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Regularity and integrator properties of variation processes of two-parameter martingales with jumps

by

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Abstract

With a view towards a general Ito formula, the main aim of this paper is to study the regularity and stochastic integrator properties of the principal variation processes of the stochastic calculus of two-parameter martingales with jumps. By considering its elementary jump components and continuous part separately, we first show that any $L \log^{+} L$-integrable martingale possesses one-directional quadratic variations, which are right continuous and have left limits in the two-parameter sense. Square integrable martingales are even seen to inherit their precise continuity properties to their quadratic variations. As an application of this, we are able to identify these processes as stochastic integrators in the L^1-sense and describe their natural domains. We finally define and study the "anti-diagonal" martingale component appearing in the stochastic calculus of two-parameter martingales, as another application. It is also shown to precisely inherit the discontinuity properties of the underlying martingale.

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Introduction

The aim of this paper is to study and describe the principal variation processes appearing in the stochastic calculus of two-parameter martingales with jumps.

It is seen in the theory of one-parameter martingales and and in the special case of continuous two-parameter martingales alike (see Nualart [12], [13]) that these processes are already exhibited in the simplest form of the transformation theorem, the Ito formula for the square of a square integrable martingale. For example, if M is a square integrable continuous two-parameter martingale, M^2 is represented by roughly three kinds of processes: two martingales, one of which is given by the usual stochastic integral of M, the other one being a kind of "anti-diagonal" martingale part which will occasionally be called "mixed martingale part"; one process of bounded variation, the quadratic variation [M] of M; finally two processes which show martingale-like behaviour in one direction and are of bounded variation in the other, the "quadratic i-variations" $[M]^i$ of M, which are defined by the quadratic variations of the one-parameter processes $M_{(t_1,.)}$ resp. $M_{(.,t_2)}$ for $t=(t_1,t_2)$ fixed. Of course, if these processes are to be used in a general Ito formula, it is inevitable to consider their stochastic integrals (see Nualart [13]). Abstractly stated, this means that they have to be classified in terms of stochastic integrators in the sense of Bichteler [2]. This is an easy task for the two martingale parts and the bounded variation part (see for example Hürzeler [5] . It is considerably harder for the two processes with "mixed" behaviour. Recently, in [6], the quadratic i-variations of square integrable continuous martingales were shown to be 1-stochastic integrators and their integrands described.

For general square integrable martingales M however, which are "regular", i.e. continuous for approach from the right upper quadrant and possess limits in the remaining three, little is known so far. As will be seen below, the transformation theorem for M^2 still produces the same species of variation processes, i.e. two martingales, the quadratic variation and the two quadratic i-variations. One of the martingales, the stochastic integral of M, has been repeatedly studied and used (see Cairoli, Walsh [3], Merzbach [9]), and [M] has been described in [7]. Of course, both of these processes are stochastic integrators. The remaining three, however, seem to be known very little, in contrast to this. Although $[M]^i$ are known as families of one-parameter quadratic variations, their genuinely two-parameter properties, for example regularity and stochastic integrator properties, had not been studied. This holds a fortiori for the mixed martingale part which, for martingales with jumps, had not yet been seen to exist in general. See, however, Mishura [10],[11], for a class of martingales defined by restrictive conditions.

In this paper we investigate both regularity and stochastic integrator properties of the quadratic i-variations and the mixed martingale part of square and $L \log^+ L$-integrable martingales with jumps. Section 1 is devoted to the regularity and jump classification of $[M]^i$ for a square integrable martingale M. Guided by [7], we decompose M into its elementary compensated jump components and its continuous component, describe the quadratic i-variations of these processes explicitly as regular processes and finally use uniform convergence to extend these regularity results to M. This approach provides us, in addition, with precise information about the kinds of jumps $[M]^i$ may have. As expected, it inherits the discontinuities of M (theorem 1.1). Another uniform approximation extends the regularity result on $[M]^i$ to $L \log^+ L$ (theorem 1.2). On the background of section 1, in section 2 we are able to transfer the methods of [6] to prove that $[M]^i$ is a 1-stochastic integrator, defined on a vector space of $\bar{i}$-previsible processes ($\bar{i}$ is the complementary index of i)(theorem 2.2) for a square integrable M. In section 3 we finally derive the decomposition of M^2 given by the simple form of the transformation theorem. This is employed to study the mixed martingale part via the properties of the stochastic integral of M, the quadratic variation [M] as described in [7] and of $[M]^1$ as derived in section 1 (theorem 3.1). Its regularity properties are described in theorem 3.2. As expected, it inherits discontinuities of the three possible kinds as well as continuity from M.

Since, as mentioned above, the decomposition of M^2 given in theorem 3.1 contains the most important variation processes whose integrals appear in Ito's formula for M, the results of this paper, in particular the stochastic integrator aspect, can be considered a step towards a general transformation theorem for square integrable two-parameter martingales with jumps.

0. Notations, definitions and basics

The stochastic processes considered in this paper are parametrized by the unit square $\mathbb{I} = [0,1]^2$. $\mathbb{I}$ is ordered by "≤", which is understood to be coordinatewise linear ordering on [0,1]. Intervals with respect to this ordering are defined as usual. If J is an interval, we write s^J, t^J for its respective lower and upper corners. By a partition of a parameter interval we always mean a partition generated by a finite number of axial parallel lines (points) consisting of left open, right closed intervals (in the relative topology of $\mathbb{I}$ ([0,1])). A 0-sequence of partitions is a sequence of partitions which is increasing with respect to fineness and the mesh of which goes to 0. To denote components of points (intervals, partitions) in $\mathbb{I}$, we use lower indices. For example, $t = (t_1,t_2)$ ($J = J_1 \times J_2$, $\mathbb{K} = \mathbb{K}_1 \times \mathbb{K}_2$) for $t \in \mathbb{I}$ (an interval J in $\mathbb{I}$, a partition $\mathbb{K}$ of $\mathbb{I}$). We sometimes write $t = (t_i, t_{\overline{i}})$ regardless of whether $i = 1$ or 2, where $\overline{i}$

denotes the complementary index 3-i of i. Given an interval J in $\mathbb{I}$, $J^1 =]s_1^J, t_1^J] \times [0, s_2^J]$ resp. $J^2 = [0, s_1^J] \times]s_2^J, t_2^J]$ is the "1-shadow" resp. "2-shadow" of J. Given a function f: [0,1] → $\mathbb{R}$, the increment of f over an interval J in [0,1] will be written $\Delta_J f$. This also applies to functions f: $\mathbb{I} \to \mathbb{R}$. Here

$$\Delta_J f = f(t^J) - f(s_1^J, t_2^J) - f(t_1^J, s_2^J) + f(s^J).$$

f is called increasing, if $\Delta_J f \geq 0$ for all intervals J, regular, if

$$\lim_{s \uparrow t} f(s) = f(t), \lim_{s \downarrow t} f(s), \lim_{s_1 \uparrow t_1, s_2 \downarrow t_2} f(s), \lim_{s_1 \downarrow t_1, s_2 \uparrow t_2} f(s)$$

exist; for $t \in \mathbb{I}$.

On our basic probability space $(\Omega, \mathfrak{F}, P)$, $\mathfrak{F}$ is assumed to be complete with respect to P. The filtration $\mathbb{F} = (\mathfrak{F}_t)_{t \in \mathbb{I}}$ which is also fixed throughout the paper, is supposed to satisfy some basic assumptions: it is right continuous, i.e. $\mathfrak{F}_t = \bigcap_{s>t} \mathfrak{F}_s$, it is complete, i.e. $\mathfrak{F}_t$ contains all P-zero sets, and, for convenience, $\mathfrak{F}_t$ is trivial whenever $t \in \mathbb{I} \cap \partial \mathbb{R}_+^2$. The most important hypothesis, however, is the "conditional independence" of the filtrations $\mathbb{F}_1 = (\mathfrak{F}_{t_1}^1)_{t_1 \in [0,1]}$ and $\mathbb{F}_2 = (\mathfrak{F}_{t_2}^2)_{t_2 \in [0,1]}$, where $\mathfrak{F}_{t_i}^i = \mathfrak{F}_{(t_i,1)}$, i=1,2. It states that for all $t \in \mathbb{I}$, the σ-algebras $\mathfrak{F}_{t_1}^1$ and $\mathfrak{F}_{t_2}^2$ are conditionally independent given $\mathfrak{F}_t$, and is often referred to as the (F4)-condition of Cairoli, Walsh [3]. Stochastic processes are a priori no more than families of random variables. A stochastic process X on $\Omega \times \mathbb{I}$ defines two families of one-parameter processes: for $t_{\bar{i}} \in [0,1]$, $X_{(.,t_{\bar{i}})}$ is the process $(\omega, t_i) \to X_t(\omega)$, i=1,2. Two processes X and Y are considered as being equal, if the differ on a zero set, as being versions of each other, if $X_t = Y_t$, a.s. for all t. A process X is called increasing (regular), if for all $\omega \in \Omega$ the trajectories $X(\omega,.)$ are increasing (regular). Besides the usual Banach spaces of random variables $L^p(\Omega, \mathfrak{F}, P)$, $p \geq 1$, we will have to deal with the "Orlicz space" L $\log^+$L, i.e. the topological vector space of random variables ξ for which $E(|\xi| \log^+ |\xi|) < \infty$. This space is topologized by the functional

$$\|\xi\|_{L \log^+ L} = \inf \{\lambda > 0: E(|\xi|/\lambda \log^+(|\xi|/\lambda)) \leq 1\}.$$

By far the most important measurability concepts for stochastic processes are evoked by the words "optionality" and "previsibility". There are several notions related to them which are relevant to us. We recall the notations used here, but refer the reader to [7] for their definitions and basic properties. For i=1,2, the i-optional (i-previsible) sets on $\Omega\times\mathbb{I}$ are denoted by $\mathfrak{O}^1$ ($\mathfrak{P}^1$), the optional (previsible) sets by $\mathfrak{O}$ ($\mathfrak{P}$). Due to conditional independence, we have $\mathfrak{O} = \mathfrak{O}^1\cap\mathfrak{O}^2$, $\mathfrak{P} = \mathfrak{P}^1\cap\mathfrak{P}^2$. For i=1,2, the dual i-previsible projection of an integrable increasing process A is denoted by $A^{\pi i}$, its previsible projection by A^{π}. We have $A^{\pi 1 \pi 2} = A^{\pi 2 \pi 1} = A^{\pi}$.

To analyze the jumps of regular processes, the following concepts of "thin" optional sets introduced in [7] will be of interest. A set $T\in\mathfrak{O}$ is called 0-simple, if $\omega \rightarrow |T_\omega|$ is integrable, 1-simple, if T_ω consists of finitely many vertical open line segments whose upper boundary is on $\partial\mathbb{I}$ for $\omega\in\Omega$, the number of which constitutes an integrable random variable, 2-simple, if an analogous statement for horizontal line segments can be made. For $p\geq 1$, a simple set is called p-integrable, if the respective random number of points (lines) is p-integrable. A 0-simple set T is sometimes studied by means of its associated increasing process $\Gamma(T)$, defined by

$$\Gamma(T)_t(\omega) = |T_\omega\cap[0,t]|, \; (\omega,t) \in \Omega\times\mathbb{I}.$$

$\Gamma(T)_t$ just counts the number of points in T up to t. In analogy to the graphs of stopping times in the classical theory, simple sets can be decomposed by simple sets of different "accessibility" degrees. A 0-simple set T is called i-previsible, $\bar{\text{i}}$-inaccessible (totally inaccessible) if for any 0-simple $S\in\mathfrak{P}$ ($S\in\mathfrak{P}^1\cup\mathfrak{P}^2$) the intersection $S\cap T$ is evanescent. Similarly, an i-simple set T is said to be inaccessible if the intersection with any previsible i-simple set is evanescent, i=1,2. A 0-simple set is called pure, if it is given by a (disjoint) union of four 0-simple sets of the four different accessibility degrees. In an analogous way, pure i-simple sets are defined, i=1,2. Theorems on the decomposition of simple sets by inaccessible/previsible simple sets are presented in [7].

The most important class of processes we will have to discuss here are the martingales. An integrable, adapted process M on $\Omega\times I$ is called martingale, if for s,t$\in I$, s$\le$t, we have $E(M_t|\mathfrak{F}_s) = M_s$. Due to conditional independence, M is a martingale iff $M_{(\cdot,t_{\bar{i}})}$ is an $\mathbb{F}_i$-martingale for any $t_{\bar{i}}\in[0,1]$, i=1,2. A martingale M is said to be L $\log^+$L-integrable resp. p-integrable, if $M_1\in L\log^+L$ resp. $L^p(\Omega,\mathfrak{F},P)$, p≥1. Accoding to the regularity theorem of Bakry, Millet and Sucheston (see [7]), any L $\log^+$L-integrable martingale M possesses a version with regular trajectories. For a regular process X, the following three kinds of jumps are well defined and will prove to be relevant. A point $(\omega,t)\in\Omega\times I$ is called 0-jump, if

$$\Delta_t X(\omega) = \lim_{s\uparrow t} \Delta_{]s,t]}X(\omega) \neq 0,$$

i-jump, if

$$\Delta_t X(\omega) = 0 \text{ and } \Delta_{t_i}X_{(\cdot,t_{\bar{i}})} = \lim_{s_i\uparrow t_i} \Delta_{]s_i,t_i]}X_{(\cdot,t_{\bar{i}})} \neq 0,\ i=1,2.$$

Any regular increasing process A can be uniquely decomposed by

$$A = A^o + A^1 + A^2 + A^c,$$

where A^i is its i-jump part, i=0,1,2, A^c its continuous part (see [7], p. 107). It is shown in [7], pp. 120-123, that the set of discontinuities of a regular martingale M is contained in a countable union of simple sets. If, moreover, M is 2-integrable (square integrable), it can be decomposed by three jump parts M^i, i=0,1,2, consisting of the compensated jumps of M of the respective kind and a continuous part M^c (see [7], p. 156). The most general existence theorem for quadratic variation (see [7], p. 161) states that any L $\log^+$L-integrable martingale M possesses a quadratic variation [M]. For any t$\in I$,

$$[M]_t = \lim_{m\to\infty} (\text{in prob}) \sum_{J\in\mathbb{K}_m} (\Delta_{J\cap[0,t]}M)^2$$

along any sequence $(\mathbb{K}_m)_{m\in\mathbb{N}}$ of partitions of I. For more information on [M] see [7]. By $[M]^i_{(\cdot,t_{\bar{i}})}$ we denote the quadratic variation of the one-parameter process $M_{(\cdot,t_{\bar{i}})}$, $t_{\bar{i}}\in[0,1]$. We call the two-parameter process $[M]^i$ quadratic i-variation of M, i=1,2. Occasionally, we will have to consider the i-previsible process $\langle M\rangle^i$, and the previsible process $\langle M\rangle$ associated with M in the Doob-Meyer decomposition of M^2 (see [7], p. 96). Note that $\langle M\rangle^i$ is also defined if M is only a martingale in direction i, i=1,2. We finally emphasize that, for convenience of notation, all martingales to be considered are assumed to vanish on $\mathbb{I}\cap\partial\mathbb{R}_+^2$.

1. The quadratic i-variations

In [7], the quadratic variation process of a square integrable martingale possessing all possible kinds of jumps was described in terms of its jump components and regularity properties were derived. They were shown to be exactly the same as for the martingale itself. More precisely, the continuity degree of the quadratic variation coincides with the martingale's continuity degree. One clearly expects the same behaviour for the quadratic i-variations. But since they are of mixed type, i.e. show martingale-like behaviour in one direction and are increasing in the other, unlike for the purely increasing quadratic variation, they are not quite as easy to handle. In this section, we will show that any regular L $\log^+$L-integrable martingale possesses quadratic i-variations which are regular. A more precise discussion of their regularity properties is given for square integrable martingales.

The method we will use to obtain the results are of the same kind as in [7]. We will describe the quadratic i-variations for simple jump components and pass to the general case by applying a uniform convergence argument which is prepared by the following inequalities. Due to the fact that any L $\log^+$L-integrable martingale can be approximated by a sequence of martingales which are integrable in any order, there are two relevant versions. We first state the L^p-version for p>1.

Proposition 1. Let $i=1,2$. For any $p>1$ there exists a constant c_p such that for any pair (M,N) of regular p-integrable martingales such that $[M]^i$, $[N]^i$ are regular

$$\| \sup_{t\in\mathbb{I}} | [M]^i_t - [N]^i_t| \|_{p/2} \le c_p \, \|(M+N)_1\|_p \, \|(M-N)_1\|_p \, .$$

Proof: For any $t\in\mathbb{I}$, the definition of the quadratic i-variation and the inequality of Cauchy-Schwarz give

$$| [M]^i_t - [N]^i_t| \le \{ [M-N]^i_t \, [M+N]^i_t\}^{1/2}.$$

Now quadratic i-variations are increasing in direction i. Therefore

$$(1) \quad \sup_{t\in\mathbb{I}} | [M]^i_t - [N]^i_t| \le \sup_{t\in\mathbb{I}} \{ [M-N]^i_t \, [M+N]^i_t \}^{1/2}$$
$$\le \sup_{t_i\in[0,1]} \{ [M-N]^i_{(1,t_i)} \, [M+N]^i_{(1,t_i)}\}^{1/2}.$$

By the regularity of $[M]^i$, $[N]^i$ and $[M-N]^i_{(1,.)}$, $[M+N]^i_{(1,.)}$ (the latter can be assumed since the one-parameter processes are submartingales) we are allowed to integrate (1). We find

$$\| \sup_{t\in\mathbb{I}} | [M]^i_t - [N]^i_t| \|_{p/2}$$

$$\le \| \sup_{t_i\in[0,1]} ([M-N]^i_{(1,t_i)})^{1/2} \|_p \; \| \sup_{t_i\in[0,1]} ([M+N]^i_{(1,t_i)})^{1/2} \|_p$$

(Cauchy-Schwarz, Hölder)

$$\le c^1_p \, \|([M-N]^i_1)^{1/2}\|_p \, \|([M+N]^i_1)^{1/2}\|_p$$

(proposition 1.1 of [6], Doob's inequality)

$$\le c^2_p \, \|(M-N)_1\|_p \, \|(M+N)_1\|_p$$

(Burkholder's inequality)

with universal constants c^1_p, c^2_p. This is the desired result. □

If M and N are only $L \log^+ L$-integrable, we have the following inequality.

Proposition 2. Let $i=1,2$. There is a constant c_1 such that for any pair (M,N) of regular $L \log^+ L$-integrable martingales such that $[M]^i$, $[N]^i$ are regular, any $\delta,\lambda>0$

$$P(\sup_{t\in\mathbb{I}} | [M]^i_t - [N]^i_t | > \delta)$$
$$\le c_1 \{ \lambda/\delta \, \|(M-N)_1\|_{L \log^+ L} + 1/\lambda \, \|(M+N)_1\|_{L \log^+ L}\}.$$

Proof: For any pair (X,Y) of nonnegative random variables, any $\delta,\lambda>0$ we have

$$P(XY>\delta) = P(X\le\lambda,\ XY>\delta) + P(X>\lambda,\ XY>\delta)$$
$$\le P(Y>\delta/\lambda) + P(X>\lambda).$$

Hence by (1) for $\delta,\lambda>0$

$$P(\sup_{t\in I} |\,[M]^1_t - [N]^1| > \delta)$$
$$\le P(\{\sup_{t_1\in[0,1]} [M-N]^1_{(1,t_1)} \sup_{t_1\in[0,1]} [M+N]^1_{(1,t_1)}\}^{1/2} > \delta)$$
$$\le P(\sup_{t_1\in[0,1]} ([M-N]^1_{(1,t_1)})^{1/2} > \delta/\lambda) + P(\sup_{t_1\in[0,1]} ([M+N]^1_{(1,t_1)})^{1/2} > \lambda)$$
$$\le \lambda/\delta\ E(([M-N]^1_1)^{1/2}) + 1/\lambda\ E(([M+N]^1_1)^{1/2})$$

(proposition 16.1 of [7], Doob's inequality)

$$\le c^1_1\{\lambda/\delta\ \| \sup_{t_1\in[0,1]} |M_{(1,t_1)} - N_{(1,t_1)}|\,\|_1 + 1/\lambda\ \| \sup_{t_1\in[0,1]} |M_{(1,t_1)} - N_{(1,t_1)}|\,\|_1\}$$

(Davis' inequality)

$$\le c^2_1\ \{\ \lambda/\delta\ \|(M-N)_1\|_{L\ \log^+ L} + 1/\lambda\ \|(M+N)_1\|_{L\ \log^+ L}\}$$

(Doob's inequality).

This completes the proof. □

We now concentrate on the simple jump parts of regular square integrable martingales and consider the regularity properties and jump sets of their quadratic i-variations, i=1,2. In order to describe the discontinuities of the quadratic i-variations of a compensated 0-jump part, we need some information about the discontinuities of the compensator. We know from [7], that the compensators of jumps on pure 0-simple sets have at most 1- or 2-jumps. In the following proposition we will show that they occur in the "shadows" of the pure 0-simple sets considered. Hereby we will use the following notation introduced in [7]. For a regular process X and a 0-simple set S let

$$X(S) = \int_{[0,.]} \Delta_{.} X\ d\Gamma(S),$$

where $\Gamma(S)$ is the increasing process associated with S (see section 0). We call X(S) the "jump process of X on S". Recall that for a square integrable martingale M and a pure 0-simple set S which is p-integrable for some $p>2$, $M(S)$ can be compensated by a process C which is described in [7], § 15, and has at most 1- or 2-jumps. Recall further that the 1-shadow $\sigma_1(S)$ of S is the random set of open (relative to $\mathbb{I}$) vertical line segments connecting the points of S to the boundary of $\mathbb{I}$, and the 2-shadow $\sigma_2(S)$ of S the corresponding random set of horizontal line segments.

Proposition 3. Let M be a square integrable regular martingale, S a pure 0-simple set which is p-integrable for $p\geq 1$, C the compensator of $M(S)$ according to [7], pp. 126, 127. Then the i-jumps of C are contained in $\sigma_i(S)$, $i=1,2$.

Proof: As a pure set, S is composed of a totally inaccessible, a 1-previsible, 2-inaccessible, a 2-previsible, 1-inaccessible, and a previsible 0-simple set, which are pairwise disjoint. To concentrate on the most difficult case, assume that S is totally inaccessible. First of all,

$$C = \overline{M}(S)^{\pi 1} + \overline{M}(S)^{\pi 2} - \overline{M}(S)^{\pi} - (\underline{M}(S)^{\pi 1} + \underline{M}(S)^{\pi 2} - \underline{M}(S)^{\pi})$$

([7], theorem 15.1), where

$$M(S) = \overline{M}(S) - \underline{M}(S)$$

is the decomposition of the jump process $M(S)$ into its positive and negative parts. We will show that

(2) $\overline{M}(S)^{\pi 1}$, $\underline{M}(S)^{\pi 1}$ have at most 2-jumps contained in $\sigma_2(S)$,

(3) $\overline{M}(S)^{\pi 2}$, $\underline{M}(S)^{\pi 2}$ have at most 1-jumps contained in $\sigma_1(S)$,

(4) $\overline{M}(S)^{\pi}$, $\underline{M}(S)^{\pi}$ are continuous.

To do this, we have to analyze the jump parts of the integrable increasing processes appearing in (2)-(4) (see section 0). We know from theorem 15.1 of [7] that

$$A^{o} = 0 \quad \text{for } A = \overline{M}(S)^{\pi 1},\ \underline{M}(S)^{\pi 1},\ \overline{M}(S)^{\pi 2},\ \underline{M}(S)^{\pi 2},\ \overline{M}(S)^{\pi},\ \underline{M}(S)^{\pi}.$$

To see that $(\overline{M}(S)^{\pi 1})^{1} = 0$, assume that T is a 1-previsible 1-simple set. Then T is previsible and we have

$$(5)\ E(\int_{\mathbb{I}} 1_T\, d(\Gamma(S)^{\pi_1})^1) = E(\int_{\mathbb{I}} 1_T\, d(\Gamma(S)^{\pi_1})) \qquad ((\Gamma(S)^{\pi_1})^o=0,\ [(\Gamma(S)^{\pi_1})^2+(\Gamma(S)^{\pi_1})^c](T)=0)$$

$$= E(\int_{\mathbb{I}} 1_T\, d\Gamma(S)) \qquad (T\in\mathfrak{P}^1)$$

$$= 0 \qquad (S\cap T = \emptyset \text{ P-a.s}).$$

This obviously implies $(\Gamma(S)^{\pi_1})^1 = 0$. Therefore, proposition 13.1 of [7] gives

$(\bar{M}(S)^{\pi_1})^1 = (\underline{M}(S)^{\pi_1})^1 = 0.$

To see that $(\bar{M}(S)^{\pi_1})^2$ has no mass outside $\sigma_2(S)$, let T be an arbitrary 2-simple set such that $T\cap\sigma_2(S) = \emptyset$ P-a.s. Then, since S is totally inaccessible and $T\in\mathfrak{P}^1$, we have $\bar{M}(S)(T) = 0$ and consequently

$$E(\int_{\mathbb{I}} 1_T\, d\bar{M}(S)^{\pi_1}) = E(\int_{\mathbb{I}} 1_T\, d(\bar{M}(S) - \bar{M}(S)^{\pi_1})) = 0.$$

Since analogously $(\underline{M}(S)^{\pi_1})^2(T) = 0$, (2) follows. (3) is proved by interchanging the roles of the coordinates in the arguments just given. Finally, observe that by

$$\Gamma(S)^{\pi} = \Gamma(S)^{\pi_1\pi_2} = \Gamma(S)^{\pi_2\pi_1},$$

(5) goes through for both $(\Gamma(S)^{\pi})^1$ and $(\Gamma(S)^{\pi})^2$. Therefore, another application of proposition 13.1 of [7] gives

$(\bar{M}(S)^{\pi})^i = (\underline{M}(S)^{\pi})^i = 0,\ i=1,2.$

This finally entails (4). □

Proposition 4. Let M be a square integrable regular martingale, S a pure 0-simple set which is p-integrable for all $p\geq 1$, M^o the compensated jump process of M on S. Then

$$[M^o]^i_t = \int_{[0,t]} (\Delta^i_{s_i} M_{(\cdot,t_{\bar{i}})} - \Delta^i_{s_i} M_{(\cdot,s_{\bar{i}}-)})^2\, d\Gamma(S)_s\ ,\ t\in\mathbb{I},\ i=1,2.$$

In particular, $[M^o]^i$ has a regular version with discontinuities contained in $S\cup\sigma_i(S)$, $i=1,2$.

Proof: Following proposition 3, the martingale $M_{(\cdot,t_{\bar{i}})}$ has its jumps on $S\cup\sigma_i(S)\cap(\Omega\times[0,1]\times\{t_{\bar{i}}\})$, $t_{\bar{i}}\in[0,1]$, $i=1,2$. Moreover, it is of bounded variation. This clearly implies the desired formula, the right hand side of which is a regular process. □

We next consider 1- and 2-jumps. Hereby we will use the appropriate notation introduced in [7]. For a regular process X and an i-simple set S let the "jump process of X on S" be given by

$$X(S)_t = \int_{[0,t]} (\Delta^i_{s_i} X_{(.,t_{\bar{i}})} - \Delta^i_{s_i} X_{(.,s_{\bar{i}}-)})\, d\Gamma(\partial S)_s \ , \ t \in \mathbb{I}, \ i=1,2,$$

where ∂S is the random set of lower (left) boundary points of S, and $\Gamma(\partial S)$ is the increasing process associated with this 0-simple set. Note that if X has no 0-jumps, the minus sign in the definition of X(S) indicating a left limit in direction $\bar{i}$ can be omitted. Recall that for a square integrable martingale M without 0-jumps and a pure i-simple set which is p-integrable for some $p>2$, M(S) can be compensated by a continuous process C given by theorem 15.3 of [7] .

To see the regularity of the quadratic i-variation of a compensated $\bar{i}$-jump part, we prove that it is of bounded variation.

Proposition 5. Let $i=1,2$, M a regular square integrable martingale without 0-jumps and S a pure i-simple set which is p-integrable for any $p \geq 1$, M^i the compensated jump process of M on S. Then $[M^i]^{\bar{i}}$ is of bounded variation.

Proof: Since S is the union of a previsible and an inaccessible i-simple set, which are disjoint, and since for previsible i-simple sets the associated jump process of M is a martingale, the quadratic $\bar{i}$-variation of which is obviously of bounded integrable variation, we need only consider the case S inaccessible. Now

$$M^i = M(S) - C,$$

and by definition and proposition 14.2 of [7] $[M(S)]^{\bar{i}}$ is of bounded integrable variation. Hence it is enough to show that $[C]^{\bar{i}}$ is of bounded integrable variation. We argue for $i=1$. Let $M(S) = \overline{M}(S) - \underline{M}(S)$ be the decomposition of the jump process by its positive resp. negative part. Accordingly let $C = \overline{C} - \underline{C}$, where $\overline{C}$, $\underline{C}$ are compensators of $\overline{M}(S)$, $\underline{M}(S)$ respectively (see theorem 15.3 of [7]). Now fix an interval J_1 in [0,1].

Then by continuity of C,

$$\Delta_{J_1}[C]^2_{(.,.)} = \Delta_{J_1}\langle C\rangle^2_{(.,.)} = \Delta_{J_1}\langle\overline{C}\rangle^2_{(.,.)} - \Delta_{J_1}\langle\underline{C}\rangle^2_{(.,.)}.$$

But since $\Delta_{J_1}\overline{M}(S)$, $\Delta_{J_1}\underline{M}(S)$ are submartingales in direction 2, we see that both $\Delta_{J_1}\langle\overline{C}\rangle^2_{(.,.)}$, $\Delta_{J_1}\langle\underline{C}\rangle^2_{(.,.)}$ are increasing processes of the second parameter. Therefore the (two-parameter) variation of $[C]^2$ is dominated by the (one-parameter) variation of

$$\langle\overline{C}\rangle^2_{(.,1)} + \langle\underline{C}\rangle^2_{(.,1)}.$$

But this process is increasing, so that an upper bound of the variation of $[C]^2$ is given by

$$\langle\overline{C}\rangle^2_1 + \langle\underline{C}\rangle^2_1,$$

which is integrable (see theorem 4.3 of [7]). □

We are ready to describe the quadratic j-variations of simple i-jump parts more precisely.

Proposition 6. Let i=1,2, M a regular square integrable martingale without 0-jumps and S a pure i-simple set which is p-integrable for any p≥1, M^i the compensated i-jump process of M on S. Then

$$[M^i]^i_t = \int_{[0,t]} (\Delta^i_{s_i} M_{(.,t_{\overline{i}})} - \Delta^i_{s_i} M_{(.,s_i)})^2 \, d\Gamma(\partial S)_s \ , \ t\in \mathbb{I},$$

$[M^i]^{\overline{i}}$ is of bounded integrable variation.

In particular, $[M^i]^j$ has a regular version with i-jumps contained in S, j=1,2.

Proof: For $[M^i]^i$, see part 1 of the proof of theorem 17.2 of [7]. Now let us discuss the case i=1. By approximation on S, we may assume that M is p-integrable for any p≥1, and by proposition 5, that $[M^1]^2$ is regular. Moreover, since for any fixed $t_1\in[0,1]$ we have

$$[M^1]^2_{(t_1,.)} = \langle M^1\rangle^2_{(t_1,.)},$$

we may even assume that

(6) $[M^1]^2 = \langle M^1\rangle^2$

is regular and of bounded integrable variation. Now suppose that $[M^1]^2$

has, say nonnegative, 0-jumps on a 0-simple set T which is p-integrable for all $p\geq 1$. Since $\langle M^1\rangle^2$ is 2-previsible, (6) shows that it is enough to consider $T\in\mathfrak{P}^2$. Now $(M^1)^2 - [M^1]^2$ is a 2-martingale, hence also

$((M^1)^2 - [M^1]^2)(T)$ is a 2-martingale,

where we use the notation explained before proposition 5. This implies that

$$E([M^1]^2(T)) = E(([M^1]^2 - (M^1)^2)(T)) \qquad (\text{ } M^1 \text{ has no 0-jumps})$$
$$= 0.$$

This excludes the possibility that $[M^1]^2$ has 0-jumps.

Next suppose that $[M^1]^2$ has 2-jumps, say on a 2-simple set U which is p-integrable for all $p\geq 1$. Now U is 1-previsible by definition. Hence

$$([M^1]^2 - [M^1])(U) = [M^1]^2(U) \qquad ([M^1] \text{ has no 2-jumps})$$

is a nonnegative martingale in direction 1, since $[M^1]^2 - [M^1]$ is. But this forces $[M^1]^2(U) = 0$ and thus excludes the possibility of 2-jumps.

Finally, suppose that V is a 1-simple set which is p-integrable for all $p\geq 1$, and such that $V\cap S = \emptyset$ P-a.s. Then, as above,

$$- [M^1]^2(V) = ((M^1)^2 - [M^1]^2)(V)$$

is a continuous martingale in direction 2, which is of bounded variation. This, again, forces it to be zero.

Summarizing, we have shown that $[M^1]^2$ has at most 1-jumps contained in S. This completes the proof. □

We finally consider martingales without any jumps.

Prposition 7. Let M be a continuous square integrable martingale. Then $[M]^1$, $[M]^2$ have continuous versions.

Proof: 1. For $n\in\mathbb{N}$ let M^n be a regular version of the bounded martingale $E((-n)\vee(M_1\wedge n)|\mathfrak{F}.)$, $(M^n)^c$ its continuous part according to proposition 19.3 of [ ]. Then $(M^n)^c$ is p-integrable for any $p\geq 1$ and

$$\|((M^n)^c - M)_1\|_2 \le c_1 \| [(M^n)^c - M]_1\|_1^{1/2}$$
$$\le c_1 \| [M^n - M]_1\|_1^{1/2} \qquad \text{(proposition 20.2 of [7])}$$
$$\le c_2 \|(M^n - M)_1\|_2 \to 0 \ (n \to \infty),$$

where c_1, c_2 are universal constants. With an appeal to proposition 1, we can therefore assume M to be p-integrable for any $p \ge 1$.

2. Let us concentrate on $[M]^1$. First of all,

$$[M]^1_{(.,t_2)} = \langle M\rangle^1_{(.,t_2)} \quad \text{for all } t_2 \in [0,1],$$

by a well known one-parameter result for continuous martingales. Next,

$$\langle M\rangle^1_{(.,1)} - \langle M\rangle^1_{(.,1)} \text{ is continuous}$$

(see Dellacherie, Meyer [4], p. 376). Therefore, a slight extension of theorem 6.1 of [7], to processes which are p-integrable for all $p \ge 1$, instead of bounded ones, shows that the optional projection N of $\langle M\rangle^1_{(.,1)} - \langle M\rangle^1_{(.,1)}$ in direction 2 is a process which possesses at most 2-jumps. Moreover, theorem 11.2 of [7] proves that $\langle M\rangle^1 - \langle M\rangle$ is a 2-martingale. Hence we may and do assume

$$N = \langle M\rangle^1 - \langle M\rangle.$$

Since $\langle M\rangle$ itself is continuous, we deduce that $\langle M\rangle^1$ is regular and possesses at most 2-jumps. Of course, the same is true for the process $M^2 - \langle M\rangle^1$, which is a martingale in direction 1 and p-integrable for any $p \ge 1$. Now suppose S to be a 2-simple set which is p-integrable for any $p \ge 1$. The jump process

$$(M^2 - \langle M\rangle^1)(S) = - \langle M\rangle^1(S) \qquad \text{(M is continuous)}$$

is a continuous 1-martingale. But, since $\langle M\rangle^1$ is increasing in direction 1, it is also of bounded variation. Hence $\langle M\rangle^1(S) = 0$. Thus $\langle M\rangle^1$ possesses no jumps. Remember that $\langle M\rangle^1$ is a version of $[M]^1$ to conclude. □

Remark. For an alternative proof of proposition 7 which makes no use of either simple sets or the classification of jumps, see Nualart [12].

We now come back to the principal aim of this section. It consists in showing that any regular square integrable martingale M possesses regular quadratic i-variations and describing their discontinuities. First of all, M can be decomposed according to the formula

$$M = M^o + M^1 + M^2 + M^c,$$

where M^i is the i-jump part of M, i=0,1,2, and M^c its continuous part. The jump parts are orthogonal sums of related simple jump parts as discussed above, and, like the continuous part, can be treated separately by what we already know. Unfortunately, this does not carry over immediately to M itself. The reason is this: in general, we do not have

$$[M]^i = [M^o]^i + [M^1]^i + [M^2]^i + [M^c]^i,$$

and "mixed" variations may appear. Using the polarization identity, however, we can represent them by quadratic i-variations of sums and differences of single components. So we are lead to investigate $[M^j+M^k]^i$ for j,k=0,1,2,c, i=1,2. It turns out that only M^o can interfere with either M^1 or M^2. All other combinations of components have indeed "orthogonal i-variation", i=1,2.

Proposition 8. Let i=1,2, M,N square integrable regular martingales, N without 0-jumps. Let further S be a pure 0-simple set, T a pure i-simple set, both of which are p-integrable for any p≥1, M^o the compensated 0-jump process of M on S, N^i the compensated i-jump process of N on T. Then

$$[M^o + N^i]^i = \int_{[0,.]} (\Delta^i_{s_i}(M^o+N^i)_{(.,t_{\bar{i}})} - \Delta^i_{s_i}(M^o+N^i)_{(.,s_{\bar{i}}-)})^2 \, d\Gamma(S\cup T)_s.$$

In particular, $[M^o + N^i]^i$ has a regular version with discontinuities contained in $S\cup\sigma_i(S)\cup T$.

Proof: Using proposition 3, one proceeds as in the proofs of the propositions 4 and 6. □

The other interesting case is the quadratic i-variation of a continuous martingale and a compensated i-jump process.

Proposition 9. Let $i=1,2$, M,N square integrable regular martingales such that M is continuous, N without 0-jumps. Let further S be a pure i-simple set which is p-integrable for any $p\geq 1$, N^i the compensated i-jump process of N on S. Then $[M + N^i]^{\bar{i}}$ possesses a regular version which has at most i-jumps contained in S.

Proof: Let $i=1$. In complete analogy to the second part of the proof of proposition 7 we first derive that

$$[M + N^1]^2 = \langle M + N^1\rangle^2$$

may be assumed to be regular. Now we proceed as in the proof of proposition 6 after equation (6) to conclude. □

In the remaining cases, the quadratic i-variations of the respective components are orthogonal.

Proposition 10. Let $i=1,2$, M,N square integrable regular martingales such that M is of bounded variation in direction i, N continuous in direction i. Then

$$[M + N]^i = [M]^i + [N]^i.$$

Proof: For each $t_{\bar{i}}\in[0,1]$, we clearly have

$$[M + N]^i_{(\cdot,t_{\bar{i}})} = [M]^i_{(\cdot,t_{\bar{i}})} + [N]^i_{(\cdot,t_{\bar{i}})}.$$ □

Corollary. Let $i=1,2$, M a square integrable regular martingale, S^j a pure j-simple set which is p-integrable for any $p\geq 1$, N^j the compensated j-jump process of M $(M-M^o)$ on S^j, $j=0$ $(j=1,2)$. Then

$$[N^j + N^k]^i = [N^j]^i + [N^k]^i \quad \text{for } (j,k),\ (k,j)\in\{(0,\bar{i}),\ (i,\bar{i})\},$$

$$[M^c + N^k]^i = [M^c]^i + [N^k]^i \quad \text{for } k=0,i.$$

Proof: N^o is of bounded variation, N^k of bounded variation in direction k and continuous in direction $\bar{k}$, M^c continuous. Hence proposition 10 applies in all cases stated. □

The following diagram illustrates the results of propositions 8-10 and the preceding corollary. As usual, i=1,2. Combinations of components, for which a mixed i-variation may exist, are indicated by "1", whereas "0" means that they do not interfere.

Diagram

M^o				
M^i	1			
$M^{\bar{i}}$	0	0		
M^c	0	0	1	
	M^o	M^i	$M^{\bar{i}}$	M^c

We are ready to state our first main result.

Theorem 1. Let M be a regular square integrable martingale. Then $[M]^1$ and $[M]^2$ are integrable and have regular versions. Moreover, the set of discontinuities of $[M]^1$, $[M]^2$ is contained in the set of discontinuities of M, and

i) $[M]^j$ has no 0-jumps, if M has no 0-jumps,

ii) $[M]^j$ has at most i-jumps, if M has at most i-jumps, i=1,2,

iii) $[M]^j$ is continuous, if M is continuous,

j=1,2.

Proof: By [7], p. 156, we can and do choose a sequence $(M_n^i)_{n\in\mathbb{N}}$ of compensated i-jump processes of M on pure i-simple sets which are p-integrable for all p≥1, such that

$$\| \sup_{t\in\mathbb{T}} |(M_n^i)_t - M_t^i| \|_2 \to 0 \quad (n \to \infty),$$

where M^i is the i-jump component of M, i=0,1,2. According to propositions 8-10 and the corollary we may assume that the processes

(7) $[M_n^o + M_n^1 + M_n^2 + M^c]^j = [M_n^o + M_n^j]^j + [M_n^{\bar{j}} + M^c]^j$, j=1,2,

are regular and have their jumps prescribed by M_n^i, i=0,1,2. Therefore, the first part of the assertion follows by applying proposition 1 for $p=2$. The rest is obvious from the representation (7) and the description of the jump species which may appear along with the respective compensated jumps. □

Finally we extend the regularity result of theorem 1 to $L\log^+L$.

Theorem 2. Let M be a regular $L\log^+L$-integrable martingale. Then $[M]^1$, $[M]^2$ possess regular versions.

Proof: Let M^n, $n\in\mathbb{N}$, be as in proposition 7. Then

$$\|(M^n - M)_1\|_{L\log^+L} \to 0 \quad (n\to\infty).$$

By theorem 1, $[M^n]^i$, $n\in\mathbb{N}$, $i=1,2$, has a regular version. Now apply proposition 2 to see that for any $\delta,\lambda>0$

$$(8)\quad P(\sup_{t\in I} |[M^n]^i_t - [M]^i_t| > \delta)$$
$$\le c_1 \{ \lambda/\delta \; \|(M^n - M)_1\|_{L\log^+L} + 1/\lambda \; \|(M^n + M)_1\|_{L\log^+L}\}.$$

For $\delta,\varepsilon>0$ fixed, choose λ big enough to ensure that the second term on the right hand side of (8) is smaller than $\varepsilon/2$ for all $n\in\mathbb{N}$. Then choose n big enough to force the first term below $\varepsilon/2$. This completes the proof. □

Remark. Theorem 1 should extend completely to $L\log^+L$. It appears as if this could be shown by associating a "dual" Orlicz space to $L\log^+L$ and deriving an "orthogonal" decomposition for $L\log^+L$-integrable martingales. But this seems to involve methods which are out of our scope here.

2. The stochastic integrator properties of quadratic i-variations

In [6], the quadratic i-variations of continuous square integrable martingales were shown to be 1-stochastic integrators, operating on vector spaces of 1-previsible (for $i=2$) or 2-previsible (for $i=1$) processes. The approach taken hereby goes over to the quadratic i-variations of arbitrary regular martingales with jumps without essential changes, and needs only the results of the preceding section as additional information. We therefore generalize the theory of [6] by stating the extended results and justifying only those, which require some extra thought. The main difference consists in the fact that, due to the appearance of discontinuities, the martingale inequalities basic to the construction of dominating processes for the quadratic i-variations, are no longer valid for $p<1$. This forces us to concentrate on square integrable martingales for most of the section. The analogue of proposition 2.1 of [6], however, can be stated for $L \log^+ L$.

Proposition 1. Let $J \subset \mathbb{I}$ be an interval. Then for any $L \log^+ L$-integrable martingale M

$$\Delta_{J_i}[\Delta_{J_{\bar{i}}}[M]^i_{(\cdot,\cdot)}]^{\bar{i}} \leq 4\, \Delta_J[M] \sup_{s\in[0,1]} \Delta_{J_i}[M]^i_{(\cdot,s)}, \quad i=1,2.$$

Proof: Let us consider the case $i=1$. The proof is essentially the same as the proof of proposition 2.1 of [6]. But there is one place at which we used the continuity of M. It was not essential to do so, and here we show why. Let $(\mathbb{K}_m)_{m\in\mathbb{N}}$ resp. $(\mathbb{L}_n)_{n\in\mathbb{N}}$ be a 0-sequence of partitions of J_1 resp. J_2. Then, with limits in probability,

$$\Delta_{J_2}[\Delta_{J_1}[M]^1_{(\cdot,\cdot)}]^2 = \lim_{n\to\infty} \lim_{m\to\infty} \sum_{K_2\in\mathbb{L}_n} \Big(\sum_{K_1\in\mathbb{K}_m} \Delta_{K_2}(\Delta_{K_1} M_{(\cdot,\cdot)})^2\Big)^2$$

$$= \lim_{n\to\infty} \lim_{m\to\infty} \sum_{K_2\in\mathbb{L}_n} \Big(\sum_{K_1\in\mathbb{K}_m} \Delta_K M\, (\Delta_{K_1} M_{(\cdot,t_2^K)} + \Delta_{K_1} M_{(\cdot,s_2^K)})\Big)^2$$

$$\leq 2 \lim_{n\to\infty} \lim_{m\to\infty} \Big\{\sum_{K_2\in\mathbb{L}_n} \Big\{\Big(\sum_{K_1\in\mathbb{K}_m} \Delta_K M\, \Delta_{K_1} M_{(\cdot,t_2^K)}\Big)^2 + \Big(\sum_{K_1\in\mathbb{K}_m} \Delta_K M\, \Delta_{K_1} M_{(\cdot,s_2^K)}\Big)^2\Big\}\Big\}.$$

From this point on, we can continue as in the proof of proposition 2.1 of [6], remarking that for the desired estimation it is immaterial whether we take suprema over s_2^K or over t_2^K. This completes the proof. □

In the following proposition, due to the presence of jumps, we have, contrary to [6], to restrict the domain of validity of the desired inequality to L^p for $p \geq 1$.

Proposition 2. For $p \geq 1$ there is a constant $a_p > 0$, such that for any pair (X,Y) of regular processes satisfying

i) $X \geq 0$, X is increasing in direction 1, $\Delta_{J_1} X_{(.,.)}$ is a submartingale in direction 2 for any interval $J_1 \subset [0,1]$,

ii) $Y \geq 0$, Y is increasing, $\Delta_{J_1}(X-Y)_{(.,.)}$ is a martingale in direction 2 for any interval $J_1 \subset [0,1]$,

iii) $\Delta_{J_2}[\Delta_{J_1} X_{(.,.)}]^2 \leq \Delta_J Y \sup_{s \in [0,1]} \Delta_{J_1} X_{(.,s)}$ for any interval $J \subset \mathbb{I}$,

and any partition $\mathbb{K}$ of $[0,1]$ we have

$$\| \sum_{K \in \mathbb{K}} \sup_{s \in [0,1]} \Delta_K X_{(.,s)} \|_p \leq a_p \, \| Y_1 \|_p .$$

An analogous statement holds, if the roles of the coordinates are interchanged.

Proof: In the proof of proposition 4 of [6] replace the application of propositions 1.1 and 1.2 of [6] by an application of Bakry's [1], p. 364, inequality, which holds for families of regular one-parameter martingales in case $p \geq 1$. □

As a corollary of the preceding two propositions, we have

Proposition 3. For $p \geq 1$ there is a constant a_p such that for any square integrable regular martingale M, any partition $\mathbb{K}$ of $[0,1]$

$$\| \sum_{K \in \mathbb{K}} \sup_{s \in [0,1]} \Delta_K [M]^1_{(.,s)} \|_p \leq a_p \, \| [M]_1 \|_p ,$$

$$\| \sum_{K \in \mathbb{K}} \sup_{s \in [0,1]} \Delta_K [M]^2_{(s,.)} \|_p \leq a_p \, \| [M]_1 \|_p .$$

The abstract criterion for the integrator property of processes which are of bounded variation in one direction generalizes as follows.

Theorem 1. For $p,q,r \geq 1$ such that $1/r + 1/q = 1$ there is a constant $b_{p,q,r}$, such that for any quadruple (W,X,Y,Z) of regular processes which satisfy

i) W is adapted and a martingale in direction 2,

ii) $X \geq 0$, X is increasing,

iii) (Y,Z) fulfills i)-iii) of proposition 2,

iv) $\Delta_{J_2}[\Delta_{J_1} W_{(\cdot,\cdot)}]^2 \leq \Delta_J X \sup_{s \in [0,1]} \Delta_{J_1} Y_{(\cdot,s)}$ for any interval $J \subset \mathbb{I}$,

and any 2-previsible elementary process Y_o we have

$$\| \sup_{t \in \mathbb{I}} | \int_{[0,t]} Y_o \, dW | \|_p \leq b_{p,q,r} \, \|Z_1\|_{pq/2}^{1/q} \, \| \int_{\mathbb{I}} Y_o^2 \, dX \, \|_{pr/2}^{1/r}.$$

A similar statement holds, if the roles of the coordinates are interchanged.

Proof: In the proof of theorem 1.1 of [6] the following changes are necessary. Replace the applications of propositions 1.1 and 1.2 of that paper with Bakry's [1], p. 364, inequality and proposition 1.4 with proposition 2 of the present paper. □

We are now ready to state the integrator properties of the quadratic i-variations of square integrable martingales.

Theorem 2. Let M be a regular square integrable martingale. Then the elementary stochastic integrals of $[M]^1$ resp. $[M]^2$, defined on the linear spaces of 2-previsible resp. 1-previsible elementary processes, can be linearly and continuously extended to $L^2(\Omega \times \mathbb{I}, \mathfrak{P}^2, P \times [M])$ resp. $L^2(\Omega \times \mathbb{I}, \mathfrak{P}^1, P \times [M])$ and for $p \geq 1$ we have

$$\| \sup_{t \in \mathbb{I}} | \int_{[0,t]} Y \, d[M]^i | \|_p \leq (1+b_{p,2,2}) \, \|[M]_1\|_p^{1/2} \, \| \int_{\mathbb{I}} Y^2 \, d[M] \|_p^{1/2} ,$$

$Y \in L^2(\Omega \times \mathbb{I}, \mathfrak{P}^{\bar{i}}, P \times [M])$, $i=1,2$, where $b_{p,2,2}$ is given by theorem 1.

Proof: The results of section 1 allow us to assume that $[M]^i$ is regular, $i=1,2$. Therefore, theorem 1 applies and we can run the short proof of theorem (2.1) of [6]. The inequalities extend immediately from the respective spaces of previsible elementary functions to the L^2-spaces. □

Remark. Theorem 2 implicitly proves that for regular square integrable martingales the quadratic i-variations are 1-stochastic integrators in the sense of Bichteler [2] or Hürzeler [5]. As was pointed out in [6], this property is essential for the development of a stochastic calculus for square integrable martingales.

3. The mixed martingale part in the decomposition of the square of a square integrable martingale

Let M be a regular square integrable martingale, $\mathbb{K}$ a partition of $\mathbb{I}$ by intervals. Then for any $t\in\mathbb{I}$ we have the following decomposition of M_t^2

$$
\begin{aligned}
(1)\quad M_t^2 &= \sum_{J\in\mathbb{K}} \Delta_{J\cap[0,t]} M^2 \\
&= 2\sum_{J\in\mathbb{K}} M_{s_J}\, \Delta_{J\cap[0,t]}M + 2\sum_{J\in\mathbb{K}} \Delta_{J^1\cap[0,t]}M\, \Delta_{J^2\cap[0,t]}M \\
&+ 2\sum_{J\in\mathbb{K}} \Delta_{J^1\cap[0,t]}M\, \Delta_{J\cap[0,t]}M + 2\sum_{J\in\mathbb{K}} \Delta_{J^2\cap[0,t]}M\, \Delta_{J\cap[0,t]}M \\
&+ \sum_{J\in\mathbb{K}} (\Delta_{J\cap[0,t]}M)^2.
\end{aligned}
$$

The processes appearing in the first line on the extreme right hand side of (1) are martingales. Given a 0-sequence $(\mathbb{K}_n)_{n\in\mathbb{N}}$ of partitions of $\mathbb{I}$, we want to study the limit martingales of the two corresponding martingale sequences. Now the first one converges to a well known stochastic integral of M. We are mainly interested in the limit of the second one which we call "mixed martingale part" of M^2. Since the left hand side of (1) is constant in n, this process can be studied via the limits of the remaining 4 sequences appearing on the right hand side. As it happens, sections 1 and 2 give us enough information about the last three. Indeed, the fifth sequence converges in L^1 to $[M]_t$, $t\in\mathbb{I}$, the quadratic variation of M. Moreover, since

$$
\sum_{J\in\mathbb{K}} \Delta_{J^1\cap[0,t]}M\, \Delta_{J\cap[0,t]}M = \sum_{J_1\in\mathbb{K}_1}(\Delta_{J_1\cap[0,t_1]}M)^2 - \sum_{J\in\mathbb{K}}(\Delta_{J\cap[0,t]}M)^2,
$$

the L^1-limit of the third sequence is given by $[M]_t^1 - [M]_t$, and analogously the fourth one produces $[M]_t^2 - [M]_t$, $t\in\mathbb{I}$. Therefore, the regularity results of section 1, of [7], and of the following simple proposition about the stochastic integral of M, should give us complete information about the continuity properties of the mixed martingale part of M^2. For any square integrable martingale M, we will denote by I^M the stochastic integral (process) defined on $L^2(\Omega\times\mathbb{I},\mathfrak{P},P\times[M])$ associated with M.

Proposition 1. Let M be a regular square integrable martingale. Then

(2) $I^M = I^{M^0} + I^{M^1} + I^{M^2} + I^{M^c}$,

where M^i, $i=0,1,2,c$, is the i-jump component resp. continuous component of M. Moreover, for any $Y\in L^2(\Omega\times\mathbb{I},\mathfrak{P},P\times[M])$, the set of discontinuities of $I^M(Y)$ is contained in the set of discontinuities of M, and

i) $I^M(Y)$ has no 0-jumps, if M has no 0-jumps,

ii) $I^M(Y)$ has at most i-jumps, if M has at most i-jumps, $i=1,2$,

iii) $I^M(Y)$ is continuous, if M is continuous.

Proof: For elementary previsible Y, the assertion is obvious. But this is all we need to know, since we may approximate a given $Y\in L^2(\Omega\times\mathbb{I},\mathfrak{P},P\times[M])$ by a sequence $(Y^n)_{n\in\mathbb{N}}$ of elementary processes and use the uniform convergence on $\mathbb{I}$ of $(I^M(Y^n))_{n\in\mathbb{N}}$ to $I^M(Y)$ granted by Doob's inequality. □

It is clear from the discussion preceding proposition 1 that the sequence $(\sum_{J\in\mathbb{K}_n} \Delta_{J^1\cap[0,t]}M\, \Delta_{J^2\cap[0,t]}M)_{n\in\mathbb{N}}$ converges to the mixed martingale part of M^2 at t for any $t\in\mathbb{I}$. The convergence is even uniform on $\mathbb{I}$, as will be concluded from the following proposition.

Proposition 2. 1. For any $p>1$ there is a constant c_p such that for any pair (M,N) of regular 2p-integrable martingales, any partition $\mathbb{K}$ of $\mathbb{I}$

$$\| \sup_{t\in\mathbb{I}} | \sum_{J\in\mathbb{K}} \Delta_{J^1\cap[0,t]}M\, \Delta_{J^2\cap[0,t]}N| \|_p \le c_p\, \|M_1\|_{2p}\, \|N_1\|_{2p}.$$

2. There is a constant c_1 such that for any pair (M,N) of regular square integrable martingales, any partition $\mathbb{K}$ of $\mathbb{I}$, any $\lambda>0$

$$\lambda\ P(\sup_{t\in\mathbb{I}} |\sum_{J\in\mathbb{K}} \Delta_{J^1\cap[0,t]}M\ \Delta_{J^2\cap[0,t]}N| > \lambda) \le c_1\ \|M_1\|_2\ \|N_1\|_2.$$

Proof: We argue for the slightly more complicated second assertion. Let $\mathbb{K}$ be a partition of $\mathbb{I}$ and observe that the quadratic variation of the martingale $\sum_{J\in\mathbb{K}} \Delta_{J^1\cap[0,.]}M\ \Delta_{J^2\cap[0,.]}N$ is given by

$$[\sum_{J\in\mathbb{K}} \Delta_{J^1\cap[0,.]}M\ \Delta_{J^2\cap[0,.]}N] = \sum_{J\in\mathbb{K}} \Delta_{J^1\cap[0,.]}[M]^1\ \Delta_{J^2\cap[0,.]}[N]^2,$$

a fact which follows straight from the definition. Hence we find constants $a_1,\dots,a_4$ such that for any $\lambda>0$

$$\lambda\ P(\sup_{t\in\mathbb{I}} |\sum_{J\in\mathbb{K}} \Delta_{J^1\cap[0,t]}M\ \Delta_{J^2\cap[0,t]}N| > \lambda)$$

$$\le E(\sup_{t_1\in[0,1]} |\sum_{J\in\mathbb{K}} \Delta_{J^1\cap[0,t_1]\times[0,1]}M\ \Delta_{J^2\cap[0,t_1]\times[0,1]}N|) \qquad \text{(Doob's inequality)}$$

$$\le a_1\ E(([\sum_{J\in\mathbb{K}} \Delta_{J^1\cap[0,.]}M\ \Delta_{J^2\cap[0,.]}N]^1_1)^{1/2}) \qquad \text{(Davis' inequality)}$$

$$\le a_2\ E(([\sum_{J\in\mathbb{K}} \Delta_{J^1\cap[0,.]}M\ \Delta_{J^2\cap[0,.]}N]_1)^{1/2}) \qquad \text{([8], theorem 4)}$$

$$= a_2\ E((\sum_{J\in\mathbb{K}} \Delta_{J^1}[M]^1\ \Delta_{J^2}[N]^2)^{1/2})$$

$$\le a_2\ E((\sum_{J_1\in\mathbb{K}_1} \sup_{s\in[0,1]} \Delta_{J_1}[M]^1_{(.,s)} \sum_{J_2\in\mathbb{K}_2} \sup_{s\in[0,1]} \Delta_{J_2}[N]^2_{(s,.)})^{1/2})$$

$$\le a_2\ E(\sum_{J_1\in\mathbb{K}_1} \sup_{s\in[0,1]} \Delta_{J_1}[M]^1_{(.,s)})^{1/2}\ E(\sum_{J_2\in\mathbb{K}_2} \sup_{s\in[0,1]} \Delta_{J_2}[N]^2_{(s,.)})^{1/2} \qquad \text{(Cauchy-Schwarz)}$$

$$\le a_3\ E([M]_1)^{1/2}\ E([N]_1)^{1/2} \qquad \text{(proposition 2.3)}$$

$$\le a_4\ E(M_1^2)^{1/2}\ E(N_1^2)^{1/2} \qquad \text{(Burkholder's inequality).}$$

This completes the proof. □

We can now state our main result about the existence of the mixed martingale part in the decomposition of a square integrable martingale.

Theorem 1. Let M be a regular square integrable martingale. Then there exists a regular martingale $\tilde{M}$ such that for any 0-sequence $(\mathbb{K}_n)_{n\in\mathbb{N}}$ of prtitions of $\mathbb{I}$, the sequence of martingales

$(\sum_{J\in\mathbb{K}_n} \Delta_{J^1\cap[0,.]}M \, \Delta_{J^2\cap[0,.]}M)_{n\in\mathbb{N}}$ converges to $\tilde{M}$ in $L^1(\Omega,\mathfrak{F},P)$ and uniformly on $\mathbb{I}$ in probability.

If M is 2p-integrable for some p>1, then the convergence is uniform on $\mathbb{I}$ in $L^p(\Omega,\mathfrak{F},P)$. Moreover, we have the following representation

$$M^2 = 2\, I^M(M^{--}) + 2\,\tilde{M} + [M]^1 + [M]^2 - [M],$$

where $M^{--} = \lim_{s\uparrow .} M_s$.

Proof: For fixed $t\in\mathbb{I}$, convergence is clear from (1) and the discussion preceding proposition 1. The only thing we need to check is whether the convergence is uniform on $\mathbb{I}$ in the asserted sense. If this is done, the representation formula for M^2 also follows. For p>1, the uniform convergence is a consequence of pointwise convergence and Doob's inequality. In case p=1, to show uniform convergence in probability, let $(M^m)_{m\in\mathbb{N}}$ be the sequence of martingales associated with M, defined in the proof of proposition 1.7. Then

$$\|(M^m - M)_1\|_2 \to 0 \quad (m\to\infty).$$

Now for $\lambda>0$, $k,l,m\in\mathbb{N}$ we have

$$\begin{aligned}
(3)\quad & \lambda P(\sup_{t\in\mathbb{I}} |\sum_{J\in\mathbb{K}_k} \Delta_{J^1\cap[0,t]}M \, \Delta_{J^2\cap[0,t]}M \\
& \qquad - \sum_{J\in\mathbb{K}_l} \Delta_{J^1\cap[0,t]}M \, \Delta_{J^2\cap[0,t]}M \,| > \lambda) \\
\le\ & \lambda P(\sup_{t\in\mathbb{I}} |\sum_{J\in\mathbb{K}_k} \Delta_{J^1\cap[0,t]}(M-M^m) \, \Delta_{J^2\cap[0,t]}M \,| > \lambda) \\
& + \lambda P(\sup_{t\in\mathbb{I}} |\sum_{J\in\mathbb{K}_k} \Delta_{J^1\cap[0,t]}M^m \, \Delta_{J^2\cap[0,t]}(M-M^m)| > \lambda) \\
& + \lambda P(\sup_{t\in\mathbb{I}} |\sum_{J\in\mathbb{K}_k} \Delta_{J^1\cap[0,t]}M^m \, \Delta_{J^2\cap[0,t]}M^m \\
& \qquad - \sum_{J\in\mathbb{K}_l} \Delta_{J^1\cap[0,t]}M^m \, \Delta_{J^2\cap[0,t]}M^m| > \lambda) \\
& + \lambda P(\sup_{t\in\mathbb{I}} |\sum_{J\in\mathbb{K}_l} \Delta_{J^1\cap[0,t]}(M^m-M) \, \Delta_{J^2\cap[0,t]}M^m| > \lambda) \\
& + \lambda P(\sup_{t\in\mathbb{I}} |\sum_{J\in\mathbb{K}_l} \Delta_{J^1\cap[0,t]}M \, \Delta_{J^2\cap[0,t]}(M^m-M)| > \lambda) \\
\le\ & c_1 \{ \|(M^m - M)_1\|_2 \, (\|M^m_1\|_2 + \|M_1\|_2)\} \qquad \text{(proposition 2)} \\
& + \lambda P(\sup_{t\in\mathbb{I}} |\sum_{J\in\mathbb{K}_k} \Delta_{J^1\cap[0,t]}M^m \, \Delta_{J^2\cap[0,t]}M^m \\
& \qquad - \sum_{J\in\mathbb{K}_l} \Delta_{J^1\cap[0,t]}M^m \, \Delta_{J^2\cap[0,t]}M^m| > \lambda),
\end{aligned}$$

with a constant c_1 which does not depend on λ, k,l,m. Since the first term after the last inequality sign in (3) does not depend on k,l, and since for $m\in\mathbb{N}$ the martingale sequence

$(\sum_{J\in\mathbb{K}_k} \Delta_{J^1\cap[o,.]}M^m \, \Delta_{J^2\cap[o,.]}M^m)_{k\in\mathbb{N}}$ converges uniformly in $L^p(\Omega,\mathfrak{F},P)$ for any $p\geq 1$, according to what we already proved, the left hand side of (3) is seen to converge to 0 for any $\lambda>0$. This is what we had to show. □

Our final result is concerned with the continuity properties of the mixed martingale part of a square integrable martingale. Given the representation formula of theorem 1 and the fact that we know about the continuity properties of all other processes appearing therein, this is an easy task.

Theorem 2. Let M be a regular square integrable martingale, $\tilde{M}$ the regular martingale according to theorem 1. Then the set of discontinuities of $\tilde{M}$ is contained in the set of discontinuities of M and

i) $\tilde{M}$ has no 0-jumps, if M has no 0-jumps,

ii) $\tilde{M}$ has at most i-jumps, if M has at most i-jumps, $i=1,2$,

iii) $\tilde{M}$ is continuous, if M is continuous.

Proof: This follows from theorem 1, theorem (1.1) and proposition 1. □

Remark. It is possible to describe the jump components and the continuous component of $\tilde{M}$ by the respective components of M. We will refrain from doing so here.

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