}$$

with a similar expression for B_t.

By Proposition 2.4 with $(x,y) = (X_t(\omega), Y_t(\omega))$, we get

$$E^{(0,s)}(A_\infty - A_t|\mathcal{F}_t) \le c_4 E^{(0,s)}(B_\infty - B_t|\mathcal{F}_t) \le c_4 E^{(0,s)}(B_\infty|\mathcal{F}_t). \tag{2.13}$$

So by Lemma 2.3, with p replaced by $p/2$.

$$E^{(0,s)} A_\infty^{p/2} \le c_8(p) E^{(0,s)} B_\infty^{p/2}. \tag{2.14}$$

Now by the Burkholder-Davis-Gundy inequalities (see [7] or [8]),

$$
\begin{aligned}
(2.15) \qquad E^{(0,s)}|Tf(X_\tau)|^p &= E^{(0,s)}|M_\tau^{Tf}|^p \\
&\le cE^{(0,s)}|M_0^{Tf}|^p + cE^{(0,s)} <M^{Tf}>_\tau^{p/2} \\
&= cE^{(0,s)}|M_0^{Tf}|^p + cE^{(0,s)} A_\infty^{p/2}.
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
(2.16) \qquad E^{(0,s)} B_\infty^{p/2} &= E^{(0,s)} <M^f>_\tau^{p/2} \le E^{(0,s)}(|M_0^f|^2 + <M^f>_\tau)^{p/2} \\
&\le cE^{(0,s)}|M_\tau^f|^p = cE^{(0,s)}|f(X_\tau)|^p.
\end{aligned}
$$

Putting (2.14), (2.15), and (2.16) together,

$$
(2.17) \qquad E^{(0,s)}|Tf(X_\tau)|^p \le c|E^{(0,s)}Tf(X_\tau)|^p + cE^{(0,s)}|f(X_\tau)|^p.
$$

Note

$$
sE^{(0,s)}|f(X_\tau)|^p = \int |f(x)|^p sw_s(x)dx \to c_\alpha \int |f(x)|^p\, dx
$$

as $s \to \infty$ by monotone convergence. Similarly, $sE^{(0,s)}|Tf(X_\tau)|^p \to c_\alpha \int |Tf(x)|^p dx$.

Finally, since $T \in L^1$ and $f \in C_K^\infty$, $Tf \in L^1$. Then

$$
\begin{aligned}
s|E^{(0,s)}Tf(X_\tau)|^p &= s^{1-p}|\int Tf(x)sw_s(x)dx|^p \\
&\le cs^{1-p}(\int |Tf(x)|dx)^p \to 0
\end{aligned}
$$

as $s \to \infty$, since $p \ge 2 > 1$.

So multiplying (2.17) by s and letting $s \to \infty$ gives the required result when $p \ge 2$. Since $T^* = -T$, we also have $\|T^*\|_{L^p(dx)} \le c_5(p)$ for $p \ge 2$. The usual duality argument (see [17], p.33) gives (2.2) for $p \in (1, 2]$. □

Actually the above proof gives us more.

Corollary 2.6. *Under the hypotheses of Theorem 2.1, there exists c_9 depending only on α and c_4 and not the L^1 norm of T such that*

$$
\|Tf\|_{H^1(\mathbb{R})} \le c_9\|f\|_{H^1(\mathbb{R})}, \qquad \|Tf\|_{BMO(\mathbb{R})} \le c_9\|f\|_{BMO(\mathbb{R})}
$$

Proof. By an argument very similar to that in [8], the BMO norm of f is equivalent to $\sup_{x,y} E^{(x,y)}[f(X_\tau) - E^{(x,y)}f(X_\tau)]^2$. The BMO boundedness of T follows from (2.6), and the H^1 boundedness follows by duality. □

We also give an analytic proof of Theorem 2.1. Although short, the proof uses the Calderón-Zygmund decomposition implicitly when interpolating between BMO and L^2.

Proof of Theorem 2.1 (Analytic) As in the proof of Corollary 2.6, T is a bounded operator on BMO. Arguing as in the probabilitic proof of Theorem 2.1 (the paragraphs following (2.17)), we multiply (2.1) by r and let $r \to \infty$ to get that

$$\|Tf\|_{L^2(dx)} \leq c_4 \|f\|_{L^2(dx)}$$

for $f \in C_K^\infty$. This says that T is a bounded operator on $L^2(dx)$. Interpolation between L^2 and BMO gives Theorem 2.1 for $p \in [2, \infty)$, and the result for $p \in (1, 2]$ follows by duality. □

3. L^2 theory

In this section we prove the following

Theorem 3.1. *Suppose T is an odd integrable function and suppose that there exist $c_1, c_2 > 0$, and $\delta \in (0,1)$ such that (1.1) and (1.2) hold. Suppose $\alpha \in (0, \delta)$. Then there exists a constant c_4 independent of r and the L^1 norm of T such that (2.1) holds.*

In fact, more is true. It is not hard to show that $w_r(x)$ is an A_2 weight (see [18]), and therefore

$$\|Tf\|_{L^2(w_r(x)dx)} \leq c\|f\|_{L^2(w_r(x)dx)}. \tag{3.1}$$

The proof that (3.1) follows from w_r being an A_2 weight is not elementary.

The inequality (2.1) may be shown to be equivalent to the $L^2(dx)$ boundedness of an operator related to T. This operator, does not, however, satisfy the hypotheses of the "$T1$" theorem of David-Journé [6].

The main tool we use to prove Theorem 3.1 is Cotlar's lemma:

Theorem 3.2. *Suppose $\mathcal{H}$ is a Hilbert space and that $T_j, j = -N, \ldots, 0, \ldots, N$ are bounded operators on $\mathcal{H}$. Suppose $a : \mathbb{Z} \to [0, \infty)$ satisfies*

$$c_9 = \sum_{i=-\infty}^{\infty} a^{1/2}(i) < \infty \quad \textit{and} \quad \|T_j^* T_k\|_{\mathcal{H}} + \|T_j T_k^*\|_{\mathcal{H}} \leq a(j-k) \textit{ for all } -N \leq j,k \leq N.$$

Then $\|\sum_{j=-N}^{N} T_j\|_{\mathcal{H}} \leq c_9$.

The proof of Cotlars lemma is both elementary and short. See [18, pp.285–286], for example.

We will also use the following well-known lemma.

Lemma 3.3. *Suppose $V(x,y)$ is a nonnegative kernel with respect to μ, a σ-finite measure. Suppose*

$$\sup_x \int V(x,y)\mu(dy) \le c_{10}, \quad \sup_y \int V(x,y)\mu(dx) \le c_{11}.$$

Then $\|V\|_{L^2(\mu(dx))} \le (c_{10}c_{11})^{1/2}$.

Proof. By Cauchy-Schwartz,

$$|Vf(x)| = |\int V(x,y)f(y)\mu(dy)| \le (\int V(x,y)\mu(dy))^{1/2}(\int V(x,y)f^2(y)\mu(dy))^{1/2}.$$

Then

$$\begin{aligned}\int |Vf(x)|^2\mu(dx) &\le c_{10}\int\int V(x,y)f^2(y)\mu(dy)\mu(dx)\\ &\le c_{10}c_{11}\int f^2(y)\mu(dy). \qquad \square\end{aligned}$$

Let $\varphi = \varphi_0$ be a nonnegative even C^∞ function with support in $[-1,1]$ satisfying $\int \varphi(x)dx = 1$. Let $\varphi_j(x) = 2^{-j}\varphi(x2^{-j}), j \in \mathbb{Z}$.

Define

$$T_j = T * \varphi_j - T * \varphi_{j+1}.$$

Define the operator U_j^r by

$$U_j^r f(x) = T_j f(x) - \int T_j f(v)w_r(v)dv. \tag{3.2}$$

Since $\int T_j f(v)w_r(v)dv = -\int T_j w_r(v)f(v)dv$, we see that

$$U_j^r f(x) = \int U_j^r(x,y)f(y)w_r(y)dy],$$

where

$$U_j^r(x,y) = -\frac{T_j(y-x) - T_j w_r(y)}{w_r(y)}. \tag{3.3}$$

The key estimate we need is the following. We defer its proof until Section 4.

Proposition 3.4. *There exist constants c_{12} and $\beta_1 > 0$ depending only on α, δ, c_1, and c_2 (and not on the L^1 norm of T) such that*

$$\sup_{r,x}\int |U_0^r(x,y)|w_r(y)dy \le c_{12}; \tag{3.4}$$

$$\sup_{r,y} \int |U_0^r(x,y)| w_r(x) dx \le c_{12}; \tag{3.5}$$

$$\sup_{r,x} \int |(U_0^r)^* U_k^r(x,y)| w_r(y) dy \le c_{12} 2^{-k\beta_1}, \quad k > 0; \tag{3.6}$$

and

$$\sup_{r,y} \int |U_0^r (U_k^r)^*(x,y)| w_r(x) dx \le c_{12} 2^{-k\beta_1}, \quad k > 0. \tag{3.7}$$

With this proposition we can now prove Theorem 3.1.

Proof of Theorem 3.1. By Lemma 3.3, (3.4), and (3.5), we get

$$\sup_{r} \|U_0^r\|_{L^2(w_r(x)dx)} \le c_{12}. \tag{3.8}$$

Fix j for the moment and let $\widetilde{T}(x) = 2^j T(x2^j)$. Observe that $\widetilde{T}$ satisfies the hypotheses of Theorem 3.1 with the same constants c_1, c_2, δ. Define $\widetilde{U}_0^r$ in terms of $\widetilde{T}$ the same way U_0^r was defined in term of T (see (3.2)). A simple scaling argument (i.e., a linear change of variables) shows that $\|U_j^r\|_{L^2(w_r(x)dx)} = \|\widetilde{U}_0^{r2^{-j}}\|_{L^2(w_{r2^{-j}}(x)dx)}$. Applying (3.8) to $\widetilde{U}_0^\bullet$ yields

$$\sup_{r} \|U_j^r\|_{L^2(w_r(x)dx)} \le c_{12}. \tag{3.9}$$

Similarly,

$$\sup_{r,x} \int |U_j^r(x,y)| w_r(y) dy \le c_{12} \tag{3.10}$$

and

$$\sup_{r,y} \int |U_j^r(x,y)| w_r(x) dx \le c_{12}. \tag{3.11}$$

Next, observe that by Fubini, (3.4), (3.10), and (3.11),

$$\begin{aligned} \int |(U_0^r)^* U_k^r(x,y)| w_r(x) dx &\le \int \int |U_0^r(v,x)| \ |U_k^r(v,y)| w_r(v)\ dv\ w_r(x)\ dx \\ &\le c_{12} \int |U_k^r(v,y)| w_r(v) dv \le c_{12}^2. \end{aligned}$$

By Lemma 3.3, (3.6), and (3.12),

$$\sup_{r} \|(U_0^r)^* U_k^r\|_{L^2(w_r(x)dx)} \le c_{12}^{3/2} 2^{-k\beta_1/2}. \tag{3.13}$$

Scaling as in the derivation of (3.9), if $j \leq k$,

$$\|(U_j^r)^* U_k^r\|_{L^2(w_r(x)dx)} \leq c_{12}^{3/2} 2^{-(k-j)\beta_1/2}. \tag{3.14}$$

Observing that if $j > k$,

$$\begin{aligned}\|(U_j^r)^* U_k^r\|_{L^2(w_r(x)dx)} &= \|((U_j^r)^* U_k^r)^*\|_{L^2(w_r(x)dx)} \\ &= \|(U_k^r)^* U_j^r\|_{L^2(w_r(x)dx)} \leq c_{12}^{3/2} 2^{-(j-k)\beta_1/2}\end{aligned}$$

by (3.14).

So in any case we have

$$\|(U_j^r)^* U_k^r\|_{L^2(w_r(x)dx)} \leq c_{12}^{3/2} 2^{-|j-k|\beta_1/2}. \tag{3.15}$$

Similarly, starting with (3.7), we get

$$\|U_j^r (U_k^r)^*\|_{L^2(w_r(x)dx)} \leq c_{12}^{3/2} 2^{-|j-k|\beta_1/2}. \tag{3.16}$$

We now apply Cotlar's lemma (Theorem 3.2) and obtain

$$\|\sum_{j=-N}^{N} U_j^r\|_{L^2(w_r(x)dx)} \leq c_{13}, \tag{3.17}$$

c_{13} independent of N and r.

Finally, observe that

$$\begin{aligned}(3.18)\quad -\sum_{j=-N}^{N} U_j^r f(x) &= [(T * \varphi_{-N})f(x) - \int (T * \varphi_{-N})f(v)w_r(v)dv] \\ &\quad - [(T * \varphi_N)f(x) - \int (T * \varphi_N)f(v)w_r(v)dv].\end{aligned}$$

So to conclude the proof, it suffices to show that for $f \in C_K^\infty$, the right side of (3.18) converges to $Tf(x) - \int Tf(v)w_r(v)dv$ in $L^2(w_r(x)dx)$ norm.

If $f \in C_K^\infty$, then $f \in L^2(dx)$, and since $T \in L^1(dx)$, $Tf \in L^2(dx)$. So $(T*\varphi_{-N})f = Tf * \varphi_{-N} \to Tf$ in $L^2(dx)$ norm. Since $w_r(\cdot)$ is bounded above by a constant, it is not hard to see that this implies that the first term on the right of (3.18) converges to $Tf - \int Tf(v)w_r(v)dv$ as $N \to \infty$ as desired.

On the other hand,

$$T * \varphi_N(z)| \leq \int |T(z-v)| 2^{-N} \varphi(v 2^{-N}) dv \leq 2^{-N} \|\varphi\|_{L^\infty} \|T\|_{L^1(dx)} \to 0$$

as $N \to \infty$, which shows that the second term on the right of (3.18) converges to 0 in $L^2(w_r(x)dx)$ norm. □

Proof of Theorem 1.1. Immediate from Theorem 2.1 and 3.1. □

Remarks. 1. We have shown that T is a bounded operator on $L^p(dx)$ with a bound independent of the L^1 norm of T. So one could dispense with the hypothesis that T is integrable by a suitable limiting process (cf. [17]).

2. Operators such as the truncated Hilbert transform T^ε ([17], p. 38) can be written as a sum $T_1^\varepsilon + T_2^\varepsilon$, where T_1^ε satisfies the hypotheses of Th.1.1 with constants c_1 and c_2 independent of ε and T_2^ε is integrable with L^1 norm independent of ε. Hence Theorem 1.1 and (2.11) shows such operators are bounded on $L^p(dx)$.

3. Only minor modifications are needed to handle the case $T : \mathbb{R}^d \to \mathbb{R}$, $d > 1$. The condition that T be odd gets replaced by

$$\int_{R_1<|x|<R_2} T(x)dx = 0 \quad \text{for all } 0 < R_1 < R_2.$$

4. The bounds in (3.15) and (3.16) are much stronger than are necessary to obtain convergence in Cotlar's lemma. It would be interesting to see whether our method could be extended to the case where (1.2) is replaced by Hörmander's condition:

$$\sup_{y>0} \int_{|x|\geq 2|y|} |K(x-y) - K(x)|dx \leq c_2.$$

5. Necessary and sufficient conditions are know on a weight function w for T to be a bounded operator on $L^p(w(x)dx)$. Can such theorems be proved by our method? One would need to replace our Brownian motion X_t be another diffusion whose invariant measure is $w(x)dx$.

4. Estimates

In this section we prove Proposition 3.4. It is here that (1.1) and (1.2), unused so far, come into play. Define

$$\rho(x) = (1 + |x|^{1+\delta})^{-1},$$

let $\rho_j(x) = 2^{-j}\rho(x2^{-j})$, and define

$$M(x) = 1 \wedge |x|.$$

We start with an elementary lemma.

Lemma 4.1.
(4.1) $\int M(z)w_r(z)dz \leq cM(r^\alpha)$;
(4.2) $\int \rho(z)M(z2^{-k})dz \leq c2^{-k\beta}$;
(4.3) if $|x|, r \leq 2^{k/2}$, *then* $\int M(\frac{z-x}{2^k})w_r(z)dz \leq c2^{-k\beta}$.

Proof. The only one requiring comment, perhaps, is (4.3). If $|x| \leq 2^{k/2}$ and $|z| \leq 2^{2k/3}$, then $M(\frac{z-x}{2^k}) \leq c2^{-k/3}$. So

$$\int M(\frac{z-x}{2^k})w_r(z)dz \leq c2^{-k/3}\int_{|z|\leq 2^{2k/3}} w_r(z)dz + \int_{|z|>2^{2k/3}} w_r(z)dz$$
$$\leq c2^{-k/3} + \int_{|z|>r^{-1}2^{2k/3}} w_1(z)dz \leq c2^{-k\beta},$$

since $r^{-1}2^{k/3} \geq 2^{k/6}$. □

Next, we have

Lemma 4.2.

$$U_j^r 1 \equiv 0; \tag{4.4}$$
$$(U_j^r)^* 1 \equiv 0. \tag{4.5}$$

Proof. Since T is odd, using Fubini gives

$$T_j 1(x) = \int T_j(y)dy = \int\int T(y-z)(\varphi_j - \varphi_{j+1})(z)dz\,dy = 0.$$

Substituting in (3.2) gives (4.4),

As for (4.5), recalling (3.3) we have

$$(U_j^r)^*1(x) = -\int \frac{T_j(x-y) - T_j w_r(x)}{w_r(x)} w_r(y)dy$$
$$= -w_r(x)^{-1}[T_j w_r(x) - T_j w_r(x)\int w_r(y)dy] = 0. \quad \square$$

The next three lemmas give the required estimates on T_j.

Lemma 4.3. *(cf. [6], Lemma 4)*
(4.6) $|T_j(x)| \leq c\rho_j(x)$;
(4.7) $|T_j(x) - T_j(y)| \leq cM(\frac{x-y}{2^j})[\rho_j(x) + \rho_j(y)]$.

Proof. We will do the case $j = 0$. The case when $j \neq 0$ can be reduced to this one by scaling, as in the derivation of (3.9).

Suppose first that $|x| \geq 8$. Since $\int(\varphi_0 - \varphi_1)(y)dy = 0$ and the support of $\varphi_0 - \varphi_1$ is contained in $[-4, 4]$, we have, using (1.2),

$$\begin{aligned}|T_0(x)| &= |\int T(x-y)(\varphi_0 - \varphi_1)(y)dy| \\ &= |\int [T(x-y) - T(x)](\varphi_0 - \varphi_1)(y)dy| \\ &\leq c \int_{|y|\leq 4} \frac{|y|^\delta}{|x|^{1+\delta}} \|\varphi_0 - \varphi_1\|_{L^\infty}\, dy \leq c\rho(x).\end{aligned} \tag{4.8}$$

Suppose now that $|x| \leq 8$. Since T is odd and $(\varphi_0 - \varphi_1)(x-y) = 0$ if $|y| > 16$, we have

$$\begin{aligned}|T_0(x)| &= |\int T(y)(\varphi_0 - \varphi_1)(x-y)dy| \\ &= |\int T(y)[(\varphi_0 - \varphi_1)(x-y) - (\varphi_0 - \varphi_1)(x)1_{(|y|\leq 16)}]dy| \\ &\leq \int_{|y|\leq 16} |T(y)|\ \ |y|\ \|(\varphi_0 - \varphi_1)'\|_{L^\infty}\, dy \leq c,\end{aligned} \tag{4.9}$$

using (1.1). Putting (4.8) and (4.9) together gives (4.6).

To prove (4.7), again when $j = 0$, we observe that if $|x - y| \geq 1$, then

$$|T_0(x) - T_0(y)| \leq |T_0(x)| + |T_0(y)| \leq c[\rho(x) + \rho(y)]$$

by (4.6). If $|x - y| \leq 1$, (with $x < y$, say)

$$|T_0(x) - T_0(y)| \leq |x - y| \sup_{v\in[x,y]} |T_0'(v)|. \tag{4.10}$$

Now $T_0' = T * (\varphi_0 - \varphi_1)'$ and repeating the proof of (4.6) with $\varphi_0 - \varphi_1$ replaced by $(\varphi_0 - \varphi_1)'$, we get

$$\begin{aligned}\sup_{v\in[x,y]} |T_0'(v)| &\leq c \sup_{v\in[x,y]} \rho(v) \\ &\leq c[\rho(x) + \rho(y)],\end{aligned}$$

since $|x - y| \leq 1$. □

The most technical lemma is

Lemma 4.4.

$$|T_0 w_r(x) - T_0(x)| \le cM(r^\alpha)\rho(x). \tag{4.11}$$

Proof. If $|x| \le 64$, the proof is easy. Using (4.1),

$$\begin{aligned}|T_0 w_r(x) - T_0(x)| &\le \int |T_0(x-y) - T_0(x)| w_r(y)dy \\ &\le c\int M(y)[\rho(x-y)+\rho(x)]w_r(y)dy \\ &\le cM(r^\alpha),\end{aligned}$$

since ρ is bounded by 1.

So suppose $|x| > 64$, and without loss of generality, assume $x > 0$. Define s_r, t_r, u_r as follows. For $y \in [3x/4, 5x/4]$, let $t_r(y) = w_r(y)$. Define t_r so as to be nonnegative, 0 on $[x/2, 3x/2]^c$, and with $|t_r(y)| \le cw_r(x)$, $|t_r'(y)| \le cx^{-1}w_r(x)$, and $|t_r''(y)| \le cx^{-2}w_r(x)$ for y in $[x/2, 3x/4]$ and $[5x/4, 3x/2]$. Let $s_r(y) = [w_r(y) - t_r(y)]1_{(-x,x)}(y)$ and $u_r(y) = w_r(y) - t_r(y) - s_r(y)$.

Now write

$$\begin{aligned}T_0 w_r(x) - T_0(x) &= \int [T_0(x-y) - T_0(x)]w_r(y)dy \\ &= \int [T_0(x-y) - T_0(x)]s_r(y)dy - \int T_0(x)[w_r(y) - s_r(y)]dy \\ &\quad + \int T_0(x-y)u_r(y)dy + \int T_0(x-y)t_r(y)dy \\ &= I_1 + I_2 + I_3 + I_4.\end{aligned}$$

By the definitions of s_r and t_r, we have $s_r(y) = 0$ unless $y \in [-x, 3x/4]$. For y in this range, $\rho(x-y) \le c\rho(x)$, and so by (4.1)

$$\begin{aligned}|I_1| &\le c\int_{-x\le y\le 3x/4} M(y)[\rho(x-y)+\rho(x)]w_r(y)dy \le c\rho(x)\int M(y)w_r(y)dy \\ &\le cM(r^\alpha)\rho(x).\end{aligned}$$

For all r

$$|I_2| \le c\rho(x)\int_{|y|>x/2} w_r(y)dy \le c\rho(x).$$

And if $r \le 1$,

$$|I_2| \le c\rho(x)\int_{|y|>x/2} w_r(y)dy \le c\rho(x)r^\alpha \int_{|y|>x/2} \frac{dy}{y^{1+\alpha}} \le c\rho(x)r^\alpha.$$

Since $u_r(y) = 0$ unless $y \le -x$ or $y > 5x/4$, and $\rho(x-y) \le c\rho(x)$ for y in this range,

$$|I_3| \le c \int_{y\in[-x,5x/4]^c} \rho(x-y) w_r(y)dy \le c\rho(x) \int_{|y|\ge x} w_r(y)dy$$
$$\le c\rho(x) M(r^\alpha),$$

as in bounding I_2.

Finally, we look at I_4. Write $\overline{\varphi}$ for $\varphi_0 - \varphi_1$. Since t_r is supported in $[x/2, 3x/2]$, then $\overline{\varphi} * t_r$ is supported in $[x/4, 7x/4]$. Since T is odd,

$$\begin{aligned} |I_4| &= |\int T_0(x-y)t_r(y)dy| = |\int T(y)(\overline{\varphi} * t_r)(x-y)dy| \\ &= |\int_{|y|\le 3x/4} T(y)(\overline{\varphi} * t_r)(x-y)dy| \\ &= |\int_{|y|\le 3x/4} T(y)[\overline{\varphi} * t_r(x-y) - \overline{\varphi} * t_r(x)]dy| \\ &\le \int_{|y|\le 3x/4} |T(y)|\, |y| \sup_{z\in[x/4,7x/4]} |(\overline{\varphi} * t_r)'(z)|dy \\ &\le cx \sup_{z\in[x/4,7x/4]} |(\overline{\varphi} * t_r)'(z)|, \end{aligned} \tag{4.12}$$

using (1.1).

Since $\int \overline{\varphi}(y)dy = 0$ and $\overline{\varphi}$ has support in $[-4, 4]$,

$$\begin{aligned} |(\overline{\varphi} * t_r)'(z)| &= |\int \overline{\varphi}(y) t_r'(z-y)dy| \\ &= |\int \overline{\varphi}(y)[t_r'(z-y) - t_r'(z)]dy| \\ &\le \int |\overline{\varphi}(y)||y|dy \sup_{z-4\le v\le z+4} |t_r''(v)|. \end{aligned}$$

So

$$\sup_{z\in[x/4,7x/4]} |(\overline{\varphi} * t_r)'(z)| \le c \sup_{x/8\le v\le 2x} |t_r''(v)| \le cx^{-2} w_r(x) \vee w_r''(x). \tag{4.13}$$

Plugging (4.13) into (4.12) and estimating $w_r''(x)$ (do the cases $x \le r$ and $x > r$ separately), we get $|I_4| \le cM(r^\alpha)\rho(x)$.

Summing our bounds for I_1, I_2, I_3, and I_4 proves the lemma. □

The final estimate we need is

Lemma 4.5. *If* $r \geq 1$,

$$|T_0 w_r(x)| \leq cr^{-1}(r^{-\beta} + M(\frac{|x|}{r})). \tag{4.14}$$

Proof. Since T is odd and φ is even, T_0 is an odd function. Recalling that c_α is the normalizing constant for w_r (see (1.4)), we have

$$T_0 w_r(x) = \int_{|y|\geq r^{1/2}} T_0(y) w_r(x-y)dy + \int_{|y|\leq r^{1/2}} T_0(y)[w_r(x-y) - c_\alpha r^{-1}]dy. \tag{4.15}$$

Since w_r is bounded by c_α/r, the first term on the right of (4.15) is bounded by $cr^{-1-\delta/2}$, using (4.6). For similar reasons, the second term on the right of (4.15) is bounded by c/r for all x. But if in addition $|x| \leq r$, then the elementary inequality

$$(1+a^2)^{\frac{1+\alpha}{2}} \leq 1 + 4a \quad \text{for } a \in [0,2]$$

yields

$$[w_r(x-y) - c_\alpha r^{-1}] \leq cr^{-2}|x-y| \leq cr^{-2}|x| + cr^{-3/2}$$

when $|y| \leq r^{1/2}$. Substituting this better bound into (4.15) when $|x| \leq r$ and using (4.6) again completes the proof of (4.14). □

We are now ready to prove Proposition 3.4. We break the proof into a number of steps.

Proof of Proposition 3.4.

PROOF OF (3.4). By (4.11) and (4.6),

$$|T_0 w_r(x)| \leq |T_0 w_r(x) - T_0(x)| + |T_0(x)| \leq c\rho(x). \tag{4.16}$$

Using the definition of $U_0^r(x,y)$ in (3.3), (4.6), and (4.16),

$$\int |U_0^r(x,y)| w_r(y)dy \leq \int [|T_0(y-x)| + |T_0 w_r(y)|]dy \leq \int [\rho(y-x) + \rho(y)]dy \leq c,$$

which is (3.4).

PROOF OF (3.5). Since $\sup_{r,y}(\ldots) \leq \sup_{r\leq 1,y}(\ldots) + \sup_{r\geq 1,y}(\ldots)$, it suffices to look at the cases $r \leq 1$ and $r \geq 1$ separately. Suppose $r \leq 1$.

By (4.7) and (4.11),

$$|T_0(y-x)-T_0w_r(y)| \le |T_0(y-x)-T_0(y)|+|T_0(y)-T_0w_r(y)| \le cM(x)[\rho(y-x)+\rho(y)]+cM(r^\alpha)\rho(y). \tag{4.17}$$

Then

$$\begin{aligned}\int |U_0^r(x,y)|w_r(x)dx &\le cw_r(y)^{-1}\int_{|x|\le|y|/2} M(x)\rho(y-x)w_r(x)dx \\ &\quad + cw_r(y)^{-1}\int_{|x|>|y|/2} M(x)\rho(y-x)w_r(x)dx \\ &\quad + c\rho(y)w_r(y)^{-1}\int M(x)w_r(x)dx + cr^\alpha w_r(y)^{-1}\rho(y)\int w_r(x)dx \\ &= I_1+I_2+I_3+I_4.\end{aligned}$$

Treating these in reverse order, we see that

$$I_4 = cr^\alpha w_r(y)^{-1}\rho(y) \le c$$

by looking at the cases $|y|\le r$ and $|y|>r$ separately and recalling that $\alpha<\delta$.

By (4.1), I_3 reduces to I_4.

When $|x|>|y|/2$, $w_r(x)/w_r(y)$ is bounded by a constant independent of r, and so

$$I_2 \le c\int \rho(y-x)dx \le c.$$

Finally, when $|x|\le|y|/2$, $\rho(y-x)/\rho(y)$ is bounded by a constant. So

$$I_1 \le cw_r(y)^{-1}\rho(y)\int M(x)w_r(x)dx \le cI_3.$$

Summing gives (3.5) when $r\le 1$.

Now suppose $r>1$. In place of (4.17), we write

$$|T_0(y-x)-T_0w_r(y)| \le |T_0(y-x)|+|T_0w_r(y)| \le \rho(y-x)+c(r^{-1}\wedge\rho(y)), \tag{4.18}$$

using (4.6) and either (4.14) or (4.16). Then

$$\int |U_0^r(x,y)|w_r(x)dx \le cw_r(y)^{-1}\int \rho(y-x)w_r(x)dx + c(r^{-1}\wedge\rho(y))w_r(y)^{-1}\int w_r(x)dx = I_5+I_6.$$

If $|y| \geq r$, we break up the range of integration in I_5 into $|x| \leq |y|/2$ and $|x| > |y|/2$, we handle the first range similarly to the way we bounded I_1 and we do the second range similarly to the way we bounded I_2. If $|y| \leq r$, we simply observe that $w_r(x)/w_r(y)$ is bounded. To bound I_6, consider the cases $|y| \geq r$ and $|y| < r$ separately.

PROOF OF (3.6), $r \leq 1$.

By (4.5),

$$(4.19)\quad \int |(U_0^r)^* U_k^r(x,y)| w_r(y)dy = \int \Big| \int U_0^r(z,x)[U_k^r(z,y) - U_k^r(x,y)] w_r(z)dz \Big| w_r(y)dy$$
$$\leq \int\int |U_0^r(z,x)|\, |U_k^r(z,y) - U_k^r(x,y)| w_r(z) w_r(y) dy\, dz.$$

Substituting from (3.3), we see that we must suitably bound

$$(4.20)\quad I_7 = w_r(x)^{-1} \int\int |T_0(z-x) - T_0 w_r(x)|\, |T_k(z-y) - T_k(x-y)| dy\ w_r(z)dz.$$

Bounding the first factor of the integrand as in (4.17) and the second factor using (4.7), we have

$$(4.21)\quad I_7 \leq c w_r(x)^{-1} \int \{M(z)[\rho(z-x) + \rho(x)] + M(r^\alpha)\rho(x)\} M(\frac{z-x}{2^k}) \times$$
$$\int [\rho(z-y) + \rho(x-y)] dy w_r(z)dz$$
$$\leq c w_r(x)^{-1} \int \{M(z)[\rho(z-x) + \rho(x)] + M(r^\alpha)\rho(x)\} M(\frac{z-x}{2^k}) w_r(z)dz$$
$$\leq c w_r(x)^{-1} \int_{|z|>|x|/2} M(z)\rho(z-x) M(\frac{z-x}{2^k}) w_r(z)dz$$
$$+ c w_r(x)^{-1} \int_{|z|\leq|x|/2} M(z)\rho(z-x) M(\frac{z-x}{2^k}) w_r(z)dz$$
$$+ c\rho(x) w_r(x)^{-1} \int M(z) M(\frac{z-x}{2^k}) w_r(z)dz$$
$$+ c r^\alpha \rho(x) w_r(x)^{-1} \int M(\frac{z-x}{2^k}) w_r(z)dz$$
$$= I_8 + I_9 + I_{10} + I_{11}.$$

When $|z| > |x|/2$, $w_r(z)/w_r(x) \leq c$ independently of r, and so

$$I_8 \leq c \int \rho(z-x) M(\frac{z-x}{2^k}) dz \leq c 2^{-k\beta}$$

by (4.2)

When $|z| \le |x|/2$, $\rho(z-x) \le c\rho(x)$, and so $I_9 \le cI_{10}$.

We turn to I_{10}. If $|x| \ge 2^{k/2}$, then $\rho(x)/w_r(x) \le c|x|^{\alpha-\delta}r^{-\alpha} \le c2^{-k\beta}r^{-\alpha}$ and in this case

$$I_{10} \le c2^{-k\beta}r^{-\alpha}\int M(z)w_r(z)dz \le c2^{-k\beta}$$

by (4.1). If $|x| \le 2^{k/2}$, then $\rho(x)/w_r(x) \le cr^{-\alpha}$. But

$$\int M(z)M(\frac{z-x}{2^k})w_r(z)dz \le c2^{-k\beta}\int_{|z|\le 2^{2k/3}} M(z)w_r(z)dz + c\int_{|z|\ge 2^{2k/3}} w_r(z)dz$$
$$\le c2^{-k\beta}r^{\alpha}.$$

by (4.1). So for $|x|$ in this range also, we have $I_{10} \le c2^{-k\beta}$.

Finally, we look at I_{11}. If $|x| \ge 2^{k/2}$,

$$I_{11} \le c|x|^{\alpha-\delta}\int w_r(z)dz \le c2^{-k\beta}.$$

If $|x| < 2^{k/2}$,

$$I_{11} \le c\int M(\frac{z-x}{2^k})w_r(z)dz \le c2^{-k\beta}$$

by (4.3).

PROOF OF (3.6), $r \ge 1$.

We bound $|T_0(z-x) - T_0w_r(x)|$ by $\rho(z-x) + |T_0w_r(x)|$. Using this bound and arguing as in (4.19), (4.20), and (4.21), we see that it suffices to bound

$$\begin{aligned} I_{12} &= w_r(x)^{-1}\int[\rho(z-x)+|T_0w_r(x)|]M(\frac{z-x}{2^k})w_r(z)dz \\ &= w_r(x)^{-1}\int_{|z|>|x|/2}\rho(z-x)M(\frac{z-x}{2^k})w_r(z)dz \\ &\quad + w_r(x)^{-1}\int_{|z|\le|x|/2}\rho(z-x)M(\frac{z-x}{2^k})w_r(z)dz \\ &\quad + |T_0w_r(x)|w_r(x)^{-1}\int M(\frac{z-x}{2^k})w_r(z)dz \\ &= I_{13}+I_{14}+I_{15}. \end{aligned} \tag{4.22}$$

When $|z| > |x|/2$, $w_r(z)/w_r(x) \le c$, and so $I_{13} \le c2^{-k\beta}$ by (4.2).

Next we look at I_{14}. If $|x| \le r$, again $w_r(z)/w_r(x) \le c$, and we bound I_{14} as we did I_{13}. When $|z| \le |x|/2$, $\rho(z-x) \le c\rho(x)$.

If $r \le |x| \le 2^{k/2}$, then by (4.3),

$$I_{14} \le c\rho(x)w_r(x)^{-1}\int M(\frac{z-x}{2^k})w_r(z)dz \le c2^{-k\beta}.$$

If $|x| \geq 2^{k/2} \vee r$, then

$$I_{14} \leq c\rho(x)w_r(x)^{-1} \leq c|x|^{\alpha-\delta} \leq c2^{-k\beta}.$$

In any case we have the desired estimate for I_{14}.

Finally, look at I_{15}. First consider the case $|x| \geq r$. Using (4.16),

$$|T_0 w_r(x)|/w_r(x) \leq c\rho(z)/w_r(x) \leq c|x|^{\alpha-\delta}. \tag{4.23}$$

If $|x| \leq 2^{k/2}$, $I_{15} \leq c2^{-k\beta}$ by (4.3). And if $|x| \geq 2^{k/2}$, then $I_{15} \leq c2^{-k\beta}$ by (4.23).

Next consider the case $|x| \leq r$. If $r \leq 2^{k/2}$, we use (4.14) to see that $|T_0 w_r(x)|/w_r(x) \leq c$, and then use (4.3) to get $I_{15} \leq c2^{-k\beta}$. If $|x| \leq r^{1-\delta/4}$ and $r \geq 2^{k/2}$, we use (4.14) to see that

$$|T_0 w_r(x)|/w_r(x) \leq c(r^{-\beta} + |x|/r) \leq c2^{-k\beta}. \tag{4.24}$$

And lastly, if $r \geq 2^{k/2}$ and $r^{1-\delta/4} \leq |x| \leq r$, then by (4.16),

$$|T_0 w_r(x)|/w_r(x) \leq cr\rho(x) \leq cr|x|^{-(1+\delta)} \leq cr^{1-(1-\delta/4)(1+\delta)} \leq c2^{-k\beta}. \tag{4.25}$$

So in any case $I_{15} \leq c2^{-k\beta}$, and the proof of (3.6) is complete.

PROOF OF (3.7), $r \leq 1$.

We write, using (3.3),

$$\begin{aligned}
&\int \Big| \int U_0^r(x,z)(U_k^r)^*(z,y)w_r(z)dz\Big| w_r(x)dx \\
&\leq \int_{|x|\leq 1} \int |U_0^r(x,z)U_k^r(y,z)| w_r(z)w_r(x)dz\,dx \\
&\quad + \int \Big| \int T_0(z-x)U_k^r(y,z)dz \\
&\qquad - \int T_0 w_r(z)U_k^r(y,z)dz\Big| w_r(x)dx \\
&\leq \int_{|x|\leq 1} \int |U_0^r(x,z)U_k^r(y,z)| w_r(z)w_r(x)dz\,dx \\
&\quad + \int_{|x|>1} \int |T_0 w_r(z)|\,|U_k^r(y,z)| w_r(x)dz\,dx \\
&\quad + \int_{|x|>1} \Big| \int_{|z-x|>2^{k/8}} T_0(z-x)U_k^r(y,z)dz\Big| w_r(x)dx \\
&\quad + \int_{|x|>1} \Big| \int_{|z-x|\leq 2^{k/8}} T_0(z-x)U_k^r(y,z)dz\Big| w_r(x)dx \\
&= I_{16} + I_{17} + I_{18} + I_{19}.
\end{aligned} \tag{4.26}$$

By (4.17),

$$I_{16} \le c \int_{|x|\le 1} \int \{M(x)[\rho(z-x)+\rho(z)]+M(r^\alpha)\rho(z)\} w_r(z)^{-1}[\rho_k(y-z)+\rho_k(z)]dz w_r(x)dx.$$

Since $|x| \le 1$, $\rho(z-x) \le c\rho(z)$, and by (4.1),

$$\begin{aligned} \text{(4.27)} \qquad I_{16} &\le cM(r^\alpha) \int \rho(z) w_r(z)^{-1}[\rho_k(y-z)+\rho_k(z)]dz \\ &\le cM(r^\alpha) \int_{|z|\le 2^{k/2}} + cM(r^\alpha) \int_{|z|>2^{k/2}} . \end{aligned}$$

Since $r^\alpha \rho(z)/w_r(z) \le c$ and ρ_k is bounded by 2^{-k}, the first term on the right of (4.27) is bounded by $c2^{-k/2}$. When $|z| \ge 2^{k/2}$, $r^\alpha\rho(z)/w_r(z) \le c|z|^{\alpha-\delta} \le c2^{-k\beta}$. Since $\rho_k(y-z)$ and $\rho_k(z)$ are integrable, the second term on the right of (4.27) is also bounded by $c2^{-k\beta}$; hence so is I_{16}.

Since $\int_{|x|\ge 1} w_r(x)dx \le cr^\alpha$, then using (4.16),

$$\begin{aligned} I_{17} &\le c \int_{|x|>1} \int \rho(z) w_r(z)^{-1}[\rho_k(y-z)+\rho_k(z)]dz w_r(x)dx \\ &\le cr^\alpha \int \rho(z) w_r(z)^{-1}[\rho_k(y-z)+\rho_r(z)]dz. \end{aligned}$$

But this is bounded by $c2^{-k\beta}$ by (4.27).

Next,

$$\begin{aligned} \text{(4.28)} \qquad I_{18} &\le c \iint_A \rho(z-x) w_r(z)^{-1}[\rho_k(y-z)+\rho_k(z)] w_r(x)dx\, dz \\ &\le c \iint_{A\cap(|x|>|z|/2)} + c \iint_{A\cap(|x|\le|z|/2)} , \end{aligned}$$

where $A = (|x| > 1, |z-x| \ge 2^{k/8})$. When $|x| > |z|/2$, $w_r(x)/w_r(z) \le c$, and so the first term on the right of (4.28) is bounded by

$$c \int [\int_{|z-x|\ge 2^{k/8}} \rho(z-x)dx][\rho_k(y-z)+\rho_k(z)]dz \le c2^{-k\beta}.$$

If $|x| \le |z|/2$, then $\rho(z-x) \le c\rho(z)$, $|z| \ge 2 \ge r$, and $|z| \ge c2^{k/8}$. So the second term on the right of (4.28) is

$$\begin{aligned} &\le c \int_{|z|\ge c2^{k/8}} \rho(z) w_r(z)^{-1}[\rho_k(y-z)+\rho_k(z)][\int_{|x|>1} w_r(x)dx]dz \\ &\le cr^\alpha \int_{|z|\ge 2^{k/8}} \rho(z) w_r(z)^{-1}[\rho_k(y-z)+\rho_k(z)]dz, \end{aligned}$$

which is $\le c2^{-k\beta}$ as in (4.27).

We now turn to I_{19}. In the proof of Lemma 4.5 we showed that T_0 is odd. So $\int_{|z-x|\le 2^{k/8}} T_0(z-x)g(x,y)dz = 0$ for any function $g(x,y)$. Hence

$$I_{19} = \int_{|x|>1} |\int_{|z-x|\le 2^{k/8}} T_0(z-x)[U_k^r(y,z) - U_k^r(y,x)]dz|w_r(x)dx$$
$$\le \int_{|x|>1}\int_{B_1} \rho(z-x)|U_k^r(y,z) - U_k^r(y,x)|w_r(x)dz\,dx$$
$$+ \int\int_{B_2} \rho(z-x)|U_k^r(y,z) - U_k^r(y,x)|w_r(x)dz\,dx = I_{20} + I_{21},$$

where $B_1 = (|z-x| \le 2^{k/8}, |z| \le 2^{k/4})$ and $B_2 = (|z-x| \le 2^{k/8}, |z| > 2^{k/4})$. When $|z| \le 2^{k/4}$ and $|z-x| \le 2^{k/8}$, then $|x| \le c2^{k/4}$, $w_r(z)^{-1} \le 1 + \frac{|z|^{1+\alpha}}{r^\alpha} \le c2^{k/2}r^{-\alpha}$, and similarly for $w_r(x)^{-1}$. Since T_k is bounded by $c2^{-k}$, we get

$$I_{20} \le c\int_{|x|>1}\int_{|z|\le 2^{k/4}} 2^{-k/2}r^{-\alpha}w_r(x)dz\,dx \le c2^{-k\beta}.$$

The last integral to bound is I_{21}. We have

(4.29)
$$\left|\frac{T_k(y-z)}{w_r(z)} - \frac{T_k(y-x)}{w_r(x)}\right| \le \frac{|T_k(y-z) - T_k(y-x)|}{w_r(z)} + \frac{|T_k(y-x)||w_r(x) - w_r(z)|}{w_r(x)w_r(z)}.$$

When $|z-x| \le 2^{k/8}$ and $|z| \ge 2^{k/4}$, the first term on the right of (4.29) is bounded by

$$cM(\frac{z-x}{2^k})\frac{\rho_k(y-z) + \rho_x(y-x)}{w_r(z)} \le c2^{-k\beta}\frac{\rho_k(y-x) + \rho_k(y-x)}{w_r(z)}.$$

Routine estimates show that the second term on the right of (4.29) is bounded by $c2^{-k\beta}\rho_k(y-x)/w_r(x)$. Also $w_r(x)/w_r(z) \le c$. Then

$$(4.30)\int\int_{B_2} \rho(z-x)\left|\frac{T_k(y-z)}{w_r(z)} - \frac{T_k(y-x)}{w_r(x)}\right| w_r(x)dx\,dz$$
$$\le c2^{-k\beta}\int\int_{B_2} \rho(z-x)\left\{\frac{\rho_k(y-z) + \rho_k(y-x)}{w_r(z)} + \frac{\rho_k(y-x)}{w_r(x)}\right\} w_r(x)dx\,dz \le c2^{-k\beta}.$$

Similarly, using

$$|T_k w_r(z) - T_k w_r(x)| = |\int [T_k(z-v) - T_k(x-v)]w_r(v)dv|$$
$$\le M(\frac{z-x}{2^k})\int [\rho_k(z-v) + \rho_k(x-v)]w_r(v)dv,$$

we get

$$\int\int_{B_2} \rho(z-x)\left|\frac{T_k w_r(z)}{w_r(z)} - \frac{T_k w_r(x)}{w_r(x)}\right| w_r(x)dx \le c2^{-k\beta}. \tag{4.31}$$

Together (4.30) and (4.31) bound T_{21}.

PROOF OF (3.7), $r \ge 1$.

Similarly to (4.26), we write

$$\begin{aligned}\int\Big|\int U_0^r(x,z)(U_k^r)^*(z,y)w_r(z)dz\Big|w_r(x)dx &\le \int\int |T_0 w_r(z)||U_k^r(y,z)|w_r(x)dz\,dx \\ &\quad+\int_{|z-x|\ge 2^{k/8}}\int |T_0(z-x)||U_k^r(y,z)|w_r(x)dz\,dx \\ &\quad+\int\Big|\int_{|z-x|\le 2^{k/8}} T_0(z-x)U_k^r(y,z)dz\Big|w_r(x)dx = I_{22}+I_{23}+I_{24}\,.\end{aligned} \tag{4.32}$$

For I_{22}, we have

$$\begin{aligned} I_{22} &\le \int |T_0 w_r(z)|w_r(z)^{-1}[\rho_k(y-z)+\rho_k(z)]dz \\ &= \int_{|z|\le 2^{k/2}} + \int_{|z|>2^{k/2}}\,. \end{aligned} \tag{4.33}$$

Using either (4.14) or (4.16), $|T_0 w_r(z)|/w_r(z) \le c$. Since ρ_k is bounded by 2^{-k}, the first term on the right of (4.33) is bounded by $c2^{-k/2} = c2^{-k\beta}$. Since $\rho_k(y-z)+\rho_k(z)$ is integrable, to bound the second term on the right of (4.33), it suffices to bound $|T_0 w_r(z)|/w_r(z)$ for $|z| \ge 2^{k/2}$. If $|z| \ge r$, we use (4.23). If $|z| < r$, we use (4.24) and (4.25).

We turn to I_{23}. We see that

$$\begin{aligned} I_{23} &\le c\int\int_{|z-x|\ge 2^{k/8}} \rho(z-x)\frac{\rho_k(y-z)+\rho_k(z)}{w_r(z)}w_r(x)dx\,dz \\ &= c\int\int_{C_1} + c\int\int_{C_2}, \end{aligned} \tag{4.34}$$

where $C_1 = (|z-x| \ge 2^{k/8}, |x| \ge |z|/2)$ and $C_2 = (|z-x| \ge 2^{k/8}, |x| < |z|/2)$. When $|x| \ge |z|/2$, $w_r(x) \le cw_r(z)$, and the first term on the right of (4.34) is

$$\le c\int\int_{|z-x|\ge 2^{k/8}} \rho(z-x)[\rho_k(y-z)+\rho_k(z)]dx\,dz \le c2^{-k\beta}.$$

When $|x| < |z|/2$, $\rho(z-x) \le c\rho(z)$ and $|z| \ge c2^{k/8}$, so the second term on the right of (4.34) is

$$\le c \iint_D \rho(z) w_r(z)^{-1}[\rho_k(y-z) + \rho_k(z)] w_r(x) dx\, dz,$$

where $D = (|x| < |z|/2, |z| \ge c2^{k/8})$. When $|z| \ge r$ and $|z| \ge c2^{k/8}$, then $\rho(z)/w_r(z) \le c|z|^{\alpha-\delta} \le c2^{-k\beta}$. When $|z| \le r$, then $w_r(x)/w_r(z) \le c$, and

$$\int_{D\cap(|z|\le r)} \int \rho(z)[\rho_k(y-z) + \rho_k(z)] dx\, dz \le \int_{|z|\ge c2^{k/8}} |z|\rho(z)[\rho_k(y-z) + \rho_k(z)] dz \le c2^{-k\beta},$$

since $|z|\rho(z) \le |z|^{-\delta} \le c2^{-k\beta}$. So the second term on the right of (4.34), hence I_{23} also, is bounded by $c2^{k\beta}$.

As with I_{19},

$$\begin{aligned} I_{24} &= \int \Big| \int_{|z-x|\le 2^{k/8}} T_0(z-x)[U_k^r(y,z) - U_k^r(y,x)] dz \Big| w_r(x) dx \\ &\le c \int_{|z-x|\le 2^{k/8}} \rho(z-x)|U_k^r(y,z) - U_k^r(y,x)| w_r(x) dz\, dx \\ &= c \iint_{E_1} + c \iint_{E_2} = I_{25} + I_{26}. \end{aligned}$$

where $E_1 = (|z-x| \le 2^{k/8}, |z| \le 2^{k/4})$ and $E_2 = (|z-x| \le 2^{k/8}, |z| \ge 2^{k/4})$. When $|z| \le 2^{k/4}$ and $|z-x| \le 2^{k/8}$, then $|x| \le c2^{k/4}$, and both $w_r(z)^{-1}$ and $w_r(x)^{-1}$ are bounded by $cr + c(2^{k/4})^{1+\alpha} \le cr + c2^{k/2}$. Let $F = (|z| \le 2^{k/4}, |x| \le c2^{k/4})$. Since T_k is bounded by 2^{-k},

$$\begin{aligned} I_{25} &\le c2^{-k} r \iint_F \rho(z-x) w_r(x) dx\, dz + c2^{-k/2} \iint_F \rho(z-x) w_r(x) dx\, dz \qquad (4.35) \\ &\le c2^{-k\beta}, \end{aligned}$$

using the fact that $w_r(x) \le cr^{-1}$ to handle the first term on the right of (4.35).

The final term, I_{26}, is handled just as I_{21} was. □

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