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THE BEST ESTIMATION OF A RATIO INEQUALITY FOR CONTINUOUS MARTINGALES

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Let $M = (M_t)$ be a continuous local martingale with $M_0 = 0$. In this note, we deal with such a [local] martingale only. In [2] we have proved that if $\alpha < 1$, then the ratio inequality

$$E[\langle M\rangle_\infty^p \exp\{\alpha(\frac{\langle M\rangle_\infty^{\frac{1}{2}}}{M^*})^2\}] \leqq C_{\alpha,p} E[\langle M\rangle_\infty^p]$$

is valid for every $p > 0$. Our aim here is to establish <u>the inequality corresponding to the case where</u> $\langle M\rangle_\infty^{\frac{1}{2}}$ <u>and</u> M^* <u>are interchanged</u> in the above. For this purpose, we need a good-λ inequality involving $\langle M\rangle_\infty^{\frac{1}{2}}$ and M^*.

<u>Theorem</u>. i) Let $0 < \alpha < \frac{1}{2}$ and $p > 0$. Then, for any continuous local martingale $M = (M_t)$, we have

$$E[M^{*p} \exp\{\alpha(\frac{M^*}{\langle M\rangle_\infty^{\frac{1}{2}}})^2\}] \leqq C_{\alpha,p} E[M^{*p}], \tag{1}$$

where $C_{\alpha,p}$ is a universal constant depending on α and p only.

ii) If $\alpha \geqq \frac{1}{2}$, the inequality (1) is false for any $p > 0$.

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First of all, we shall exemplify ii). Let $B = (B_t)$ be a one dimensional Brownian motion starting at 0, and let $M_t = B_{t\wedge 1}$ $(0 \leqq t \leqq \infty)$. Since $\exp(\frac{1}{2}B_1^2)$ is not integrable and $\langle M \rangle_\infty = \langle B \rangle_1 = 1$, we find

$$\infty = E[\exp(\frac{1}{2}B_1^2) : |B_1| \geqq 1] \leqq E[M^{*p} \exp\{\frac{1}{2}(\frac{M^*}{\langle M \rangle_\infty^{\frac{1}{2}}})^2\}] .$$

On the other hand, it is clear that $M^* \in L^p$ for each p. It follows that the inequality (1) fails for any $p > 0$ and $\alpha \geqq \frac{1}{2}$.

Now, we shall prove i) of the theorem. For that, we need two lemmas. The following one is analogous to Corollary 1 in [3], and refers an integral inequality to a distribution function inequality

Lemma 1. Let X and Y be positive random variables. If there are two constants $a > 0$ and $c > 0$ such that

$$P\{X > \gamma\lambda , Y \leqq \lambda\} \leqq c \exp\{-a(\gamma - 1)^2\} P\{X > \lambda\}$$

holds for every $\gamma > 1$ and $\lambda > 0$, then for each $p > 0$ and $\alpha < a$ we have

$$E[X^p \exp\{\alpha(\frac{X}{Y})^2\}] \leqq C_{\alpha,p} E[X^p]$$

where $C_{\alpha,p}$ is a constant depending on α and p.

Proof. Note that $X = 0$ a.s. on $\{Y = 0\}$ by the assumption. Let $1 < \delta < (\frac{a}{\alpha})^{\frac{1}{4}}$, and for each $i \in \mathbb{Z}$ and $j \in \mathbb{N}$ let

$$\Lambda_{ij} = \{\delta^{i-1} < X \leqq \delta^i , \delta^{i-j-1} < Y \leqq \delta^{i-j}\} .$$

Since the complement of $\bigcup_{i\in\mathbb{Z}} \bigcup_{j\in\mathbb{N}} \Lambda_{ij}$ is included in $\{ Y = 0 \} \cup \{ X \leqq \delta Y \}$, we have

$$E[X^p \exp\{\alpha(\frac{X}{Y})^2\}] \leqq \sum_{i,j} E[X^p \exp\{\alpha(\frac{X}{Y})^2\} 1_{\Lambda_{ij}}] + \exp(\alpha\delta^2) E[X^p] ,$$

so that, it is sufficient to estimate the sum of the expectations in the above. By elementary computation, we have

$$\sum_{i,j} E[X^p \exp\{\alpha(\frac{X}{Y})^2\} 1_{\Lambda_{ij}}]$$

$$\leqq \sum_{i,j} \delta^{pi} \exp(\alpha\delta^{2j+2}) P\{ X > \delta^{i-1} , Y \leqq \delta^{i-j} \}$$

$$\leqq c \sum_{i,j} \delta^{pi} \exp\{\alpha\delta^{2j+2} - a(\delta^{j-1} - 1)^2\} p\{ X > \delta^{i-j} \}$$

$$= c \sum_{i,j} \delta^{pj} \exp\{ (\alpha\delta^4 - a)\delta^{2j-2} + 2a\delta^{j-1} - a\}\delta^{p(i-j)} P\{ X > \delta^{i-j} \}$$

$$= c[\sum_{j\in\mathbb{N}} \delta^{pj} \exp\{ (\alpha\delta^4 - a)\delta^{2j-2} + 2a\delta^{j-1} - a \}][\sum_{m\in\mathbb{Z}} \delta^{pm} P\{ X > \delta^m \}].$$

The first series of the right side converges and the second one is dominated by $\delta^p(\delta^p - 1)^{-1} E[X^p]$, so the proof is completed.

We are now going to prove the inequality:

$$(2) \qquad P\{ M^* > \gamma\lambda , \langle M \rangle_\infty^{1/2} \leqq \lambda \} \leqq c \exp\{-\frac{1}{2} (\gamma - 1)^2\} P\{ M^* > \lambda \}$$

for every $\gamma > 1$ and $\lambda > 0$. This result is essentially due to R. Bañuelos [1]. We shall give a more simple proof of it.

<u>Lemma 2.</u> Let $M = (M_t)$ be a continuous martingale such that $\langle M \rangle_\infty \leqq K$ a.s. Then the inequality:

$$(3) \qquad P\{ M^* > \lambda \} \leqq c \exp(-\frac{\lambda^2}{2K})$$

holds for every $\lambda > 0$.

<u>Proof</u>. Consider the process $Z_t = \exp(M_t - \frac{1}{2}\langle M \rangle_t)$, which is clearly a uniformly integrable martingale. Noticing that $Z_0 = 1$, we have $E[\exp(M_\infty - K/2)] \leqq E[Z_0] = 1$, so that $E[\exp(M_\infty)] \leqq \exp(K/2)$. Since this is valid for $-M$, we have

$$E[\exp(|M_\infty|)] \leqq 2\exp(\frac{K}{2})$$

and replacing M by $\frac{\lambda}{K}M$, we obtain

$$E[\exp(\frac{\lambda}{K}|M_\infty|)] \leqq 2\exp(\frac{\lambda^2}{2K})$$

On the other hand, we have $E[M^{*n}] \leqq 4E[|M_\infty|^n]$ for $n \geqq 2$ by Doob's inequality and $E[\frac{\lambda}{K}M^*] \leqq \text{const.} \exp(\lambda^2/2K)$ by Davis' inequality. Thus, expanding exp., we find

$$E[\exp(\frac{\lambda}{K}M^*)] \leqq 2\exp(\frac{\lambda^2}{2K})$$

Then (3) follows at once from Chebyshev's inequality.

By conditioning (3), we obtain

$$(3') \qquad P\{ M^* - M^*_T > \lambda \ , \ T < \infty \} \leqq c \exp(-\frac{\lambda^2}{2K}) P\{ T < \infty \}$$

for each stopping time T, where $M^*_t = \sup_{s \leqq t} |M_s|$. We are now ready to prove (2)

<u>Proof of i) of the theorem</u>. For each fixed $\lambda > 0$, we define

the two stopping times σ and τ as follows:

$$\sigma = \inf\{ t \geqq 0 : \langle M \rangle_t^{\frac{1}{2}} > \lambda \} \quad , \quad \tau = \inf\{ t \geqq 0 : M_t^* > \lambda \}$$

Obviously we have $\langle M^\sigma \rangle_\infty = \langle M \rangle_\sigma \leqq \lambda^2$ a.s., where M^σ denotes the stopped martingale $(M_{t\wedge\sigma})$. Hence by (3′) we obtain

$$\begin{aligned} P\{ M^* > \gamma\lambda \ , \ \langle M \rangle_\infty^{\frac{1}{2}} \leqq \lambda \} &\leqq P\{ M^* - M_\tau^* > (\gamma - 1)\lambda \ , \ \sigma = \infty \ , \ \tau < \infty \} \\ &\leqq P\{ (M^\sigma)^* - (M^\sigma)_\tau^* > (\gamma - 1)\lambda \ , \ \tau < \infty \} \\ &\leqq c \exp\{ - \frac{1}{2} (\gamma - 1)^2 \} P\{ \tau < \infty \} \end{aligned}$$

which is just (2).

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