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A Zero-One Law for Integral Functionals of The Bessel Process*

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Abstract. *In this paper, we find necessary and sufficient conditions for the finiteness of the integral functionals of the Bessel process:* $\int_0^t f(R_s)\,ds$, $0 \le t < \infty$. *They are in the form of a zero-one law and can be regarded as a counterpart of the Engelbert-Schmidt (1981) results, in the case of the Bessel process with dimension* $n \ge 2$.

Let $(W_t,\ \ t \ge 0)$ be a Brownian motion in R^n starting at x. Let $R_t = |W_t|$ be the radial part of W_t; then $R \triangleq (R_t,\ \ t \ge 0)$ is a *Bessel process with dimension* n, and if $n \ge 2$, the stochastic differential equation

$$R_t = r_0 + \int_0^t \frac{n-1}{2R_s}\,ds + B_t, \qquad 0 \le t < \infty \tag{1}$$

is satisfied, where $(B_t, t \ge 0)$ is a standard, one dimensional Brownian motion, and $r_0 = |x|$. We are interested in finding conditions which will guarantee the finiteness of integral functionals:

$$\int_0^t f(R_s)\,ds; \qquad 0 \le t < \infty, \tag{2}$$

where f: $[0,\infty) \to [0,\infty)$ is a Borel measurable function. When $n = 1$, such conditions are provided as special cases of the well-known Engelbert-Schmidt zero-

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one law for integral functionals of Brownian motion (see [1] or [5], section 3.6). When $n \geq 2$ and $r_0 > 0$, Engelbert & Schmidt state necessary and sufficient conditions for the finiteness of (2) in their recent paper [3]. When $n \geq 2$ and $r_0 = 0$ things are different, because the origin is then an entrance boundary; in this case, Pitman & Yor [7] obtain necessary and sufficient conditions for the finiteness of (2) in which f has a support in a right neighbourhood of the point 0 and is locally bounded on $(0, \infty)$, and Engelbert & Schmidt [3] obtain a sufficient condition when $n \geq 3$.

In this paper, we shall provide necessary and sufficient conditions for the finiteness of (2) when $n \geq 2$ and $r_0 = 0$ (Proposition 2 and Corollary 2) and when $n = 2$ and $r_0 > 0$ for Bessel processes defined by (1), where the dimension $n \geq 2$ is a *real* number (Remark 4) . These conditions are in the form of a zero-one law, and can be regarded as a counterpart of the Engelbert-Schmidt (1981) results in the case of the Bessel process with dimension $n \geq 2$. We will give a counterexample which shows that *the zero-one law fails when* $n > 2$ *and* $r_0 > 0$ (Remark 5). We also show that Engelbert-Schmidt zero-one laws for integral functionals of Brownian motion, and for those of the Bessel Process with $n = 2$ and $r_0 > 0$, are two special cases of a zero-one law for integral functionals of semimartingales (Proposition 3).

It is also of interest to investigate under what conditions

$$\int_0^\infty f(R_s)\, ds$$

will be finite. Engelbert & Schmidt [3] provide zero-one laws in the case $n = 2$ and in the case $n \geq 3$ and $r_0 > 0$. They also give some conditions in the case $n \geq 3$ and $r_0 = 0$. In this paper, we establish a zero-one law for the case $n > 2$ and $r_0 = 0$ (Corollary 4).

The continuity of *local time* $\{L_t(a),(t,a) \in [0,\infty) \times [0,\infty)\}$ for the Bessel process $(R_t, t \geq 0)$ will play an important role in our paper, so we start with a direct statement of this fact. It is well-known that for $P-$ a.e. $\omega \in \Omega$,

$$\int_0^t f(R_s(\omega))\,ds = \int_0^\infty f(r)L_t(r,\omega)\,dr, \quad \forall \quad 0 \leq t < \infty. \tag{3}$$

From now on, we assume n is a real number and $n \geq 2$.

Proposition 1. *Let $\{L_t(a); \quad (t,a) \in [0,\infty) \times [0,\infty)\}$ be the local time for the Bessel process $(R_t, \quad t \geq 0)$. Then $L_t(a)$ is $P-a.s.$ continuous in (t,a).*

Proof: By the semimartingale representation (1), this follows immediately from Corollary 1 in [9] (see also Exercise 3.7.10 and the proof of Theorem 3.7.1, in [5]). ◇

Proposition 2. *Suppose $R_0 = r_0 = 0$, and $f\colon [0,\infty) \longrightarrow [0,\infty)$ is a Borel measurable function. Then the following conditions are equivalent:*

(i) $P\left\{\int_0^t f(R_s)\,ds < \infty, \quad \forall \quad 0 \leq t < \infty\right\} > 0$;

(ii) $P\left\{\int_0^t f(R_s)\,ds < \infty, \quad \forall \quad 0 \leq t < \infty\right\} = 1$;

(iii) $f(r)$ *is locally integrable on* $(0,\infty)$ *and*

(a) $\int_0^c f(r)r(\log\frac{1}{r})^+\,dr < \infty$, *if* $n = 2$; *or*

(b) $\int_0^c f(r)r\,dr < \infty$, *if* $n > 2$,

where c is an arbitrary positive constant.

Remark 1. If (b) holds, then $f(r)$ is locally integrable on $(0,\infty)$. ◇

The proof of the proposition depends on the following lemmas.

Define

$$T_a = \inf\{t \geq 0 : \quad R_t = a\}, \qquad a \in [0, \infty);$$

$$T = \begin{cases} T_1, & \text{if} \quad n = 2; \\ \infty, & \text{if} \quad n > 2. \end{cases} \tag{4}$$

Lemma 1. *Suppose $R_0 = r_0 = 0$. Let $(U_t, \ \ t \geq 0)$ be the square of a Bessel process with dimension 2, such that $U_0 = 0$. Then the law of $(L_T(r), \ \ r \geq 0)$ is identical to that of*

$$rU_{(\log(1/r))^+} \quad \textit{if} \quad n = 2, \quad \textit{and} \quad \frac{1}{n-2} r^{n-1} U_{r^{2-n}} \quad \textit{if} \quad n > 2.$$

This result is proved in [6] for $n = 2$, and for integers $n \geq 3$; the proof of the latter part holds also for any real number $n > 2$. ◇

The following lemma is a particular case of a result by Jeulin ([4], Application 1). For the convenience of the reader, we shall go carefully through his proof in this particular case.

Lemma 2. *Let $\mu(dr)$ be a positive measure on $(0, c\,]$, let $(V(r), \ \ r \in (0, c\,])$ be a Borel measurable, R^+- valued random process with $P\{V(r) = 0\} = 0, \ \ r \in (0, c\,]$, such that there exists a locally bounded, Borel measurable function $\Phi : (0, c\,] \to (0, \infty)$ satisfying: for every $r \in (0, c\,]$, the law of $V(r)/\Phi(r)$ is equal to the law of an integrable random variable X. Then the following are equivalent:*

(i) $P\{\int_0^c V(r)\, \mu(dr) < \infty\} > 0$;

(ii) $P\{\int_0^c V(r)\, \mu(dr) < \infty\} = 1$;

(iii) $\int_0^c \Phi(r)\, \mu(dr) < \infty$.

Proof. For the implication *(iii)* $\Rightarrow$*(ii)*, observe

$$E\int_0^c V(r)\,\mu(dr) = \int_0^c E[V(r)/\Phi(r)]\Phi(r)\,\mu(dr) = E\,[X]\cdot\int_0^c \Phi(r)\,\mu(dr) < \infty.$$

In order to show *(i)* $\Rightarrow$*(iii)*, denote $A_t \triangleq \int_0^t \Phi(r)\,\mu(dr), 0 \le t \le c$. For any given event set B,

$$\begin{aligned} E[1_B\int_0^c V(r)\,\mu(dr)] &= \int_0^c E\,[1_B(V(r)/\Phi(r))]\;dA_r \\ &= \int_0^c\int_0^\infty P[B\cap\{V(r)/\Phi(r) > u\}]\,du\,dA_r \\ &\ge \int_0^c\int_0^\infty [P(B) - P\{V(r)/\Phi(r) \le u\}]^+\;du\,dA_r \\ &= A_c\int_0^\infty [P(B) - P\{X \le u\}]^+\,du. \end{aligned} \tag{5}$$

Now (i) implies that there exists some $N > 0$ for which the event $B \triangleq \{\int_0^c V(r)\,\mu(dr) \le N\}$ has positive probability. Choosing such a B in (5), we obtain

$$A_c\int_0^\infty [P(B) - P\{X \le u\}]^+\,du \le N < \infty. \tag{6}$$

Notice that $P\{X = 0\} = 0$, therefore,

$$\int_0^\infty [P(B) - P\{X \le u\}]^+\,du > 0.$$

Whence, (6) implies that

$$A_c = \int_0^c \Phi(r)\,\mu(dr) < \infty. \qquad \diamond$$

Lemma 3. *Suppose* $R_0 = r_0 = 0$, *and* $f\colon [0,\infty) \longrightarrow [0,\infty)$ *is a Borel function which has support in the finite interval* $(0,b]$ *and is locally integrable on* $(0,\infty)$. *Then the following are equivalent:*

(i) $P\{\int_0^T f(R_s)\,ds < \infty\} > 0$;

(ii) $P\{\int_0^T f(R_s)\,ds < \infty\} = 1$;

(iii) *For every* $c > 0$,

(a) $\int_0^c f(r) r(\log(1/r))^+\,dr < \infty$, *if* $n = 2$; *or*

(b) $\int_0^c f(r) r\,dr < \infty$, *if* $n > 2$,

where T *is given in (4).*

Proof: We first show that, for any $c > 0$,

$$\left\{\int_0^T f(R_s)\,ds < \infty\right\} = \left\{\int_0^c f(r)L_T(r)\,dr < \infty\right\}, \quad \text{mod} \quad P. \tag{7}$$

In fact,

$$\begin{aligned}\int_0^T f(R_s)\,ds &= \int_0^T f(R_s)1_{\{R_s \le c\}}\,ds + \int_0^T f(R_s)1_{\{R_s > c\}}\,ds,\\ &= \int_0^T f(R_s)1_{\{R_s \le c\}}\,ds + \int_0^{T'} f(R_s)1_{\{R_s > c\}}\,ds,\\ &= \int_0^c f(r)L_T(r)\,dr + \int_c^{c\vee b} f(r)L_{T'}(r)\,dr.\end{aligned}$$

where $T' \triangleq T = T_1$ if $n = 2$, and $T' \triangleq S_b = \sup\{t \ge 0; \quad R_t = b\}$ if $n > 2$. $L_{T'}(r)$ is continuous in r, and therefore,

$$\sup_{r\in[c,c\vee b]} L_{T'}(r) \le M < \infty, \quad a.s,$$

implying $\int_c^{c\vee b} f(r)L_{T'}(r)\,dr < \infty, a.s.$ under the assumption that f is locally integrable.

It is also easy to see, by Lemma 1, when $n = 2$,

$$\left\{\int_0^c f(r)L_T(r)\,dr < \infty\right\} = \left\{\int_0^{c\wedge 1} f(r)L_T(r)\,dr < \infty\right\}, \quad \text{mod} \quad P.$$

Now let

$$\mu(dr) = r^{n-1} f(r)\, dr; \tag{8}$$

$$\Phi(r) = \begin{cases} (\log(1/r))^+, & \text{if} \quad n = 2; \\ r^{2-n}, & \text{if} \quad n > 2. \end{cases} \tag{9}$$

$$V(r) = U_{\Phi(r)}. \tag{10}$$

We can now use Lemmas 1, 2 and the relation (7) to complete the proof. ◇

Remark 2. This lemma is an extension of the criterion for the divergence of an integral functional of the Bessel process in [7] (Proposition 1). It is in the form of a zero-one law. ◇

Lemma 4. *Suppose $R_0 = r_0 = 0$. For any given $a \in (0, \infty)$,*

$$P\{L_{T_{2a}}(a) > 0\} = 1.$$

Proof. As in [8], let $J(r,t)$ denote the density of the absolutely continuous part of the sojourn time, relative to the speed measure $m(dr) = r^{n-1}\, dr$. By (3.1) in [8], we have

$$\begin{aligned} P\{L_{T_{2a}}(a) > 0\} &= P\{J(a, T_{2a}) > 0\} \\ &= P\{J(a, T_{2a}) > 0 | R_{T_{2a}} = 2a\} = 1. \quad \diamond \end{aligned}$$

We are ready for the proof of Proposition 2.

Proof of Proposition 2. *(i)⇒(iii):* For arbitrary given $a \in (0, \infty)$, choose $\omega_0 \in \{T_{2a} < \infty\} \cap \{L_{T_{2a}}(a) > 0\} \cap \{\int_0^t f(R_s) < \infty, \ \forall \ 0 \le t < \infty\}$. By the continuity of $L_t(r)$, there exist $c > 0$, $\epsilon > 0$, such that

$$0 < a - \epsilon < r < a + \epsilon \quad \text{implies} \quad L_{T_{2a}}(r, \omega_0) \ge c > 0.$$

Therefore, we have from condition (i) and (3):

$$\infty > \int_0^{T_{2a}(\omega_0)} f(R_s(\omega_0))\,ds = \int_0^\infty f(r)L_{T_{2a}(\omega_0)}(r,\omega_0)\,dr$$
$$\geq c\int_{\{|r-a|<\epsilon\}} f(r)\,dr.$$

Hence, f *is locally integrable on* $(0,\infty)$. In order to get (a) and (b) in (iii), note that from (i) we have

$$0 < P\left\{\int_0^T f(R_s)\,ds < \infty\right\} = P\left\{\int_0^T f(R_s)1_{(0,1]}(R_s)\,ds < \infty\right\}$$

for $n = 2$, as well as

$$0 < P\left\{\int_0^{S_1} f(R_s)\,ds < \infty\right\} \leq P\left\{\int_0^T f(R_s)1_{(0,1]}(R_s)\,ds < \infty\right\}$$

for $n > 2$, where $S_1 \triangleq \sup\{t \geq 0;\quad R_t = 1\}$ and T is given in (4). Using Lemma 3 for the function $f(r)1_{(0,1]}(r)$, we obtain (a) and (b) respectively.

(iii)⇒(ii): For arbitrary $t > 0$,

$$\int_0^t f(R_s)\,ds = \int_0^t f(R_s)1_{(1,\infty)}(R_s)\,ds + \int_{t\wedge T_1}^t f(R_s)1_{(0,1]}(R_s)\,ds$$
$$+ \int_0^{t\wedge T_1} f(R_s)1_{(0,1]}(R_s)\,ds$$
$$\triangleq I_1(t) + I_2(t) + I_3(t).$$

For $P-a.e.\ \omega \in \Omega$, we have, with $R_t^* \triangleq \max_{0\leq u\leq t} R_u$,

$$I_1(t,\omega) = \int_1^{1\vee R_t^*(\omega)} f(r)L_t(r,\omega)\,dr$$
$$\leq \max_{1\leq r\leq(1\vee R_t^*(\omega))} L_t(r,\omega)\cdot\int_1^{1\vee R_t^*(\omega)} f(r)\,dr < \infty;\quad \forall\quad 0 \leq t < \infty,$$

due to the local integrability of f and the continuity of $L_t(r)$ in r. Similarly,

$$I_2(t,\omega) \leq \int_{\alpha\wedge 1}^1 f(r)L_t(r,\omega)\,dr < \infty;\quad \forall\quad 0 \leq t < \infty,$$

where we set $\alpha \triangleq \min\{R_u(\omega); \quad (t \wedge T_1) \le u \le t\}$, and notice that $\alpha > 0, \;\; P-$ a.s. Finally,

$$I_3(t,\omega) \le \int_0^T f(R_s(\omega))1_{(0,1]}(R_s(\omega))\,ds < \infty; \quad \forall \quad 0 \le t < \infty,$$

where T is defined in (4), because of (iii) and Lemma 3.

This shows that for $P-$ a.e. $\omega \in \Omega$, $\int_0^t f(R_s(\omega))\,ds < \infty; \quad \forall \quad 0 \le t < \infty.$ ◇

Corollary 1. *Suppose that $R_0 = r_0 = 0$, that f is a function as in Proposition 1, and that $b \in (0,\infty)$ is fixed. Then the following are equivalent:*

(i) $P\{\int_0^{T_b} f(R_s)\,ds < \infty\} > 0$;

(ii) $P\{\int_0^{T_b} f(R_s)\,ds < \infty\} = 1$;

(iii) f *is locally integrable on* $(0,b]$ *and*

(a) $\int_0^c f(r)r(\log(1/r))^+\,dr < \infty$, *if* $n=2$; *or*

(b) $\int_0^c f(r)r\,dr < \infty$, *if* $n>2$,

for any $c \in (0,b]$.

Proof: Without loss of generality, we may assume that f has support in $(0,b]$. It is well-known that $P\{T_b < \infty\} = 1$.

(i) ⇒ *(iii):* For any $a \in (0,b]$, as in Lemma 4, $P\{L_{T_b}(a) > 0\} = 1$. Choose $\omega_0 \in \{L_{T_b}(a) > 0\} \cap \{\int_0^{T_b} f(R_s)\,ds < \infty\}$. As in the proof of Proposition 2, we can obtain that f is locally integrable. Let T be as in (4), $m \triangleq \min\{R_s;\; T \wedge T_b \le s \le T\}$. We know that $P\{m > 0\} = 1$. Therefore, with T' as in the proof of Lemma 3, we

have

$$\int_{T\wedge T_b}^{T} f(R_s)\,ds \le \int_0^{T'} f(R_s)1_{[m,b]}(R_s)\,ds$$
$$= \int_m^b f(r)L_{T'}(r)\,dr$$
$$\le \max_{m\le u\le b} L_{T'}(u)\int_m^b f(r)\,dr < \infty,\ a.s.$$

Hence (i) implies

$$P\{\int_0^T f(R_s)\,ds < \infty\} = P\{\int_0^{T\wedge T_b} f(R_s)\,ds + \int_{T\wedge T_b}^{T} f(R_s)\,ds < \infty\}$$
$$\ge P\{\int_0^{T_b} f(R_s)\,ds < \infty\} > 0.$$

Now (a) and (b) follow from Lemma 3.

(iii) ⇒ (ii): Noting that f has support in $(0,b]$, this follows immediately from Proposition 2. ◇

Corollary 2. *Under the assumptions in Proposition 2, the conditions (i)– (iii) of Proposition 2 are equivalent to the condition:*

(iv) There exists some $t > 0$, such that

$$P\{\int_0^t f(R_s)\,ds < \infty\} = 1.$$

Proof: We need only prove the implication *(iv) ⇒ (iii)*. For any $b \in (0,\infty)$, (iv) implies

$$P\{\int_0^{T_b} f(R_s)\,ds < \infty\} \ge P\{\int_0^t f(R_s)\,ds < \infty,\ \ T_b \le t\ \}$$
$$= P\{T_b \le t\} > 0.$$

Now (iii) follows from Corollary 1. ◇

Corollary 3. *Suppose $R_0 = r_0 = 0$, $d > 0$, and f is a function as in Proposition 2. Then the following are equivalent:*

(i) $P\{\int_0^t f(R_s)\,ds < \infty, \quad \forall \quad 0 \le t < T_d\} > 0;$

(ii) $P\{\int_0^t f(R_s)\,ds < \infty, \quad \forall \quad 0 \le t < T_d\} = 1;$

(iii) f is locally integrable on $(0, d)$ and

(a) $\int_0^c f(r) r (\log(1/r))^+ \,dr < \infty$ *if* $n = 2$; *or*

(b) $\int_0^c f(r) r\,dr < \infty$ *if* $n > 2$,

for every $c \in (0, d)$.

Proof: *(i) ⇒ (iii)*: For any $b \in (0, d)$, $P\{T_b < T_d\} = 1$, and thus (i) implies that

$$P\{\int_0^{T_b} f(R_s)\,ds < \infty\} > 0.$$

Using Corollary 1 and the fact that b is arbitrary in $(0, d)$, we obtain (iii).

(iii) ⇒ (ii): Consider a strictly increasing sequence $\{b_n\}_{n=1}^{\infty} \subseteq [0, d)$ with $\lim_{n\to\infty} b_n = d$. Then

$$P\{T_{b_n} < T_d \text{ for all } n \ge 1, \text{ and } \lim_{n\to\infty} T_{b_n} = T_d\} = 1.$$

By Corollary 1, (iii) implies that

$$P\{\int_0^{T_{b_n}} f(R_s)\,ds < \infty\} = 1, \quad \forall \quad n \ge 1.$$

Therefore,

$$P\{\int_0^t f(R_s)\,ds < \infty, \ \forall \ 0 \le t < T_d\} = P\{\int_0^{T_{b_n}} f(R_s)\,ds < \infty, \ \forall \ n \ge 1\} = 1. \quad \diamond$$

The following is an improvement on the Corollary to Theorem 2 in [3].

Corollary 4. *Let* (R_t) *be a Bessel process with dimension* $n > 2$ *and* $R_0 = 0$, *a.s.P. Let* $f : [0,\infty) \to [0,\infty)$ *be a Borel measurable function. Then the following are equivalent:*

(i) $P\{\int_0^\infty f(X_s)\,ds < \infty\} > 0$;

(ii) $P\{\int_0^\infty f(X_s)\,ds < \infty\} = 1$;

(iii) $\int_0^\infty r f(r)\,dr < \infty$.

Proof. It is easy to see that even if c is replaced by ∞ and $(0,c]$ is replaced by $(0,\infty)$, Lemma 2 still holds. Noticing that for $P-$ a.e. $\omega \in \Omega$,

$$\int_0^\infty f(X_s(\omega))\,ds = \int_0^\infty f(r) L_T(r,\omega)\,dr \ ,$$

where $T = \infty$ as in (4), we can use Lemma 1, $V(r), \Phi(r)$ and $\mu(dr)$ as in (8)–(10), and Lemma 2 with $c = \infty$, to obtain the results. ◇

Now we discuss integral functionals of *continuous semimartingales*. Let $X = \{X_t = X_0 + M_t + V_t,\ \mathcal{F}_t;\quad 0 \le t < \infty\}$ be a continuous semimartingale, where $M = \{M_t,\ \mathcal{F}_t;\quad 0 \le t < \infty\}$ is a continuous local martingale and $V = \{V_t,\ \mathcal{F}_t;\quad 0 \le t < \infty\}$ is the difference of two continuous, nondecreasing adapted processes with $V_0 = 0$, $P-$a.s. In [2], Engelbert & Schmidt deal with a zero-one law for the integral functionals $\int_0^t f(X_s(\omega))\,d\langle M\rangle_s(\omega);\quad 0 \le t < \infty$ for some special semimartingales, to which the Girsanov theorem can be applied. Here we deal with the same problem by another approach. We know that there exists a *semimartingale local time* $\Lambda = \{\Lambda_t(r,\omega);\quad (t,r) \in [0,\infty) \times R^1,\ \ \omega \in \Omega\}$ for X, such that

$$\int_0^t f(X_s(\omega))\,d\langle M\rangle_s(\omega) = \int_{-\infty}^\infty f(r)\Lambda_t(r,\omega)\,dr;\quad 0 \le t < \infty$$

holds for P − a.e. $\omega \in \Omega$, for every Borel measurable $f : R^1 \to [0,\infty)$ (see [5], section 3.7).

Proposition 3. *Suppose X is a continuous semimartingale, satisfying $P\{X_t \in I;\ \ 0 \le t < \infty\} = 1$ for some interval $I \subset R^1$, $P\{\omega;\ \ \Lambda_t(\cdot,\omega)$ is continuous $\} = 1$ for every $t \in [0,\infty)$, and there exists a random variable T for which*

$$P\{\omega \in \Omega;\ 0 \le T(\omega) < \infty, \quad \Lambda_{T(\omega)}(r,\omega) > 0\} = 1 \tag{11}$$

holds for every $r \in I$. Also suppose that $f : R^1 \to [0,\infty)$ is a Borel measurable function. Then the following are equivalent:

(i) $P\{\int_0^t f(X_s)\,d\langle M\rangle_s < \infty, \quad \forall \quad 0 \le t < \infty\} > 0;$

(ii) $P\{\int_0^t f(X_s)\,d\langle M\rangle_s < \infty, \quad \forall \quad 0 \le t < \infty\} = 1;$

(iii) f is locally integrable on I.

Proof: *(i) ⇒(iii):* For any $x \in I$, (i) implies that

$$P\{\omega;\quad \Lambda_{T(\omega)}(x,\omega) > 0 \text{ and } \int_0^t f(X_s(\omega))\,d\langle M\rangle_s(\omega) < \infty; \quad \forall \quad 0 \le t < \infty\} > 0.$$

Choose ω_0 and a number $t_0 > T(\omega_0)$, such that $\Lambda_{t_0}(\cdot,\omega_0)$ is continuous,

$$\Lambda_{t_0}(x,\omega_0) > 0, \quad \int_0^{t_0} f(X_s(\omega_0))\,d\langle M\rangle_s(\omega_0) < \infty,$$

and

$$\int_0^{t_0} f(X_s(\omega_0))\,d\langle M\rangle_s(\omega_0) = \int_{-\infty}^{\infty} f(r)\Lambda_{t_0}(r,\omega_0)\,dr.$$

By the continuity of $\Lambda_{t_0}(\cdot,\omega_0)$, there exist $\epsilon > 0$ and $c > 0$, such that $\Lambda_{t_0}(r,\omega_0) \ge c$ for all $r \in I \cap \{a : |a - x| < \epsilon\}$. Therefore,

$$\begin{aligned}\infty &> \int_0^{t_0} f(X_s(\omega_0))\,d\langle M\rangle_s(\omega_0) = \int_{-\infty}^{\infty} f(r)\Lambda_{t_0}(r,\omega_0)\,dr \\ &\ge c\int_{I\cap\{|r-x|<\epsilon\}} f(r)\,dr.\end{aligned}$$

This implies that f is locally integrable on I.

(iii) $\Rightarrow$ *(ii):* For any $t \in [0,\infty)$, denote $m \triangleq \min\{X_s;\quad 0 \le s \le t\}$ and $l \triangleq \max\{X_s;\quad 0 \le s \le t\}$. Then,

$$\begin{aligned}\int_0^t f(X_s)\,d\langle M\rangle_s &= \int_0^t f(X_s)1_{[m,l]}(X_s)\,d\langle M\rangle_s\\ &= \int_{-\infty}^{\infty} f(r)1_{[m,l]}(r)\Lambda_t(r)\,dr\\ &\le \max_{r\in[m,l]} \Lambda_t(r)\int_{[m,l]} f(r)\,dr < \infty,\end{aligned}$$

because of the continuity of the local time $\Lambda_t(r,\omega)$ and the local integrability of f. ◇

Remark 3. A sufficient condition for the continuity of $\Lambda_t(\cdot,\omega)$ is that $|dV_\cdot(\omega)|$ be absolutely continuous with respect to $d\langle M\rangle_\cdot(\omega)$ for $P-a.e.\ \omega \in \Omega$; cf. references in the Proof of Proposition 1 . On the other hand, Professor M.Yor points out to us (personal communication) that (11) is satisfied as soon as the law of X is locally absolutely continuous with respect to that of a continuous local martingale (for instance, that of its continuous martingale part). ◇

Remark 4. This Proposition has two important consequences:

(i) If X is a *Brownian motion*, then we obtain the Engelbert-Schmidt zero-one law (see [1] or [5], section 3.6).

(ii) If X is a *Bessel process* with dimension $n = 2$ and $X_0 = r_0 > 0$, then $I = (0,\infty)$. By Proposition 1, the local time $L_t(r)$ is $P-$a.s. continuous. It is also well-known that $P\{T_a < \infty\} = 1$ for *every* $a \in (0,\infty)$. Therefore, similar to Lemma 4, we have $P\{L_{T_{2a}}(a,\omega) > 0\} = 1$ for $a \ge r_0$ and $P\{L_{T_{a/2}}(a,\omega) > 0\} = 1$ for $a \in (0,r_0)$, and the conditions in Proposition 3 are satisfied. Hence we can obtain the zero-one law for X as Theorem 1 in [3]. But it is not possible to obtain a zero-one law for the Bessel process R with dimension $n > 2$ and $R_0 = r_0 > 0$ from this Proposition, because (11) fails for $r \in (0, r_0)$. We shall discuss this situation in Remark 5. $\diamond$

Remark 5. For a Bessel process R with dimension $n \geq 2$ and $R_0 = r_0 > 0$, and a Borel measurable function $f : (0,\infty) \to [0,\infty)$, consider the statements

(i) $P\{\int_0^t f(R_s)\,ds < \infty, \ \forall\ 0 \leq t < \infty\} > 0$;

(ii) $P\{\int_0^t f(R_s)\,ds < \infty, \ \forall\ 0 \leq t < \infty\} = 1$;

(iii) f is locally integrable on $(0,\infty)$.

For $n = 2$, (i)–(iii) are equivalent (see Remark 4(ii)). However, *the zero-one law* (i.e. the equivalence $(i) \Leftrightarrow (ii)$) *does not hold when* $n > 2$. Here is a counterexample.

Let R be a Bessel process with dimension $n > 2$ and $R_0 = r_0 > 0$. Let I be an open interval such that $I \subset (0, r_0/2)$. Let $(L_t(r);\ (t,r) \in [0,\infty) \times (0,\infty))$ be the local time for R, which is $P-$a.s. continuous in (t,r), due to Proposition 1. Given any $t > 0$, we know that there exists an $a \in I$, such that

$$P\{L_t(a) > 0\} > 0. \tag{12}$$

Now define

$$f(r) = |\frac{1}{r-a}|\ 1_{I\setminus\{a\}}(r), \quad r \in (0,\infty).$$

For every $\omega \in \{L_t(a,\omega) > 0\}$, by the continuity of $L_t(r)$,

$$\int_0^t f(R_s(\omega))\,ds = \int_0^\infty f(r)L_t(r,\omega)\,dr = \infty,$$

and this leads, in conjunction with (12), to

$$P\{\int_0^t f(R_s)\,ds < \infty, \quad \forall \quad 0 \le t < \infty\} < 1.$$

On the other hand, it is well-known that $P\{T_{r_0/2} = \infty\} = 1 - 2^{2-n} > 0$ (cf. Problem 3.3.23 in [5]). For every $\omega \in \{T_{r_0/2} = \infty\}$, we have

$$\int_0^t f(R_s(\omega))\,ds = 0, \quad \forall \quad 0 \le t < \infty.$$

Therefore,

$$P\{\int_0^t f(R_s)\,ds < \infty, \quad \forall \quad 0 \le t < \infty\} \ge P\{T_{r_0/2} = \infty\} > 0.$$

Whence the zero-one law fails. However, noting that the probability of the event in (11) is positive, we can see that (ii) and (iii) are equivalent by slightly changing the proof in proposition 3. ◇

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