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On the Azéma Martingales

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INTRODUCTION. Let $(X_t)_{t\geq 0}$ be a martingale such that $\langle X,X\rangle_t = t$. As shown by Meyer [5], the <u>iterated integral</u>

$$I_n(f) = \int_{0<t_1<\dots<t_n} f(t_1,\dots,t_n)\,dX_{t_1}\dots dX_{t_n}$$

can be defined for f square-integrable on the set $C_n = \{(t_1,\dots,t_n) \in \mathbf{R}^n : 0 < t_1 < \dots < t_n\}$, and is an isometry from the Hilbert space $L^2(C_n)$ to its image $\chi_n = I_n(L^2(C_n))$. This subspace χ_n of $L^2(\Omega)$ is called the n^{th} <u>chaos</u> generated by X; for $n = 0$, by convention, $L^2(C_0) = \mathbf{R}$, χ_0 is the space of constant random variables and I_0 is the obvious identification. These chaoses $(\chi_n)_{n\geq 0}$ are mutually orthogonal in $L^2(\Omega)$; hence they provide an orthogonal expansion for the random variables Z that belong to the Hilbert sum $\bigoplus_{n\geq 0}\chi_n$ (such random variables will be called <u>chaos-decomposable</u>). It may happen that this Hilbert space is the whole space L^2; such is the case, for instance, when X is a Brownian motion or a compensated Poisson process (and F the σ-field generated by X). When this happens, one says that X has the <u>chaotic representation property</u>. A problem raised by Meyer in [6] p. 262 is: Which martingales X have this property? We shall not answer this question, but merely add a few examples to the aforementioned ones.

Notice that this chaotic representation property is stronger than the predictable representation property, since every chaos-decomposable $Z \in L^2(\Omega)$ can always be written as $E[Z] + \int_0^\infty H_s\,dX_s$ for some predictable process H with $E\int_0^\infty H_s^2\,ds < \infty$ (if $Z = I_n(f)$ for $n \geq 1$, just choose

$$H_t = \int_{0<t_1<\dots<t_{n-1}<t} f(t_1,\dots,t_{n-1},t)\,dX_{t_1}\dots dX_{t_{n-1}} ;$$

this process has a predictable version, and any such version will do). Recall that, if the filtration $(F_t)_{t\geq 0}$ we are working in is the one generated by X, then X has the predictable representation property iff its law is an extreme

(*) Part of this work was done in Strasbourg, during uncountable conversations with P.A. Meyer; part in Vancouver, while visiting U.B.C.

point of the set of all laws of martingales (see Jacod-Yor [4]).

Before starting, seeing what happens in the discrete-time case will be helpful: the time-axis $[0,\infty)$ is replaced by the finite set $\{0,1,\ldots,N\}$, we have a martingale $(X_n)_{0\le n\le N}$, and, denoting by $Y_n = X_n - X_{n-1}$ its increments, the requirement $\langle X,X\rangle_t = t$ is naturally replaced by its discrete analogue $E[Y_n^2|F_{n-1}] = 1$. Supposing also that the initial value X_0 is deterministic, the n^{th} chaos χ_n is now the vector space generated by the random variables $\prod_{j\in J} Y_j$ where J ranges over all subsets of $\{0,\ldots N\}$ with n elements (χ_n is defined only for $n \le N$). Now, supposing that the filtration is the one generated by X, the following turn out to be equivalent:

(i) X has the predictable representation property;

(ii) the filtration is dyadic: F_n essentially consists in 2^n atoms, each of them splitting into exactly two atoms of F_{n+1};

(iii) for some Borel functions $\Phi_n:\mathbf{R}^{n-1} \to \mathbf{R}$,

$$Y_n^2 = 1 + \Phi_n(Y_1,\ldots,Y_{n-1})Y_n;$$

(iv) X has the chaotic representation property.

PROOF. The equivalence between (i) and (ii) is well known; it can be obtained by replacing (i) by the law of X being extremal, and this amounts to the conditional law of X_n given F_{n-1} being supported by at most two points (see Dellacherie [2]). As the variance of this conditional law is 1, "at most" can be replaced by "exactly" in the previous sentence; and this means that the natural filtration of X is dyadic.

To get (iii), just remark that, given F_{n-1}, Y_n can assume exactly two values; hence it must solve some quadratic equation

$$Y_n^2 = \Psi_n(Y_1,\ldots,Y_{n-1}) + \Phi_n(Y_1,\ldots,Y_{n-1})Y_n.$$

But, since $E[Y_n|F_{n-1}] = 0$ and $E[Y_n^2|F_{n-1}] = 1$, Ψ_n is identically 1. Conversely, if (iii) holds, given F_{n-1}, Y_n can take at most two values and the law of X is extremal.

Last, to get the chaotic representation property (which we know is stronger than (i)), it suffices to notice that $L^2(F_n)$ is a vector space with 2^n dimensions, whereas the subspace χ_n has dimension $\binom{N}{n}$ (if dim χ_n were smaller, some Y_p would be a linear combination of $Y_1,\dots,Y_{p-1}$); hence $\bigoplus_{n=0}^{N} \chi_n$, with dimension 2^N, must be the whole space. ∎

Remark that, given $X_0 \in \mathbf{R}$ and the functions Φ_n, there is exactly one martingale X such that (iii) holds (uniqueness being understood in law). Indeed, if you know $X_0,\dots,X_{n-1}$ the quadratic equation (iii) gives you two possible values for Y_n; since their product is -1, they are real with opposite signs, so there is exactly one way of weighting them to get a mean equal to $0 = E[Y_n | F_{n-1}]$.

STRUCTURE EQUATIONS. The continuous time analogue of (iii) will be called a <u>structure equation</u>. It has the form

$$d[X,X]_t = dt + \Phi_t dX_t; \tag{SE}$$

this is, of course, a symbolic notation for the equation

$$[X,X]_t = t + \int_0^t \Phi_s dX_s$$

(by convention, all brackets and stochastic integrals are taken null at time zero).

In this equation, X is a martingale, Φ a predictable process, and the stochastic integral $\int \Phi dX$ is a martingale. Note that no further generality would be gained by assuming only that X and $M = \int \Phi dX$ are local martingales, or even stochastic integrals with respect to local martingales: In that case, the existence of the decomposition of the increasing process $[X,X]$ into $t + M$ shows that $[X,X]$ is locally integrable (see [3]); then the uniqueness implies that M is a local martingale. So it is reduced by some stopping times T_n; and $E[[X,X]_{t \wedge T_n}] = E(t \wedge T_n]$. Taking increasing limits yields $E[[X,X]_t] = t$, and, on compact time intervals, X is a square-integrable martingale and M a uniformly integrable martingale (it is even in H^1, since $\sup_{s \le t} |M_s| \le [X,X]_t + t$).

Remark also that $\langle X,X\rangle_t = t$ is a consequence of the structure equation itself, since $[X,X]_t - t$ is a local martingale. The initial value X_0 plays no role in (SE); it would be possible to decide that it is zero; for the sake of further convenience we shall take for X_0 an arbitrary constant x_0.

The structure equation is an obvious necessary condition for X to have the predictable representation property, since $[X,X]_t - t$, being a local martingale, must be an integral with respect to X. The similar necessary condition for an L^4-martingale $(X_t)_{0\le t\le 1}$ to have the chaotic representation property, would be

$$\text{(SE')} \quad [X,X]_1 = 1 + \sum_{n\ge 1} \int_{0<t_1<\ldots<t_n<1} f_n(t_1,\ldots t_n)\,dX_{t_1}\ldots dX_{t_n}$$

for some functions $f_n \in L^2(C_n)$ with $\sum_n \|f_n\|^2 < \infty$; in such an equation, the functions f_n can be considered as the data and the martingale X as the unknown. We shall also deal with such equations in the sequel; but since they can be considered as a particular case of (SE), it will be convenient to use the latter when stating a few general properties.

The two basic examples of structural equations are obtained by taking a constant process Φ in (SE).

(a) Brownian case: $\Phi \equiv 0$. The equation is $[X,X]_t = t$, and shows that $[X,X]$ is continuous, hence also X. Being a continuous martingale with quadratic variation t, X is a Brownian motion.

(b) Poisson case: $\Phi \equiv \alpha \ne 0$. The equation is now

$$[X,X]_t = t + \alpha(X_t - X_0);$$

dividing by α shows that X has finite variation, so it is the compensated sum of its jumps. These jumps verify

$$\Delta X_t^2 = \Delta[X,X]_t = \alpha\Delta X_t,$$

so each jump has amplitude α, and the martingale $Y_t = \frac{1}{\alpha} X_{\alpha^2 t}$, a compensated sum of unit jumps with $\langle Y,Y\rangle_t = t$, must be a compensated Poisson process $N_t - t$. So the solution to the structure equation is

$$X_t = X_0 + \alpha(N_{t/\alpha^2} - \frac{t}{\alpha^2}),$$

where N is a standard Poisson process.

In the latter example, the law of X depends continuously upon α (for the Skorohod topology on $[0,\infty)$); moreover, when α tends to zero, the Brownian case is the limit (in law) of the Poisson one. This makes it possible to see (SE) as some kind of differential equation: Φ being predictable, you may heuristically consider Φ as known a very short instant before X is, and, during this minute time-interval, solve the equation by considering Φ as fixed and using (b), or (a) if $\Phi = 0$.

GENERAL PROPERTIES OF STRUCTURE EQUATIONS.

PROPOSITION 1. Let a martingale X be a solution of (SE). Then

(i) When a jump of X occurs, it is equal to Φ_t; the jump times are totally inaccessible.

(ii) The continuous and purely discontinuous parts of X are given by

$$dX^c = I_{\{\Phi = 0\}} dX; \quad dX^d = I_{\{\Phi \neq 0\}} dX.$$

(iii) If V is a neighbourhood of 0 in $\mathbb{R}$, let

$$Z_t = \int_0^t I_{\{\Phi_s \notin V\}} \frac{dX_s}{\Phi_s};$$

$$A_t = \int_0^t I_{\{\Phi_s \notin V\}} \frac{ds}{\Phi_s^2}.$$

Then $Z_t = N_{A_t} - A_t$ for some Poisson process N.

PROOF. (i) If T is any stopping time, (SE) implies

$$\Delta X_T^2 = \Delta[X,X]_T = \Phi_T \Delta X_T$$

and $\Delta X_T \in \{0,\Phi_T\}$: the jumps of X are equal to Φ.

If T is a bounded predictable stopping time,

$$E[\Delta X_T^2 | F_{T-}] = E[\Delta[X,X]_T | F_{T-}] = \Phi_T E[\Delta X_T | F_{T-}] = 0;$$

hence $\Delta X_T = 0$ and X is quasi-left-continuous.

(ii) Let $C = \int I_{\{\Phi=0\}} dX$ and $D = \int I_{\{\Phi\neq 0\}} dX$. Since

$$[C,C] = \int I_{\{\Phi=0\}} d[X,X] = \int I_{\{\Phi=0\}} dt$$

is continuous, C is a continuous martingale. Since $\int \Phi dX = [X,X] - t$ has finite variation, $\int \Phi dX^c = 0$, hence $\int \Phi^2 \, d\langle X^c,X^c\rangle = 0$, $\int I_{\{\Phi\neq 0\}} \, d\langle X^c,X^c\rangle = 0$ and $D^c = \int I_{\{\Phi\neq 0\}} dX^c = 0$; so D is a purely discontinuous martingale. As $C + D = X - X_0$, $C = X^c$ and $D = X^d$.

(iii) The quadratic variation of Z is

$$d[Z,Z]_t = I_{\{\Phi_t \notin V\}} \frac{d[X,X]_t}{\Phi_t^2} = I_{\{\Phi_t \notin V\}} \left(\frac{dt}{\Phi_t^2} + \frac{dX_t}{\Phi_t}\right) = dA_t + dZ_t \;;$$

hence, if τ is the inverse of A (defined on the interval $[0,A_\infty[$),

$$d[Z,Z]_{\tau_t} = dA_{\tau_t} + dZ_{\tau_t} = dt + dZ_{\tau_t} \;;$$

so, on $[0,A_\infty[$, the martingale Z_{τ_t} is a compensated Poisson process, $N_t - t$. Inverting again, $Z_t = N_{A_t} - A_t$. ∎

PROPOSITION 2. (Change of variable formula). Let M be a $\mathbb{R}^n$-valued local martingale of the form $M = M_0 + \int HdX$, where H is a predictable $\mathbb{R}^n$-valued process and X a martingale verifying (SE). For every $f \in C^2(\mathbb{R}^n,\mathbb{R})$,

$$f(M_t) - f(M_0) = \int_0^t U_s dX_s + \int_0^t V_s ds,$$

with

$$U_s = \sum_i \int_0^1 H_s^i D_i f(M_{s-} + \theta\Phi_s H_s)d\theta = \frac{f(M_{s-} + \Phi_s H_s) - f(M_{s-})}{\Phi_s}$$

and

$$V_s = \sum_{ij} \int_0^1 H_s^i H_s^j D_{ij} f(M_{s-} + \theta\Phi_s H_s)(1-\theta)d\theta = \frac{f(M_{s-} + \Phi_s H_s) - f(M_{s-}) - \Phi_s H_s \nabla f(M_{s-})}{\Phi_s^2}$$

In this formula, the integrals $\int UdX$ and $\int Vds$ are formal semimartingales. If the semimartingale $f \circ M$ is special, then U is integrable with respect to dX, V to dt, and the formula holds in the ordinary sense.

[Recall Schwartz' definition of formal semimartingales [10]: this simply means that, if K is any strictly positive predictable process, small enough for all the following integrals to exist, for instance $K = (1 + U^2 + V^2)^{-1}$, then

$$\int Kd(f \circ M) = \int (KU)dX + \int (KV)ds.$$

The reason why such formal integrals occur here is that $f \circ M$ may have very large jumps, too large to be the jumps of some local martingale. But the small jumps of $f \circ M$ can be compensated all right, and a formula involving only semimartingales would require cutting off the large jumps and dealing with them separately.]

Only the first given expressions for U and V are rigorous. The second ones, involving Φ in the denominator, are valid only for $\Phi_s \neq 0$; when $\Phi_s = 0$, they should be replaced by their limits $U_s = H_s \nabla f(M_{s-})$ and

$$V_s = \frac{1}{2} \text{Hess } f(M_{s-})(H_s, H_s).$$

PROOF. First, the formula holds if f is a constant $(U = V = 0)$ or a linear form $(U = \nabla f \cdot H, V = 0)$.

Second, it holds for f a polynomial. Indeed, it suffices to check it for fg, knowing that it holds for f and g. With the obvious notations U^f, V^f, etc..., one has

$$d(fg \circ M) = (f \circ M_-)(U^g dX + V^g dt) + (g \circ M_-)(U^f dX + V^f dt) + U^f U^g d[X,X]$$

(the computation rules for formal semimartingales are naturally the usual ones). Replacing $d[X,X]$ by $dt + \Phi dX$, we have to verify that

$$U^{fg} = (f \circ M)_- U^g + (g \circ M)_- U^f + \Phi U^f U^g$$

$$V^{fg} = (f \circ M)_- V^g + (g \circ M)_- V^f + U^f U^g .$$

When $\Phi_s = 0$, these formulae reduce to the identities

$$\nabla(fg) = f\nabla g + g\nabla f$$

$$\text{Hess}(fg) = f\text{Hess}g + g\text{Hess}f + \nabla f \otimes \nabla g + \nabla g \otimes \nabla f;$$

and when $\Phi_s \neq 0$, to the identities

$$\frac{f(b)g(b) - f(a)g(a)}{\phi} = f(a)\frac{g(b) - g(a)}{\phi} + g(a)\frac{f(b) - f(a)}{\phi} + \phi\frac{f(b) - f(a)}{\phi}\,\frac{g(b) - g(a)}{\phi}$$

$$\frac{f(b)g(b) - f(a)g(a) - h\nabla(fg)(a)}{\phi^2} = f(a)\frac{g(b) - g(a) - h\nabla g(a)}{\phi^2}$$

$$+ g(a)\frac{f(b) - f(a) - h\nabla f(a)}{\phi^2} + \frac{f(b) - f(a)}{\phi}\,\frac{g(b) - g(a)}{\phi} .$$

Third, to prove the formula for $f \in C^2(\mathbb{R}^n, \mathbb{R})$, it is possible to approximate it by a sequence of polynomials f_p such that $f_p - f$, $\nabla(f_p - f)$ and $\text{Hess}(f_p - f)$ tend to zero uniformly on compacts. Let K be a strictly positive, predictable process such that K, KU^f and KV^f are bounded. On the predictable set

$$A_q = \{t : ||M_{t-}|| + ||M_{t-} + H_t\Phi_t|| \leq q\},$$

U^{f_p} and V^{f_p} converge uniformly to U^f and V^f. So $I_{A_q} KU^{f_p}$ and $I_{A_q} KV^{f_p}$ are bounded, and, in the equality

$$\int I_{A_q} Kd(f_p \circ M) = \int I_{A_q} KU^{f_p} dX + \int I_{A_q} KV^{f_p} ds,$$

the integrals are true semimartingales. The index p can be dropped by taking limits (the left-hand side converges since f_p approximates f in the C^2-topology); then one gets rid of I_{A_q} by letting $q \to \infty$ and using the boundedness of K, KU^f and KV^f. So the formula is true for f.

Last, if $f \circ M$ is a special semimartingale, it can be decomposed into a local martingale N and a predictable process A with finite variation. Since, for every bounded predictable K, $\int KdN + \int KdA$ is the decomposition of $\int Kd(f \circ M)$, making K small enough gives $\int KUdX = \int KdN$ and $\int KVds = \int KdA$, hence $N = \int UdX$ and $A = \int Vds$. So these integrals exist in the usual sense. ∎

REMARK. A more general formula, that we will not use, can be shown: for a function $f(M_t,t)$ (or $f(M_t,A_t)$ with $A = A_0 + \int G_s ds$, G predictable and R^m-valued) a third term must be added, namely $\int \frac{\partial f}{\partial t}(M_{s-},s)\, ds$ (or $\int G_s.\nabla_A f(M_{s-},A_s) ds$).

PROPOSITION 3. (i) <u>On</u> $(\Omega,F,P,(F_t)_{t\geq 0})$, <u>let</u> Φ <u>be a predictable process</u>. <u>A solution</u> X <u>of</u> (SE) <u>has the predictable representation property iff</u> P <u>is an extreme point of the set of all probabilities on</u> F <u>for which</u> X <u>is a martingale and verifies</u> (SE).

(ii) <u>If</u> Φ <u>is a functional of the form</u>

$$\Phi_t = \Psi(t,(X_s, 0 \leq s < t))$$

<u>and if uniqueness in law holds for</u> (SE) <u>with initial condition</u> x_0, <u>every solution</u> X <u>has the predictable representation property with respect to its natural filtration</u>.

PROOF. (i) Owing to the result of Jacod-Yor [4], we just have to verify that P is extremal in this set S iff it is extremal in the larger set M of all probabilities such that X is a martingale. The "if" part is obvious. For the necessary condition, suppose $P = \lambda Q_1 + (1-\lambda)Q_2$ with Q_1 and Q_2 in M and $0 < \lambda < 1$. Both Q_1 and Q_2 are absolutely continuous with respect to P, so $[X,X]$ and $\int \Phi dX$, when computed with P, are also valid for Q_1 and Q_2. Since (SE) holds for P, it also holds for Q_1 and Q_2, and they belong to S. So if P is not extremal in M, it is not in S either.

(ii) Taking for Ω the canonical space of càdlàg paths and for X the canonical process, (i) shows that, among the solutions of (SE), the predictable representation property is possessed by those with an extremal law. In particular, if uniqueness holds, the set S has only one point, and extremality is trivial. ∎

(c) THE CASE WITH INDEPENDENT INCREMENTS. An important generalization of (a) and (b) is obtained when the process Φ in (SE) is a deterministic function of time.

PROPOSITION 4.1) *Let* ϕ *be a Borel function on* $[0,\infty)$; *consider the structure equation*

$$d[X,X]_t = dt + \phi(t)dX_t, \quad X_0 = x_0.$$

(i) EXISTENCE. *If* B *is a Brownian motion and* P *an independent Poisson point process on* $[0,\infty)$ *with intensity* $I_{\{\phi(t)\neq 0\}} \dfrac{dt}{\phi^2(t)}$, *the martingale*

$$X_t = x_0 + \int_0^t I_{\{\phi(s)=0\}} dB_s + M_t$$

is a solution, where M *is the purely discontinuous martingale with jump times the points of* P *and* $\Delta M_T = \phi(T)$ *if* T *is a jump time*.

(ii) UNIQUENESS. *If, on some* $(\Omega, F, P, (F_t)_{t\geq 0})$, X *is a solution, then, for* $s \geq 0$, *the process* $(X_{s+t} - X_s)_{t\geq 0}$ *is independent of* F_s *and its law depends upon the function* $\phi|_{[s,\infty)}$ *only*.

(iii) CHAOTIC REPRESENTATION PROPERTY. *If* G *is the* σ*-field generated by* X, *the chaoses are total in* $L^2(G)$.

PROOF. (i) Let $A_n = \{t : 2^n \leq |\phi(t)| < 2^{n+1}\} \subset [0,\infty)$; the family $(A_n)_{n\in\mathbb{Z}}$ is a partition of $\{\phi \neq 0\}$. The restriction P_n of P to A_n is locally finite. The process with independent increments

$$M_t^n = \sum_{s\in P_n \cap [0,t]} \phi(s) - \int_0^t I_{A_n}(s) \frac{ds}{\phi(s)}$$

has mean zero, so it is a martingale. It verifies

$$[M^n, M^n]_t = \sum_{s\in P_n\cap[0,t]} \phi^2(s) = \int_0^t I_{A_n}(s)ds + \int_0^t \phi(s)dM_s^n$$

and

$$E[[M^n,M^n]_t] = \int_0^t I_{A_n}(s)ds.$$

For fixed $\tau > 0$, each $(M_t^n)_{0\le t\le\tau}$ is a square integrable martingale, with norm $||M_t^n||_2^2 = \int_0^t I_{A_n}(s)ds$. Depending upon P_n only, the M^n are mutually independent, hence orthogonal in L^2. Since $\sum_n ||M_t^n||_2^2 < \infty$, they add up to a square integrable martingale $(M_t)_{0\le t\le\tau}$; letting $\tau \to \infty$, we get a martingale M verifying

$$d[M,M]_t = I_A(t)\ (dt + \phi(t)dM_t)$$

with $A = \bigcup_{n\in Z} A_n = \{\phi \neq 0\}$. Now, adding the independent martingale $N_t = \int_0^t I_{A^c}(s)dB_s$ yields a solution to the structure equation.

(ii) Let X be a solution. If u is a bounded real Borel function with compact support on $[0,\infty)$, since the functions $\dfrac{e^{ix}-1}{x}$ and $\dfrac{e^{ix}-1-ix}{x^2}$ are bounded for x real, the functions

$$h(t) = \begin{cases} \dfrac{e^{iu(t)\phi(t)}-1}{\phi(t)} & \text{if } \phi(t) \neq 0 \\ iu(t) & \text{if } \phi(t) = 0 \end{cases}$$

$$k(t) = \begin{cases} \dfrac{e^{iu(t)\phi(t)}-1-iu(t)\phi(t)}{\phi^2(t)} & \text{if } \phi(t) \neq 0 \\ -\frac{1}{2}u^2(t) & \text{if } \phi(t) = 0 \end{cases}$$

are also bounded and compactly supported. The change of variable formula (Proposition 2) applied to $Y_t = \exp[i\int_0^t u(s)dX_s]$ shows that

$$Y_t = 1 + \int_0^t Y_{s-}(h(s)dX_s + k(s)ds);$$

hence the bounded process $Z_t = \exp[i\int_0^t u(s)dX_s - \int_0^t k(s)ds]$ verifies the Doléans exponential equation

$$Z_t = 1 + \int_0^t Z_{s-}h(s)dX_s .$$

So Z is a martingale, and $E[Z_\infty Z_s^{-1}|F_s] = 1$:

$$E[\exp\ i\int_s^\infty u(t)dX_t|F_s] = \exp\ [-\int_s^\infty k(t)dt].$$

Taking for u a linear combination $\sum_{j=1}^{n} \alpha_j I_{[0,t_j]}$ gives the conditional characteristic function of the process $(X_{t+s} - X_s)_{t\geq 0}$ given F_s. As it is deterministic, this process is independent of F_s; taking $s = 0$ gives the law of $X - x_0$, whence uniqueness in law.

(iii) The notations are the same as in (ii). For $0 \leq t \leq \infty$, the random variables

$$Z'_t = 1 + \sum_{n\geq 1} \int_{0<t_1<\ldots<t_n<t} h(t_1)\ldots h(t_n)\,dX_{t_1}\ldots dX_{t_n}$$

are well defined in L^2 because

$$\int_{0<t_1<\ldots<t_n<t} h^2(t_1)\ldots h^2(t_n)\,dt_1\ldots dt_n = \frac{1}{n!}\,||hI_{[0,t]}||_2^{2n}$$

is the general term of a summable series; moreover $Z'_t = E[Z'_\infty|F_t]$ and Z' is a martingale. Taking the right-continuous version shows that Z' solves the same Doléans equation

$$Z'_t = 1 + \int_0^t Z'_{s-}h(s)\,dX_s$$

as Z; so $Z' = Z$. Hence the random variable

$$Z_\infty = \exp[i\int_0^\infty u(t)\,dX_t - \int_0^\infty k(t)\,dt]$$

is chaos-decomposable, and so is also its multiple $\exp[i \int_0^\infty u(t)\,dX_t]$: the chaotic representation property will be established if we prove that <u>the random variables</u> $\exp[i \sum_{j=1}^{n} \alpha_j X_{t_j}]$ <u>are total in</u> $L^2(G)$.

This is a classical consequence of the Fourier transform being injective: Take a $U \in L^2(F)$ orthogonal to every $\exp[i \sum_{j=1}^{n} \alpha_j X_{t_j}]$. The Fourier transform of the finite measure μ on $\mathbb{R}^n$ defined, for f Borel and bounded, by

$$\int f\,d\mu = E[U\, f(X_{t_1},\ldots,X_{t_n})]$$

is $\int e^{i\alpha.y}\mu(dy) = E[U \exp[i\Sigma\alpha_j X_{t_j}]] = 0$, so $\mu = 0$ and U is orthogonal to every $f(X_{t_1},\ldots,X_{t_n})$, whence to all $L^2(G)$. ∎

(d) THE "MARKOV" CASE.

In discrete time, it is clear that the solution of a structure equation is a homogeneous Markov process iff the predictable process Φ_n is a function of X_{n-1} only. This is why the special instance when the structure equation has the form

$$d[X,X]_t = dt + f(X_{t-})dX_t$$

will be referred to as the Markov case. But we shall see that uniqueness does not always hold, allowing non-Markovian solutions (whence the quotation marks above).

It is possible to depict heuristically the solutions: when $f(X_{t-}) = 0$, X behaves as a Brownian motion, when $f(X_{t-}) \neq 0$, X jumps with intensity $\frac{dt}{f^2(X_{t-})}$ and amplitude $f(X_{t-})$, and, between jumps, drifts with speed $-1/f(X_{t-})$ (as $[X,X]$ is constant between jumps, X must obey the ordinary differential equation $dt + f(x)dx = 0$). So pathological behaviours, or phenomena such as non-uniqueness will be observed for X if they already occur in this deterministic equation.

Taking $M = X$ in Proposition 2 shows that, intuitively, the generator L of the Markov process X should be, for a C^2 function g,

$$Lg(x) = \int_0^1 g''(x + \theta f(x))\,(1-\theta)d\theta = \begin{cases} \dfrac{g(x+f(x)) - g(x) - f(x)g'(x)}{f^2(x)} & \text{if} \quad f(x) \neq 0 \\ \frac{1}{2}\, g''(x) & \text{if} \quad f(x) = 0. \end{cases}$$

The following existence result is due to Meyer [7].

PROPOSITION 5. <u>If</u> f <u>is a continuous function on the real line</u>, <u>for every</u> $x \in \mathbb{R}$ <u>the structure equation</u>

$$d[X,X]_t = dt + f(X_{t-})dX_t$$

<u>has a solution with</u> $X_0 = x$, <u>defined on some</u> $(\Omega, F, P, (F_t)_{t\geq 0})$.

[Meyer's proof [7] consists in discretizing time, and solving step by step the corresponding equation. By Rebolledo's criterion [9], the set of laws of martingales X verifying $\langle X,X\rangle_t = t$ and $X_0 = x$ is tight, so passing to the limit when the mesh size tends to zero is possible using the Skorohod theorem or a nonstandard hyperfinite setting.]

When f is not continuous, Markovian structure equations may have zero, one, or infinitely many solutions. Consider for instance the equation

$$d[X,X]_t = dt - \mathrm{sign}(X_{t-})dX_t,$$

where $\mathrm{sign}(0) = 0$. If the initial value X_0 is not zero, there is a unique solution: between the jumps, X drifts away from zero with speed 1; the jumps are given by a Poisson process with intensity 1 and have amplitude $-\text{sign}(X_{t-})$: X jumps towards zero and may overpass it, but not hit it (the jumps occur at totally inaccessible times and the set $\{t : |X_{t-}| = 1\}$ is a countable union of graphs of predictable times). But if $X_0 = 0$, it is possible to choose the initial value of $\frac{dX}{dt}$ as 1, -1, or any random choice between these: uniqueness does not hold.

Now look at equation $d[X,X]_t = dt + \text{sign}(X_{t-})dX_t$. Whatever the sign of X_0, $X \text{sign} X_0$ behaves as a compensated Poisson process until it hits zero. Then it may neither leave zero (when $X_- = 0$, $\Delta X = 0$; and even if it succeeded in reaching a small non-zero value, the drift would immediately bring it back to zero) nor stay there (for on $\{X = 0\}$, $d[X,X]_t = dt$ and X must be a Brownian motion). So this equation has no solution, mainly because the deterministic equation $0 = dt + \text{sign}(x)dx$ has no solution $x(t)$ starting from 0.

(e) THE AZÉMA MARTINGALES.

The rest of this article is devoted to studying structure equations of the form

$$d[X,X]_t = dt + (\alpha + \beta X_{t-})dX_t,$$

where α and β are two constants. This is of course a sub-case of the Markov case (with f affine), but also a particular instance of (SE'), with $f_1 = \alpha$, $f_2 = \beta$ and $f_n = 0$ for $n \geq 3$.

When $\beta = 0$, the right-hand side is $dt + \alpha dX_t$, and we are back to the Poisson case (b) if $\alpha \neq 0$, and to the Brownian one (a) if $\alpha = 0$ too. So we may suppose $\beta \neq 0$; and now X is a solution of this equation (with initial condition x_0) iff $X + \frac{\alpha}{\beta}$ is a solution to

$$(*) \qquad d[X,X]_t = dt + \beta X_{t-}dX_t$$

(with initial value $x_0 + \frac{\alpha}{\beta}$). So, at the cost of changing the initial condition, we do not lose any generality by assuming that $\alpha = 0$.

For $\beta = 0$, equation (*) is just the Brownian case; but for arbitrary β, as observed by Parthasarathy, it has the same scaling property as Brownian motion: if X_t is a solution to (*), then so is also $\frac{1}{\lambda} X_{\lambda^2 t}$ for every $\lambda \neq 0$; in particular, if uniqueness holds (we shall establish it for $\beta \le 0$), the solution X_t starting from 0 and $\frac{1}{\lambda}X_{\lambda^2 t}$ have the same law. Remark that this scaling property remains true (for $\lambda > 0$ only) for the more general Markov equation

$$d[X,X]_t = dt + (\beta^+ X^+_{t-} + \beta^- X^-_{t-})dX_t$$

with two coefficients β^+ and β^-.

Besides Brownian motion, obtained for $\beta = 0$, two interesting processes can be found among the solutions of (*).

First, the process X such that $X_t^2 = t$ and

$$P[X_t = \sqrt{t}] = P[X_t = -\sqrt{t}] = \frac{1}{2},$$

with jumps (that is, changes of sign) occurring according to a Poisson point process with intensity $dt/4t$. This process is a martingale since, for $0 < s < t$,

$$X_t X_s^{-1} = \sqrt{t/s}\ (-1)^N$$

where N is independent of F_s and has a Poisson law with parameter $\frac{1}{4}\operatorname{Log}\frac{t}{s}$. It verifies (*) with $\beta = -2$ since

$$[X,X]_t = X_t^2 - 2\int_0^t X_{s-}dX_s = t - 2\int_0^t X_{s-}dX_s.$$

Using Proposition 1, one can show that this solution is the only one (another proof of uniqueness will be given below). See Parthasarthy [8] for a decomposition of Fermionic Brownian motion as the product of this X and another commutative process, with values ± 1, that does not commute with X.

Second, the martingale obtained by Azéma ([1] p.464; see also Azéma-Yor [0]) when projecting a Brownian motion B starting from 0 on the filtration of sign (B): if $G_t = \sup\{s : s \le t, B_s = 0\}$, let

$$X_t = (\operatorname{sign} B_t)\ \sqrt{2(t - G_t)}$$

(the above mentioned projection gives a multiple of this X; the constant $\sqrt{2}$ featured here is chosen so that $\langle X,X\rangle_t = t$). Assuming that X is a martingale (this can also be shown directly, without using Azéma's projection), we shall show that it verifies (*), with $\beta = -1$. First, $\int I_{\{|X_-|>\epsilon\}}dX$ clearly has finite variation for every $\epsilon > 0$; so $X^c = \int I_{\{X_-=0\}}dX^c = 0$ (the latter equality holds for all semimartingales), and $[X,X]_t = \sum_{0<s\le t} \Delta X_s^2$. But ΔX_s^2 is zero unless s is the end of some excursion of B, in which case ΔX_s^2 is twice the length of that excursion; so $[X,X]_t = 2G_t$. And

$$\int_0^t X_- dX = \frac{1}{2}X_t^2 - \frac{1}{2}\,[X,X]_t = (t - G_t) - G_t = t - [X,X]_t$$

establishes the claim.

We know by Proposition 5 that solutions to (*) can be found for every value of β. These processes will be called the Azéma martingales; formally[1] they should be Markov, with generator

$$Lg(x) = \begin{cases} \dfrac{g(x+\beta x) - g(x) - \beta x g'(x)}{\beta^2 x^2} & \text{if } \beta x \neq 0 \\ \dfrac{1}{2}\, g''(x) & \text{if } \beta x = 0. \end{cases}$$

An amusing consequence of this formula is that, when $e^{\beta} + \beta + 1 = 0$ (this happens for some β between -2 and -1) and $x_0 \neq 0$, not only the process X (we shall see that is is unique) but also $L = \mathrm{Log}|X|$ is a martingale -- and X is the stochastic exponential of L. This is easy to verify rigorously by Proposition 2, using Proposition 1 to see that the first time when X or X_- hits zero, having the same law as $\int_0^\infty \exp[-2\beta(N_t - t)]dt$, must be infinite.

PROPOSITION 6. Consider the structure equation

$$d[X,X]_t = dt + \beta X_{t-} dX_t, \quad X_0 = x_0.$$

(i) If $\beta \le 0$, the solution is unique in law and is a strong Markov process.

(ii) If $-2 \le \beta \le 0$, the solution has the chaotic representation property.

LEMMA 7. Let X be a solution to the structure equation

$$d[X,X]_t = dt + f(t)dX_t + \left(\int_0^{t-} g(s,t)dX_s\right)dX_t$$

with $X_0 = x_0$, where f and g are locally bounded, Borel functions. Every random variable of the form $Z = Q(X_{t_1},\dots,X_{t_n})$ with Q a polynomial is in L^2

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[1] This means: informally!

<u>and chaos-representable</u>. <u>More precisely, if</u> Q <u>has degree</u> k, Z <u>is in</u> $\bigoplus_{j\le k} \chi_j$.

Naturally, in this structure equation, one has to take a predictable version of the integral $\int_0^{t^-} \ldots$; so the equation actually means

$$[X,X]_t = t + \int_0^t f(s)\,dX_s + \int_{0<r<s<t} g(r,s)\,dX_r dX_s$$

and is of the type (SE').

PROOF OF LEMMA 7. As the conclusion involves only finitely many instants t_i, we shall restrict ourselves to some fixed compact $[0,T]$.

Define $\chi_n^T(a)$ as the set of all elements of χ_n of the form

$$\int_{0<t_1<\ldots<t_n<T} \phi(t_1,\ldots t_n)\,dX_{t_1}\ldots dX_{t_n}$$

for some ϕ with $||\phi||_\infty \le a$.

The conclusion clearly holds for $k \le 1$; to establish it for all k, it suffices to prove the following claim, where $k \ge 1$: <u>if</u> $M = \int_0^T u_t dX_t$ <u>for some Borel, bounded function</u> u, <u>and if</u> Z <u>is in</u> $\chi_k^T(a)$, <u>then</u> ZM <u>is in</u> $\bigoplus_{j=0}^{k+1} \chi_j^T(c_k a||u||_\infty)$, <u>where</u> c_k <u>depends on</u> f, g, T <u>but not</u> M <u>nor</u> Z.

The important point is that this implies by induction on n that every product of the form $X_{t_1}\ldots X_{t_n}$ is chaos-representable. The estimate involving a and u is not particularly interesting, but it will be useful when proving the claim (the reason why the proof may be easier with a stronger conclusion is that it goes by induction).

To prove the claim, suppose that it holds when k is replaced by any smaller value, and write

$$Z = \int_{0<t_1<\ldots<t_{k-1}<t_k} \phi(t_1,\ldots,t_k)\,dX_{t_1}\ldots dX_{t_k}$$

with $|\phi| \le a$; so the martingale $Z_t = E[Z|F_t]$ can be written $\int_0^t H_s dX_s$, with

$$H_t = \int_{0<t_1<\ldots<t_{k-1}<t} \phi(t_1,\ldots,t_{k-1},t)\,dX_{t_1}\ldots dX_{t_{k-1}}$$

(if $k = 1$, $Z = \int_0^T \phi(t)\,dX_t$ and $H_t = \phi(t)$). Similarly, the martingale $M_t = E[M|F_t]$ is $\int_0^t u_s dX_s$. The integration by parts formula gives

$$ZM = \int_0^T Z_{t-}dM_t + \int_0^T M_{t-}dZ_t + [Z,M]_T$$
$$= \int_0^T (Z_{t-}u_t + M_{t-}H_t + u_tH_t\Phi_t)dX_t + \int_0^T u_tH_tdt$$

(the given structure equation has been abbreviated as $d[X,X]_t = dt + \Phi_t dX_t$). The conclusion will be checked separately for each of those four integrals. Clearly,

$$\int_0^T u_tH_tdt = \int_{0<t_1<\ldots<t_{k-1}} [\int_{t_{k-1}}^T \phi(t_1,\ldots,t_{k-1},t)u(t)dt]dX_{t_1}\ldots dX_{t_{k-1}}$$

is in $\chi_{k-1}^T(Ta||u||_\infty)$, and

$$\int_0^T Z_{t-}u_tdX_t = \int_{0<t_1<\ldots<t_k<t} \phi(t_1,\ldots,t_k)u(t)dX_{t_1}\ldots dX_{t_k}dX_t$$

in $\chi_{k+1}^T(a||u||_\infty)$; breaking Φ_t into $f(t) + \int_0^{t^-} g(s,t)dX_s$, the first part of $\int_0^T u_tH_t\Phi_tdX_t$,

$$\int_0^T u_tf(t)H_tdX_t = \int_{0<t_1<\ldots<t_{k-1}<t} \phi(t_1,\ldots,t_{k-1},t)u_tf(t)dX_{t_1}\ldots dX_{t_{k-1}}dX_t,$$

is in $\chi_k^T(||f||_\infty a||u||_\infty)$.

Both remaining terms will be dealt with using the induction hypothesis. Since, for every t, H_t is in $\chi_{k-1}^T(a)$, the product $M_{t-}H_t = H_t\int_0^T I_{[0,t)}(s)u_sdX_s$ is in $\bigoplus_{j\le k} \chi_j^T(c_{k-1}a||u||_\infty)$; hence the integral $\int_0^T M_{t-}H_tdX_t$ is in $\bigoplus_{1\le j\le k+1} \chi_j^T(c_{k-1}a||u||_\infty)$. Similarly, the product $H_t\int_0^{t^-} g(s,t)dX_s$ is in $\bigoplus_{j\le k} \chi_j^T(c_{k-1}a||g||_\infty)$, and the last term $\int_0^T u_tH_t\int_0^{t^-} g(s,t)dX_s\, dX_t$ is in $\bigoplus_{1\le j\le k+1} \chi_j^T(c_{k-1}||g||_\infty a||u||_\infty)$. So the result holds, with $c_k = 1 + T + ||f||_\infty + c_{k-1}(1 + ||g||_\infty)$. ∎

PROOF OF PROPOSITION 6. (ii) We know by lemma 7 that for an Azéma martingale X, each polynomial $Q(X_{t_1},\ldots,X_{t_n})$ is chaos representable. The chaotic representation property follows if these polynomials are dense in $L^2(G)$ (where G denotes the σ-field generated by X). For $-2 \le \beta < 0$, this is true simply because X is bounded on each compact interval $[0,T]$. Indeed, from the structure equation

$$d[X,X]_t = dt + \beta X_{t-}dX_t$$

and the integration by parts formula

$$d(X_t^2) = 2X_{t-}dX_t + d[X,X]_t,$$

one deduces easily

$$(\beta + 2)d[X,X]_t + (-\beta)d(X_t^2) = 2dt,$$

whence $X_t^2 \le x_0^2 + \frac{2}{-\beta}t$.

(i) The proof of Lemma 7 is constructive: at each step, the expansion of ZM as a sum of iterated integrals is obtained in terms of u,f,g and the coefficients in the expansion of Z. So the chaotic expansion of $Q(X_{t_1},\dots,X_{t_n})$ involves $Q,t_1,\dots,t_n,f$ and g only. This applies in particular to the first term $E[Q(t_1,\dots,t_n)]$. So, for the structure equation of Lemma 7, uniqueness in law holds as soon as X is bounded on compacts, for in this case approximating the function $\exp[i(\alpha_1 x_1 + \dots + \alpha_n x_n)]$ uniformly on compacts by polynomials gives the characteristic function of $(X_{t_1},\dots,X_{t_n})$. As we have seen, such a boundedness holds for $-2 \le \beta < 0$, whence the uniqueness in that case. The strong Markov property follows by using the uniqueness of the conditional law of $(X_{T+t})_{t\ge 0}$ given F_T, for a stopping time T.

Now for $\beta < -2$. In that case, the above equality

$$(-\beta)d(X_t^2) = 2dt + (-\beta - 2)d[X,X]_t$$

gives $X_t^2 \ge x_0^2 + \frac{2}{-\beta}t$. We shall first study the case when $x_0 \ne 0$. The preceding equation shows that X is bounded away from zero. Proposition 1 applies, and, for some standard Poisson process N,

$$\begin{cases} X_t = x_0(1+\beta)^{N_{A_t}} e^{-\beta A_t} \\ A_t = \int_0^t \frac{ds}{\beta^2 X_s^2} . \end{cases}$$

Considering N as given, this system has a unique solution, that can be constructed pathwise from N: replacing X_s^2 in the second equation by its value from the first one yields an equation of the form $dA_t = C\exp(2\beta A_t)dt$ between the jump times. So the law of $(X_t,A_t)_{t\ge 0}$ must be the image of the law of N by this deterministic operation on paths, whence uniqueness.

Observe that the change of time A transforming the filtration of N into that of X is continuous, strictly increasing and a.s. unbounded (else, X would have a limit X_∞, contradicting $X_t^2 \geq x_0^2 + \frac{2}{-\beta}t$), realizing an isomorphism of filtered probability spaces. If T is a stopping time for X, A_T is one for N, and the conditional law on F_T of the process

$$X_{T+t} = X_T(1+\beta)^{N_{A_{T+t}} - N_{A_t}} e^{-\beta(A_{T+t}-A_t)}$$

is the law of X, with initial condition X_0 replaced by X_T; so X has the strong Markov property.

When $x_0 = 0$, uniqueness is a little more difficult. The estimate $X_t^2 \geq \frac{2}{-\beta}t$ shows that X is bounded away from zero on every interval $[T_\epsilon,\infty)$, where $T_\epsilon = \inf\{t:|X_t| \geq \epsilon\}$. On this interval, the above applies, and the law of $(X_{T_\epsilon+t})_{t\geq 0}$ can be obtained by replacing x_0 by X_{T_ϵ} in the preceding formulae (with X_{T_ϵ} and N independent). As $|X_{T_\epsilon}| \leq \epsilon|1+\beta|$ we just have to show that <u>the law of</u> X <u>starting from</u> $x_0 \neq 0$ <u>has a limit when</u> $x_0 \to 0$. Repeating the same argument, we just have to show that for each $\epsilon > 0$ the law of X_{T_ϵ} has a limit when the initial condition x_0 tends to zero (for then the law of $(X_{T_\epsilon+t})_{t\geq 0}$ will be determined, and, since $T_\epsilon \leq -\frac{1}{2}\beta(1+\beta)^2\epsilon^2$, the law of X will necessarily be its limit for $\epsilon \to 0$).

By scaling, X is a solution of our equation with initial condition x_0 iff $\frac{1}{\epsilon}X_{\epsilon^2 t}$ is a solution starting from x_0/ϵ. So it suffices to check it for $\epsilon = 1$: we have to show that <u>the law of</u> X_{T_1} <u>has a limit when</u> $x_0 \to 0$.

Using the time change A, this amount to saying that the first value exceeding 1 or -1 of $x_0(1+\beta)^{N_t}e^{-\beta t}$ has a limit law. Taking logarithms, letting $a = \frac{-\beta}{\log|1+\beta|} > 1$, $x = -\frac{\log|x_0|}{\log|1+\beta|}$ and $T = \inf\{t:N_t + at \geq x\}$, we have to check that the law of $((-1)^{N_T}, N_T + aT - x)$, carried by $\{-1,1\} \times [0,1)$, has a limit when $x \to +\infty$, invariant by the symmetry exchanging -1 and 1. This law can be expressed in terms of the two functions

$$u(x) = P[N_T \text{ even},\ N_T + aT = x]$$

$$v(x) = P[N_T \text{ odd},\ N_T + aT = x]$$

Indeed, for $x \geq 1$, $0 \leq y < y + dy \leq 1$ and dy infinitesimal,

$$\begin{aligned} P[N_T \text{ even}, \ x + y < N_T + aT \leq x + y + dy] &= \sum_{n \text{ even}} P[N_T = n, \frac{x + y - n}{a} < T \leq \frac{x + y + dy - n}{a}] \\ &= \sum_{n \text{ even}} P[N_{\frac{x + y - n}{a}} = n - 1, N_{\frac{x + y + dy - n}{a}} = n] \\ &= \sum_{n \text{ even}} P[N_{\frac{x + y - n}{a}} = n - 1] \frac{dy}{a} \\ &= \sum_{n \text{ odd}} P[N_{\frac{x + y - n - 1}{a}} = n] \frac{dy}{a} \\ &= v(x + y - 1) \frac{dy}{a} \end{aligned}$$

(the probability that more than one jump of N occurs during an interval $\frac{dy}{a}$ has been neglected as an infinitesimal of higher order).

So on $\{1\} \times [0,1)$, the probability is $u(x)\epsilon_0(dy) + \frac{1}{a} v(x + y - 1)dy$; similarly, on $\{-1\} \times [0,1)$, the probability is $v(x)\epsilon_0(dy) + \frac{1}{a} u(x + y + 1)dy$. All we have to do is prove that, when $x \to \infty$, $u(x)$ and $v(x)$ both have the same limit α; and the limit law will be

$$(\epsilon_1 + \epsilon_{-1}) \otimes \alpha[\epsilon_0(dy) + \frac{1}{a} I_{[0,1]}(y)dy].$$

This could be done by using an explicit expression of $u(x)$ and $v(x)$: they are respectively the contribution of the even and odd terms in the series

$$\sum_{n \geq 0} \frac{1}{n!} I_{\{x \geq n\}} e^{-\frac{x - n}{a}} \left(\frac{x - n}{a}\right)^n .$$

But it is quicker and simpler to use some probabilistic information on u and v. First, $u(x)$, $v(x)$ and $u(x) + v(x)$ are probabilities, so they are in $[0,1]$. Second, the law of $((-1)^{N_T}, N_T + aT - x)$ has a total mass equal to one, whence, for $x \geq 1$,

$$u(x) + v(x) + \frac{1}{a} \int_{x-1}^{x} [v(t) + u(t)]dt = 1.$$

So $\sigma(x) = u(x) + v(x) - \frac{a}{a + 1}$ verifies $\sigma(x) = - \frac{1}{a} \int_{x-1}^{x} \sigma(t)dt$ for $x > 1$. As $|\sigma(x)| \leq 1$ for $x > 0$, this implies by induction that $|\sigma(x)| \leq a^{-n}$ for $x > n$. Since $a > 1$, this gives $\sigma(x) \to 0$ for $x \to \infty$, and $(u + v)(x) \to \frac{a}{a + 1}$.

Last, using the fact that $(-1)^{N_t} |1 + \beta|^{N_t + at - x}$ is a martingale (up to the time-change A, you have recognized our Azémartingale $x_0(1+\beta)^{N_t}e^{-\beta t})$ bounded on $[0,T]$, its expectation at time T is its starting value:

$$|1+\beta|^{-x} = u(x) - v(x) - \frac{1}{a}\int_0^1 |1+\beta|^y [u(x+y-1) - v(x+y-1)]dy$$
$$= \delta(x) - \frac{1}{a}\int_{x-1}^x |1+\beta|^{t-x+1}\delta(t)dt$$

with $\delta = u - v$. Now the total mass of the measure $\frac{1}{a}|1+\beta|^{t-x-1}dt$ on the interval $[x-1,x]$ is

$$b = \frac{|1+\beta| - 1}{a\mathrm{Log}|1+\beta|} = \frac{-\beta-2}{-\beta} < 1,$$

so $c = \max(b, |1+\beta|^{-1})$ is also less than 1. By induction on $n, |\delta(x)| \le (n+1)c^n$ for $x \ge n$, so $\delta(x) \to 0$ for $x \to \infty$, and $u(x)$ and $v(x)$ have the same limit $\frac{a}{2(a+1)}$. Uniqueness is proved. ∎

REMARK. What about the chaotic representation property when β is not in $[-2,0]$? I do not know.

And uniqueness for $\beta > 0$? I do not know either; but, letting $a = \frac{\beta}{\mathrm{Log}(1+\beta)} > 1$ and $S = \sup_{t\ge 0}(N_t - at)$, it is possible to show that, for $x \ge 0$

$$(1+\beta)^{-x-1} \le P[S > x] \le (1+\beta)^{-x},$$

and that uniqueness holds if and only if $(1+\beta)^x P[S > x]$ has a limit when $x \to \infty$. I have not been able to decide this question, although the law of S can be computed explicitly: it is $\frac{a-1}{a}\sum_{n\ge 0}\mu^{n*}$, where μ is the uniform measure on $[0,1]$ with total mass $\frac{1}{a}$ (this is the (sub-probability) law of the (not everywhere defined) random variable $N_U - aU$, with U the first time when $N_t - at > 0$); alternatively, for $x \ge 0$,

$$P[S \le x] = \frac{a-1}{a}\sum_{n\ge 0}\frac{(-1)^n}{n!} I_{\{x\ge n\}} e^{\frac{x-n}{a}} \left(\frac{x-n}{a}\right)^n$$

(this function of x verifies $f'(x) = \frac{1}{a}[f(x) - f(x-1)]$ for $x > 1$); and the Laplace transform is

$$E[e^{-\lambda S}] = \frac{a-1}{a - \frac{1-e^{-\lambda}}{\lambda}}.$$

Letting $T = \inf\{t : N_t - at > x\}$, uniqueness is also equivalent to the conditional law

$$L[N_T - aT - x | T > \infty]$$

having a limit for $x \to \infty$.

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Note de la rédaction (Azéma)

L'unicité dans le cas $\beta > 0$ (page précédente du présent article) semble avoir été établie par Emery à son insu ; en effet, "l'estimation de Cramer" (Feller t.2, XI 7, p. 364) fournit précisément ce dont on a besoin : $P[S > x]$ est équivalent quand x tend vers l'infini à Ce^{-kx}, avec $K = \text{Log}(1+\beta)$.

Note de la rédaction

La proposition 4 (p. 74) a été établie indépendamment par A. Dermoune.