JOHN R. BAXTER
RAFAEL V. CHACON

Multiplicative functionals and the stable topology

Séminaire de probabilités (Strasbourg), tome 23 (1989), p. 475-489
<http://www.numdam.org/item?id=SPS_1989__23__475_0>

Multiplicative Functionals and the Stable Topology

J.R. Baxter and R.V.Chacon
Department of Mathematics, University of Minnesota, Minneapolis, MN 5455, and
Department of Mathematics, University of British Columbia, Vancouver, BC V6T 1Y4

Abstract
The notion of a randomized stopping time has various applications in probability. Here it is shown that stable compactness for randomized stopping times is especially useful in the case of randomized stopping times which happen to be multiplicative functionals . The general results on convergence of multiplicative functionals are used to simplify the analysis of the convergence of diffusions in regions with many small holes.

1. Introduction

The stable topology for stopping times and time changes was originally developed as an aid to various constructions in the study of Markov processes and martingales [2],[14]. It was later used in the study of optimal stopping problems [11]. More recently [3],[4] it has proved useful in studying the behaviour of a diffusion in a region with many small holes (see Section 7). Some additional properties of stable convergence for stopping times associated with multiplicative functionals, which are especially useful in dealing with convergence of diffusions, were found in [5]. The purpose of the present paper is partly expository, to draw together these recent results on stable convergence and explain their applications. We will also give some new results on which relate to pointwise convergence of diffusions (Section 3), and a uniqueness condition for multiplicative functionals (Section 5). This allows us to make a considerable simplification in the proofs of earlier results on convergence of diffusions (Section 7). We will also discuss the connection between stable convergence and the variational Γ-convergence (Section 6).

2. Stopping Times

Let (Ω,F,F_t,X_t,P) be a stochastic process taking values in R^d, (F_t) assumed to be right continuous, and for convenience we also take F to be countably generated mod P. Let $\underline{\Omega}=\Omega \times[0,1]$, and let $\underline{P} = P\times\lambda_1$, where λ_1 is ordinary Lebesgue measure on the Borel sets B_1 of $[0,1]$. We will consider any function on Ω to be defined on $\underline{\Omega}$ in the obvious way. Let $G = F\times\{\varnothing,[0,1]\}$, $G_t = F_t\times\{\varnothing,[0,1]\}$. We will speak of a map $\tau: \underline{\Omega} \to[0,\infty]$ which is a stopping time with respect to the fields G_t as an ordinary stopping time for the process (X_t), and a stopping time T with respect to the fields $F_t \times B_1$ will be referred to as a randomized stopping time for the process (X_t). As a convenient regularization, we will assume that for any randomized stopping time T we consider, $T(\omega, \cdot)$ is nondecreasing and left continuous for every ω, and $T(\omega, 0)=0$ for every ω. We should use $\underline{P}$ and $\underline{E}$ to refer to probabilities and expectations on $\underline{\Omega}$, but we will often just use P and E when the meaning seems clear from the context.

For each $\omega \in \Omega$, we can define a probability measure $F(\omega, \cdot)$ on the Borel sets of $[0,\infty]$ by

(2.1) $$F(\omega,[0,t]) \equiv \sup \{a : T(\omega,a) \le t\}.$$

$F(\omega,\cdot)$ is a regular version of the conditional probability distribution of T given G, which we will refer to as the path distribution of T. We will denote $F(\omega,(0,\infty])$ by $F_t(\omega)$, and will refer to the family (F_t) as the survival function of T. It is easy to see that the survival function of a randomized stopping time determines the stopping time, and the fact that F_t is G_t-measurable expresses the stopping time measurability of T. Thus the notion of survival function contains the same information as the notion of randomized stopping time. Throughout this paper we will refer to the path distribution of a randomized stopping time and its survival function by the same letter, using the t subscript to distinguish the two quantities.

Definition 2.1 Let T_n, T be a randomized stopping times. We will say that T_n converges stably to T (with respect to P) if $T_n|_A$ converges in distribution to $T|_A$ (with respect to P) for each A in G. We emphasize that stable convergence is always with respect to some probability measure P. If T_n,T are randomized stopping times with survival functions F_n,F respectively, and T_n converges stably to T with respect to P, then we will say that F_n converges stably to F with respect to P.

It is shown in [2] that there is a compact metrizable topology on the space of all randomized stopping times, which we will call the stable topology, and that stable convergence is just convergence with respect to the stable topology. We will not bother to define this topology explicity, since we will only need to deal with sequential convergence. Since stable convergence is just a slightly enhanced form of weak convergence, its properties follow a familiar pattern. Probably the clearest reference for general properties of stable convergence is Meyer's paper [14]. We will need the following result from [5], which follows easily from Theorem 7 of [14]:

Lemma 2.1 Let T(n) converge to T stably with respect to P. Let $Y : \Omega\times[0,\infty] \to R$ be given. We will write $Y(\cdot,t)$ as Y_t where convenient. Suppose Y is bounded and $G\times B$ measurable, where B denotes the Borel sets on $[0,\infty]$.

(i) Suppose $Y(\omega,\cdot)$ is upper (lower) semicontinuous for P-a.e. ω. Then
$$\limsup_{n\to\infty} (\liminf_{n\to\infty}) \int Y_{T(n)}dP \le (\ge) \int Y_T dP.$$

(ii) Suppose $Y(\omega,\cdot)$ is continuous at T, P-a.e. Then
$$\lim_{n\to\infty} \int Y_{T(n)}dP = \int Y_T dP.$$

Corollary 2.1 Let T_n, T be randomized stopping times.

(i) Let σ be an ordinary G_t-stopping time. If $T_n \to T$ stably then $T_n \wedge \sigma \to T \wedge \sigma$ stably.

(ii) Let σ_j be a sequence of ordinary G_t-stopping times, $\sigma_j \uparrow \infty$. If $T_n \wedge \sigma_j \to T \wedge \sigma_j$ stably for each j, then $T_n \to T$ stably.

Corollary 2.2 Suppose that the process X. is continuous P-a.e.. If $T(n) \to T$ stably then $X_{T(n)} \to X_T$ in distribution.

Corollary 2.1 says we can truncate stopping times if we find it convenient to do so, when studying their stable convergence. This allows us to localize arguments, as in the proof of Theorem 7.3 below.

It should be noted that the result of Corollary 2.2 actually holds for any reasonable process, not necessarily continuous [2] [14]. However, this result is not quite as elementary as Lemma 2.1, and will not be needed in this paper, so we omit it.

Corollary 2.2 is a statement about the part of the process X which is stopped at time t. It is natural to also examine the convergence of the part of the process which is not yet stopped at time t. Before doing this, let us add a little more structure.

Let $(\Omega, F, F_t, B_t, \theta_t, P^x)$ be a Brownian process taking values in R^d. Here P^x is the probability measure such that $P^x(B_0 = x) = 1$, and we will as usual let P^ν denote the probability measure such that B_0 has distribution ν with respect to P^ν. Also, θ_t denotes the usual shift operator, so that $B_t \circ \theta_s = B_{s+t}$. We do not assume that Ω is the space of continuous functions, but we do assume that B. is continuous with probability 1.

Our previous definition of stable convergence applies to the Brownian motion case, of course. Since there are many probability measures P^ν available now, we will use the following terminology: if a sequence of randomized stopping times T_n converges stably to a limit T with respect to P^ν, we will say T_n converges to T <u>ν-stably</u>, and we likewise say that the survival functions converge ν-stably. If T_n converges ν-stably to T for some ν such that Lebesgue measure m on R^d is absolutely continuous with respect to ν, then we will say that T_n converges to T <u>Lebesgue-stably</u>. If T_n converges ν-stably to T for ν equal to the point measure concentrated at x, we will say T_n converges to T <u>x-stably.</u>

<u>Theorem 2.1</u> Let T_n,T be randomized stopping times for Brownian motion with $T_n \to T$ stably with respect to P^ν. Let $\nu(n,t)$, $\nu(t)$ denote the (defective) distributions of B_t restricted to $\{T_n > t\}$, $\{T_n > t\}$, respectively. Let s_n,s be such that $s_n \to s$ and $P^\nu(T=s)=0$. Then $\nu(n,s_n) \to \nu(s)$ in total variation norm as $n \to \infty$.

Theorem 2.1 is proved in [3]. It is a straightforward consequence of the continuity of the Brownian transition probabilities. It is significant here because it provides the link between stable convergence and Problem A of Section 7.

The following trivial consequence of the definition of stable convergence is sometimes useful:

<u>Lemma 2.2</u> Suppose T_n converges stably to T with respect to a probability measure P. Let P_1 be any probability measure which is absolutely continuous with respect to P. Then T_n also converges stably to T with respect to P_1.

<u>3. Multiplicative Functionals</u>

Once again let $(\Omega, F, F_t, B_t, \theta_t, P^x)$ be a Brownian process taking values in R^d. We will say that a property which holds P^x-a.e. for all x holds <u>almost surely</u> (a.s.). A stopping τ will be called a <u>terminal time</u> if

(3.1) $$\tau = t + \tau \circ \theta_t \text{ on } \{\tau > t\} \text{ a.s.}.$$

Any first hitting time has this property. The analogous property for randomized stopping times is conveniently expressed in terms of the survival function: let T be a randomized stopping time with survival function (F_t). We will say that (F_t) is a <u>multiplicative functional</u> if for all $s \geq 0$, $t \geq 0$, (3.2) $F_{t+s} = F_t F_s \circ \theta_t$ a.s..

Naturally, the survival function of a terminal time is a multiplicative functional. Unless otherwise stated, we will always assume that a multiplicative functional is exact (see [17]), by which we mean that for every $t > 0$ and every sequence $\varepsilon(k)$ of positive real numbers with $\varepsilon(k)\downarrow 0$,

(3.3) $\lim_{k \to \infty} F_{t-\varepsilon(k)} \circ \theta_{\varepsilon(k)} = F_t$ a.s..

The survival function of a first hitting time is exact.

A general reference for multiplicative functionals is [6].

Theorem 3.1 The space of multiplicative functionals is a closed set under Lebesgue-stable convergence with respect to P^ν, in the following sense: if a survival function F is the stable limit with respect to P^ν of a sequence of multiplicative functionals, where ν is a probability measure which is mutually absolutely continuous with respect to Lebesgue measure, then we can find a multiplicative functional M, such that $F=M$, P^ν-a.e..

Theorem 3.1 is proved in [5]. Theorem 3.1 simplifies the conceptual picture of convergence of hitting times, as we shall see.

The next lemma gives a typical property of multiplicative functionals. It follows (cf.[5]) from the fact that the Brownian transition densities are absolutely continuous with respect to Lebesgue measure.

Lemma 3.1 Let T be a randomized stopping time whose survival function is a multiplicative functional. Then for any $t > 0$, $P^x(T=t)=0$ for all x.

Definition 3.1 Let $M=(M_t)$ be a multiplicative functional. A point x such that $P^x(M_0=0)=1$ is called a permanent point for M. By the Blumenthal 0-1 law, if a point x is not a permanent point for M then $P^x(M_0=1)=1$ (Note $M_0^2=1$).

Lemma 3.2 Let M(n), M be multiplicative functionals such that M(n) converges Lebesgue-stably to M. Let x be permanent point for M. Then M(n) converges x-stably to M.

Proof Let μ_t denote the distribution of B_t with respect to P^x. For $t > 0$, μ_t is absolutely continuous with respect to m. Hence, by Lemma 2.2, M(n) converges stably to M with respect to P^{μ_t}. Using Lemma 2.1, or simply noting that stable convergence implies convergence in distribution, for any $s\geq 0$ $\lim\sup_{n\to\infty} E^{\mu_t}[M(n)([s,\infty))] \leq E^{\mu_t}[M([s,\infty)]$. $E^x[M(n)_{t+s}] \leq E^{\mu_t}[M(n)([s,\infty))]$ for all n, by the multiplicative property and the Markov property, so $\lim\sup_{n\to\infty} E^x[M(n)_{t+s}] \leq E^{\mu_t}[M_r]$ for all $s > r > 0$. For any $u>0$, taking $0<s<u$, letting t approach 0, and using exactness, we have $\lim\sup_{n\to\infty} E^x[M(n)_u]=0$ for all $u> 0$. The result then follows from the definition of stable convergence, since $T(n)\to 0$ in P^x-probability.

Lemma 3.3 Let M be a multiplicative functional, T the associated randomized stopping time. Let f be bounded and continuous on $[0,\infty]$, such that with probability 1 f is continuous at T. Let Y be integrable and G-measurable. Suppose M_n is a sequence of multiplicative functionals and $M_n \to M$ Lebesgue-stably. Then for every $\varepsilon>0$ and every measure ν, $E^\nu[Y\circ\theta_\varepsilon M_t(n)\circ\theta_\varepsilon] \to E^\nu[Y\circ\theta_\varepsilon M_t\circ\theta_\varepsilon]$, for all $t>0$.

Proof Let μ be distribution of B_ε with respect to P^ν.

$E^\nu[Y\circ\theta_\varepsilon M_t(n)\circ\theta_\varepsilon]=E^\mu[Y M_t(n)] \to E^\mu[YM_t]=E^\nu[Y\circ\theta_\varepsilon M_t\circ\theta_\varepsilon]$, using Lemmas 2.1 and 3.1.

Lemma 3.4 Let M(n),M be multiplicative functionals such that $M(n)\to M$ Lebesgue-stably. Let ν be any probability on R^d. Then for any t>0, and any nonnegative integrable G-measurable function Y,

(3.4) $\limsup_{n\to\infty} E^{\nu}[YM_t(n)] \le E^{\nu}[YM_t]$.

If F is any survival function which is a stable limit point of (M(n)) with respect to P^{ν} then $F \le M$, P^{ν}-a.e.

Proof Let f be a bounded nonnegative continuous function on $(R)^d$, and let $Y=f(B_{t_1},\dots,B_{t_k})$. Fix t>0. Let $0<\varepsilon<t$.

$E^{\nu}[Y\circ\theta_\varepsilon M_t(n)] \le E^{\nu}[Y\circ\theta_\varepsilon M_{t-\varepsilon}(n)\circ\theta_\varepsilon] \to E^{\nu}[Y\circ\theta_\varepsilon M_{t-\varepsilon}\circ\theta_\varepsilon]$, by Lemma 3.3. Thus

$\limsup_{n\to\infty} E^{\nu}[Y\circ\theta_\varepsilon M_t(n)] \le E^{\nu}[Y\circ\theta_\varepsilon M_{t-\varepsilon}\circ\theta_\varepsilon]$.

Fix $\delta>0$. For all sufficiently small ε, $E^{\nu}[|Y\circ\theta_\varepsilon - Y|]<\delta$.

Hence $\limsup_{n\to\infty} E^{\nu}[YM_t(n)] \le E^{\nu}[Y\circ\theta_\varepsilon M_{t-\varepsilon}\circ\theta_\varepsilon]+\delta$ for all sufficiently small ε. Letting $\varepsilon\downarrow 0$ and using (3.3) proves (3.4) for our special choice of Y. But such Y are dense in $L^1(P^{\nu})$-norm, so (3.4) is proved. Lemma 2.2(i) gives the rest of the lemma at once, since $\liminf_{n\to\infty} E^{\nu}[YM_t(n)] \ge E^{\nu}[YF_t]$.

Lemma 3.5 Let M(n),M be multiplicative functionals such that $M(n)\to M$ Lebesgue-stably. Let ν be any probability on R^d, such that

(3.5) $\lim_{t\downarrow 0} \liminf_{n\to\infty} E^{\nu}[M_t(n)]=1$.

Then $M(n)\to M$ ν-stably.

Proof Let F be any ν-stable limit point of (M(n)). We must show that F=M, P^{ν}-a.e..

By Lemma 3.4 we have $F\le M$, P^{ν}-a.e.. Thus it is enough to show that for each t>0, $E^{\nu}[M_t] \le E^{\nu}[F_t]$. By right-continuity we may restrict ourselves to t such that $E^{\nu}[F(\{t\})]=0$. Fix $\delta>0$. Choose $\varepsilon>0$ such that $\varepsilon<t$ and $\liminf_{n\to\infty} E^{\nu}[M_\varepsilon(n)]>1-\delta$. We have

$E^{\nu}[M_t(n)]=E^{\nu}[M(n)_\varepsilon M_{t-\varepsilon}(n)\circ\theta_\varepsilon] \ge E^{\nu}[M_{t-\varepsilon}(n)\circ\theta_\varepsilon]-E^{\nu}[1-M(n)_\varepsilon]$.

Hence $E^{\nu}[M_t(n)] \ge E^{\nu}[M_{t-\varepsilon}(n)\circ\theta_\varepsilon]-\delta$ for sufficiently large n. Hence, by Lemmas 2.1 , 3.1, and 3.3, $E^{\nu}[F_t] \ge E^{\nu}[M_{t-\varepsilon}\circ\theta_\varepsilon]-\delta \ge E^{\nu}[M_t]-\delta$, and the lemma is proved.

Theorem 3.2 Let ν be any probability on R^d and let M(n),M be multiplicative functionals such that $M(n)\to M$ ν-stably and Lebesgue-stably. Then there exists a subsequence n_k such that $M(n_k)\to M$ x-stably for ν-a.e. x.

Proof Fix t>0. Let $f_n(x)=E^x[M_t(n)]$, $f(x)=E^x[M_t]$. By (3.4), $\limsup_{n\to\infty} f_n \le f$. By Lemma 2.1(ii) and Lemma 3.1, $\int f_n d\nu \to \int f d\nu$. Since f_n is uniformly bounded, it follows that $f_n\to f$ in $L^1(\nu)$-norm. Let V be a countable dense set of times t>0. We can choose a set A with $P^{\nu}(A)=1$ and a subsequence (n_k) such that for x in A, $E^x[M_t(n)]\to E^x[M_t]$ for each t in V. Let x be in A. Let F be any x-stable limit point of $(M(n_k))$. By Lemma 2.1(i), for any t in V and any s<t, $E^x[M_t] \le E^x[F_s]$. It follows by Lemma 3.4 that F=M, P^x-a.e., and the theorem is proved.

The next theorem is proved in [5], as Remark 3.2.

Theorem 3.3 Let M(n),M be multiplicative functionals such that $M(n)\to M$ Lebesgue-stably. Let ν be any probability on R^d which gives measure 0 to polar sets. Then $M(n)\to M$ ν-stably.

We now discuss the semigroup and resolvent associated with a multiplicative functional.

Definition 3.2 With any multiplicative functional $M=(M_t)$ we associate the (sub-Markov) semigroup $P_t(M)$ defined by

(3.6) $$(P_t(M)h)(x)=E^x[h(B_t)M_t],$$

for every bounded Borel function h.

We note that $P_0(M)$ is not necessarily the identity operator. If P_t denotes the usual heat semigroup associated with Brownian motion, $P_t=P_t(1)$ in the present notation.

As a consequence of the exactness of M, equation (3.3), we have for each $t > 0$,

(3.7) $$\lim_{\varepsilon\downarrow 0} P_\varepsilon P_{t-\varepsilon}(M)h = P_t h.$$

If $h \geq 0$, the limit in (3.3) is nonincreasing. As a consequence, for any $h \geq 0$, the function $P_t(M)h$ is upper semicontinuous.

Definition 3.3 We will denote the resolvent associated with the semigroup $P_t(M)$ by $R_\alpha(M)$. That is,

(3.8) $$R_\alpha(M)=\int e^{-\alpha t} P_t(M)dt.$$

We note that it is a standard result, with an easy proof, that the resolvent $(R_\alpha(M))$ (and hence the semigroup $(P_t(M))$) uniquely determine the multiplicative functional M.

Both $P_t(M)$ and $R_\alpha(M)$, $\alpha > 0$, extend from operators on bounded measurable functions to bounded linear operators on $L^p(m)$, $1\leq p\leq\infty$, where m is Lebesgue measure on R^d.

The following theorem is proved in [5]. The proof is simplified by Theorem 3.2.

Theorem 3.4 Let $(M(n))$, (M) be multiplicative functionals. The following statements are equivalent:

(i) $M(n)\to M$ Lebesgue-stably.

(ii) For each $t>0$, $P_t(M(n))\to P_t(M)$ strongly on $L^2(m)$.

(iii) For each $\alpha>0$, $R_\alpha(M(n))\to R_\alpha(M)$ strongly on $L^2(m)$.

Proof (i)⇒(ii). Since $P_t(M)\leq P_t$, we may work with a dense set of functions in $L^2(m)$, such as continuous functions with compact support. For such a function f, $P_t(M(n_k))f\to P_t(M)f$ pointwise m-a.e. for some subsequence (n_k) of any given subsequence, by Theorem 3.2 and Theorem 2.1. It follows that $P_t(M(n_k))f\to P_t(M)f$ in $L^2(m)$, and (ii) is proved.

(ii)⇒(iii) is obvious, and (iii)⇒(i) follows from the compactness of the stable topology and the fact that the resovent characterizes the multiplicative functional.

It may be useful to give a summary here of the facts that we have proved concerning Lebesgue-stable convergence:

1. Lebesgue-stable convergence obeys a selection principle (Theorem 3.1).
2. Lebesgue-stable convergence implies

(i) x-stable convergence when x is a permanent point of the limit functional (Lemma 3.2);

(ii) ν-stable convergence when (3.5) holds (Lemma 3.5);

(iii) x-stable convergence for Lebesgue a.e. x (Theorem 3.2);

(iv) ν-stable convergence for any ν which does not charge polar sets (Theorem 3.3);

(v) strong convergence of the associated semigroups (Theorem 3.4).

4. Potentials and Additive Functionals

For what follows, we need to introduce the notion of the resolvent potential operator G_α. First, let $\phi_t(x)$ denote the usual transition density for Brownian motion on R^d. That is,

(4.1) $$\phi_t(x) = [2\pi t]^{d/2} \exp(-x^2/2t),\ t > 0,\ x \text{ in } R^d.$$

Define the α-resolvent kernel g_α by

(4.2) $g_\alpha(x) = \int_{[0,\infty)} e^{-\alpha t}\phi_t(x)dt.$

For any finite measure μ we then define the resolvent potential $G_\alpha\mu$ by

(4.3) $G_\alpha\mu = g_\alpha * \mu.$

We note that $G_\alpha\mu$ is a function, defined pointwise on R^d. It is the density of the usual resolvent measure associated with μ. $G_0\mu$ is infinite for d=1 and d=2. In all other cases we define $G_\alpha\mu$ is finite quasi-everywhere, and we define $G_\alpha\gamma$ for signed measures γ by additivity. We note that G_0 is the usual Newtonian or electrostatic potential operator when d=3. G_α, $\alpha > 0$, may be regarded as a potential that has the same behaviour as G_0 at short range, but drops off rapidly at long range. The larger α, the shorter the range of G_α. We note in passing that G_α is sometimes called the "Yukawa potential" when d=3.

For any two signed measures μ and ν, we define the α-mutual energy $<\mu,\nu>_\alpha$ by

(4.4) $<\mu,\nu>_\alpha = \iint g_\alpha(x-y)\,\mu(dx)\nu(dy),$

whenever $\iint g_\alpha(x-y)|\mu|(dx)|\nu|(dy)< \infty$, where $|\mu|$ denotes the total variation measure of μ.

We turn now to the construction of particular multiplicative functionals. We note that the logarithm of a multiplicative functional should be an additive functional, where we define a (nonnegative) additive functional $A=(A_t)$ to be a map A from $\Omega \times[0,\infty)$ to $[0,\infty]$ such that for any $s,t \geq 0$, t finite, if we write $A(\cdot, t)=A_t$,

(4.5) $A_{t+s} = A_t + A_s\circ\theta_t$ a.s. .

We assume that the analogue of exactness holds, i.e. for every $t > 0$ and every sequence $\varepsilon(k)$ of positive real numbers with $\varepsilon(k)\downarrow 0$,

(4.6) $\lim_{k\to\infty} A_{t-\varepsilon(k)}\circ\theta_{\varepsilon(k)} = A_t$ a.s. .

Of course if A_0 is finite a.s., so that $A_0 = 0$, (4.6) is equivalent to $\lim_{\varepsilon\downarrow 0} A_\varepsilon = 0$ a.s. .

It is clear that if $A=(A_t)$ is an additive functional, then $M=(e^{-A_t})$ is a multipicative functional. So in order to construct multiplicative functionals, we construct additive functionals. Let μ be a finite nonnegative Borel measure on R^d. Fix $\alpha\geq 0$ (if d=1 or d=2, $\alpha>0$). Suppose that $G_\alpha\mu$ is finite everywhere. By [6] we know there is a unique additive functional $A(\mu)=(A_t(\mu))$ such that for all $\beta\geq 0$ ($\beta>0$ if d=1 or d=2),

(4.7) $G_\beta\mu(x) = E^x[\int_{[0,\infty)} e^{-\beta t}dA_t(\mu)].$

$A(\mu)$ is everywhere finite and $A.(\mu)$ is continuous.

Of course, if μ happens to have a density h with respect to Lebesgue measure on R^d, the additive functional $A(\mu)$ has a particularly simple form: $A_t(\mu)=\int_{[0,t]}h(B_s)ds$. We will sometimes write $A(h)$ for $A(\mu)$ in this case. Also, if ν has a bounded density q with respect to μ, then $A(\nu)$ is given by

(4.8) $A_t(\nu) = \int_{[0,t]}q(B_s)dA_s(\mu).$

Let $\mathcal{M}$ denote the finite measures μ such that $G_\alpha\mu$ is finite everywhere for some (and hence for all) $\alpha\geq 0$ ($\alpha>0$ if d=1 or d=2). We have defined $A(\mu)$ for μ in $\mathcal{M}$. Let $\mathcal{M}_1$ denote the set of all measures μ such that μ is absolutely continuous with respect to some measure in $\mathcal{M}$. Let $L_t(\mu)$ denote the limit of $A_t(\mu_n)$, where μ_n is an increasing sequence of measures in $\mathcal{M}$ converging to μ. Let $A_t(\mu)=L_{t+}(\mu)$. It is a straightforward matter to show that $A(\mu)$ is an exact additive functional in our sense, and that $A.(\mu)$ is a continuous function on the interval where it is finite.

Definition 4.1 For any μ in $\mathcal{M}_1$, we define the multiplicative functional $M(\mu)=(M_t(\mu))$ associated with μ by $M_t(\mu)=e^{-A_t(\mu)}$.

5. The Integrated Condition for Uniqueness

Theorem 5.1 Let $A=(A_t)$ be an additive functional in the sense of Section 4, and assume that A is finite and continuous, even at $t=\infty$. Let $M=(M_t)$ be a multiplicative functional, and let T be the randomized stopping time associated with M. Suppose $M_0=1$ a.s.. Then $M=(e^{-A_t}) \Leftrightarrow E^x[A_T]=P^x(T<\infty)$ for all x.

We note that

(5.1) $$E^x[A_T]=E^x[\int A_t M(dt)]=E^x[\int_{[0,\infty)} M_t dA_t].$$

Proof of Theorem 5.1 $\Rightarrow$ is a simple computation, so we prove only $\Leftarrow$. Let $Y_t = M_t+\int_{[0,t]}M_t dA_t$. The Markov property and the multiplicative functional property, together with the hypothesis, show easily that (Y_t) is a $\mathcal{G}_t$-martingale. Since M_t is nonincreasing and $\int_{[0,t]}M_t dA_t$ is nondecreasing, Y_t has paths of bounded variation. Y_t is clearly right continuous. Since $(\mathcal{G}_t)$ is the family of fields generated by Brownian motion, Y_t must have continuous paths, and hence Y. is constant a.s.. The theorem then follows easily.

In order to apply Theorem 5.1 conveniently, we will prove an auxiliary lemma.

Lemma 5.1 Let T be a randomized stopping time whose survival function M is a multiplicative functional. Let A be a finite and continuous additive functional on $[0,\infty]$. For each $\alpha>0$, let $V(\alpha)$ be the randomized stopping time whose multiplicative functional is $(e^{-\alpha t})$. Extending our original sample space if necessary, we will consider $V(\alpha)$ to be defined (in defiance of our usual convention) so that it is independent of T as well as $\mathcal{G}$ Suppose that for sufficiently large α,

(5.2) $$P^x(T\le V(\alpha))\le(\ge)E^x[A_{V(\alpha)}].$$

Then $P^x(T<\infty)\le(\ge)E^x[A_T]$. Furthermore, if $P^x(T\le V(\alpha))\le E^x[A_{V(\alpha)}]$ for sufficiently large α then T has no permanent points.

Proof Modifying our original sample space if necessary, we may consider an iid sequence $V_j(\alpha)$ such that all the $V_j(\alpha)$ are together independent of T as well as $\mathcal{G}$. We also assume the $V_j(\alpha)$ are unaffected by the shift θ_t. Let $H_k(\alpha)=V_1(\alpha)+...+V_k(\alpha)$. For sufficiently large α we have $P^x(T<\infty)=$
$P^x(T\le H_1(\alpha))+\Sigma_{1\le k<\infty}P^x(H_k(\alpha)<T\le H_{k+1}(\alpha))= E^x[1-M_{H_1(\alpha)}]+\Sigma_{1\le k<\infty}E^x[M_{H_k(\alpha)}-M_{H_{k+1}(\alpha)}]=$
$E^x[1-M_{H_1(\alpha)}]+\Sigma_{1\le k<\infty}E^x[M_{H_k(\alpha)}E^{B_{H_k(\alpha)}}[1-M_{H_{k+1}(\alpha)-H_k(\alpha)}]]=$
$E^x[1-M_{V_1(\alpha)}]+\Sigma_{1\le k<\infty}E^x[M_{H_k(\alpha)}E^{B_{H_k(\alpha)}}[1-M_{V_{k+1}(\alpha)}]]\le(\ge)$
$E^x[A_{V_1(\alpha)}]+\Sigma_{1\le k<\infty}E^x[M_{H_k(\alpha)}E^{B_{H_k(\alpha)}}[A_{V_{k+1}(\alpha)}]]=$
$E^x[A_{V_1(\alpha)}]+\Sigma_{1\le k<\infty}E^x[M_{H_k(\alpha)}A_{V_{k+1}(\alpha)}\circ\theta_{H_k(\alpha)}]=$
$E^x[A_{V_1(\alpha)}]+\Sigma_{1\le k<\infty}E^x[1\{T>H_k(\alpha)\}(A_{H_{k+1}(\alpha)}-A_{H_k(\alpha)})]= E^x[A_{L(\alpha)}]$, where we define $L(\alpha)$ to be the first $H_k(\alpha)$ which is greater than or equal to T, $L(\alpha)=\infty$ if no such $H_k(\alpha)$ exists.

As $\alpha\to\infty$, $L(\alpha)\to T$ in probability. Since A is finite, we can use the dominated convergence theorem to conclude that $E^x[A_{L(\alpha)}]\to E^x[A_T]$, and the first statement of the lemma is proved. The hypothesis of the second assertion says that $E^x[A_{V(\alpha)}]\ge E^x[1-M_{V(\alpha)}]$, so the lemma follows at once.

6. The Variational Approach

Definition 6.1 Let M be a multiplicative functional. Let $\alpha\ge 0$ (if d=1 or d=2, $\alpha>0$). We will define the associated M-potential operator $G_\alpha(M)$ as follows: for any finite μ, (6.1) $$G_\alpha(M)\mu = G_\alpha\mu - G_\alpha\mu(M,\alpha),$$

where we define $\mu(M,\alpha)$ as the measure such that for any Borel set D in R^d, $\mu(M,\alpha)(D)= E^\mu[\int_{[0,\infty)} e^{-\alpha t}1_D(B_t)M(dt)]$. Since μ is finite, $G_\alpha\mu$ and $G_\alpha\mu(M,\alpha)$ are finite quasi-everywhere, and $G_\alpha(M)\mu$ is defined quasi-everywhere.

Lemma 6.1 $G_\alpha(M)\mu$ is the density of $\mu R_\alpha(M)$ with respect to Lebesgue measure m.

Proof Let D be a Borel set in R^d. $\int_D G_\alpha\mu(M,\alpha)dm=E^{\mu(M,\alpha)}[\int_{[0,\infty)} 1_D(B_u)e^{-\alpha u}du]= E^\mu[\int_{[0,\infty)}E^{B_t}[\int_{[0,\infty)} 1_D(B_u)e^{-\alpha u}du]e^{-\alpha t}M(dt)]= E^\mu[\int_{[0,\infty)}\int_{[0,\infty)} 1_D(B_{u+t})e^{-\alpha u}due^{-\alpha t}M(dt)]= E^\mu[\int_{[0,\infty)}\int_{[t,\infty)} 1_D(B_v)e^{-\alpha v}dvM(dt)]= E^\mu[\int_{[0,\infty)}(1-M_t)1_D(B_v)e^{-\alpha v}dv] = \mu R_\alpha(D)-\mu R_\alpha(M)(D)$, and the lemma follows.

Since a potential is determined by its integrals over balls with respect to Lebesgue measure, Lemma 6.1 shows that if $M_1 \le M_2$, then $G_\alpha(M)\mu_1 \le G_\alpha(M)\mu_2$ quasi-everywhere.

Lemma 6.2 Let ν be any measure in M. Then

$$\int G_\alpha(M)\mu d\nu = E^\mu[\int_{[0,\infty)} e^{-\alpha t}M_t dA_t(\nu)] = E^\mu[A_{T_\alpha}(\nu)],$$

where T_α is the randomized stopping time associated with $(M_t e^{-\alpha t})$.

Proof The second equality follows at once from Fubini. For the first, let $A_t=A_t(\nu)$. Using the symmetry of G_α, $\int G_\alpha\mu(M,\alpha)d\nu=E^{\mu(M,\alpha)}[\int_{[0,\infty)}e^{-\alpha u}dA_u]= E^\mu[\int_{[0,\infty)}E^{B_t}[\int_{[0,\infty)}e^{-\alpha u}dA_u]e^{-\alpha t}M(dt)]= E^\mu[\int_{[0,\infty)}\int_{[0,\infty)}e^{-\alpha u}dA_{u+t}e^{-\alpha t}M(dt)]= E^\mu[\int_{[0,\infty)}\int_{[t,\infty)}e^{-\alpha v}dA_vM(dt)] = E^\mu[\int_{[0,\infty)}(1-M_v)e^{-\alpha v}dA_v]= \int G_\alpha\mu d\nu-E^\mu[\int_{[0,\infty)}M_ve^{-\alpha v}dA_v]$, and the lemma follows.

Lemma 6.3 Let λ be in M. Let $M=M(\lambda)$. Then for any Borel set D in R^d,
$\int_D(G_\alpha(M)\mu)d\lambda = \mu(M,\alpha)(D)$.

Proof Let $\nu=1_D\lambda$ in Lemma 6.2. We find $\int_D(G_\alpha(M)\mu)d\lambda= E^\mu[\int_{[0,\infty)} 1_D(B_t)e^{-\alpha t}M_tdA_t(\lambda)] = E^\mu[\int_{[0,\infty)} 1_D(B_t)e^{-\alpha t}M(dt)] = \mu(M,\alpha)(D)$, so the lemma is proved.

For any measure λ in M_1, we define the set of finiteness $W(\lambda)$ for λ to be the union of all finely open sets D with finite λ-measure (see [5]). Because the fine topology has the quasi-Lindelof property, $W(\lambda)$ is λ-measurable and finely open up to a polar set.

Theorem 6.1 Let μ,λ be in M_1. Let $M=M(\lambda)$. For any Borel set D in R^d,
$\int_D(G_\alpha(M)\mu)d\lambda = \mu(M,\alpha)(D\cap W(\lambda))$.

Proof Since μ can be approximated from below by measures with bounded potentials, we may assume without loss of generality that $G_\alpha\mu$ is bounded. For λ in M, the result is already given in Lemma 6.3. Let $\lambda(n)$ be a nondecreasing sequence of measures in M such that $\lambda(n)\uparrow\lambda$. Let $M(n)=M(\lambda_n)$. Then $(e^{-\alpha t}M_t(n))\to(e^{-\alpha t}M_t)$ stably. Hence for any ν in M, by Lemma 6.2 $\int G_\alpha(M_n)\mu d\nu\downarrow\int G_\alpha(M)\mu d\nu$. Hence $G_\alpha(M_n)\mu\downarrow G_\alpha(M)\mu$ quasi everywhere. Let U be a finely open set with $\lambda(U)<\infty$. We can find fine continuous functions f_k with $0\le f_k\le 1$, $f_k=0$ on U^c, and $f_k\uparrow 1_U$ quasi everywhere. Since $f_k(B_t)$ is a continuous function of t, by Lemma 2.1 $\int f_k d\mu(M_n,\alpha)\to\int f_k d\mu(M,\alpha)$ as $n\to\infty$. By the dominated convergence theorem we have $\int f_kG_\alpha(M_n)\mu d\lambda(n)\to \int f_kG_\alpha(M)\mu d\lambda$. Thus $\int f_kG_\alpha(M)\mu d\lambda = \int f_k d\mu(M,\alpha)$ for all k, and so $\int 1_U G_\alpha(M)\mu d\lambda = d\mu(M,\alpha)(U)$. This gives the statement of the theorem provided that we restrict ourselves to D contained in $W(\lambda)$. But since $\int G_\alpha(M)\mu d\lambda(n) \le \int G_\alpha(M_n)\mu d\lambda(n) \le 1$ for all n, we have $\int G_\alpha(M)\mu d\lambda \le 1$. Since $G_\alpha(M)\mu$ is fine continuous, we see that $G_\alpha(M)\mu=0$ quasi-everywhere on the complement of $W(\lambda)$, and the theorem is proved.

Remark It would probably be better to study the complement of the set of permanent points of $M(\lambda)$ instead of $W(\lambda)$ (cf. Lemma 6.3 of [5]), but that would lead us further into technicalities.

Let λ in $\mathcal{M}_1$ be fixed. For each $\alpha>0$, define $R_\alpha(\lambda)$ on $L^2(dm)$ by $R_\alpha(\lambda)f=g$, where g is the weak solution in $H_0^1 \cap L^2(\lambda)$ of

(6.2) $$-\Delta g+(\lambda+\alpha m)g = f.$$

We interpret (6.2) to mean that

(6.3) $$\int \nabla g \cdot \nabla h dm + \int g h d(\alpha m+\lambda) = \int f h dm,$$

for all h in $H_0^1 \cap L^2(\lambda)$ with compact support. The existence of a unique solution to a weak equation of this form is shown in [8] (see also [5]).

The following is an easy variation on a standard fact (cf. [13]):

Lemma 6.4 Let μ be in $\mathcal{M}$, with $\int G_\alpha \mu d\mu<\infty$. Let $u=G_\alpha\mu$. Then u is in H_0^1, and for any h in H_0^1, $\int \nabla u \cdot \nabla h dm+\alpha\int uhdm=\int hd\mu$.

We wish to show

Theorem 6.2 If f in $L^2(m)$ is the density of a finite measure μ then $R_\alpha(\lambda)f=G_\alpha(M(\lambda))\mu$.

Proof Let $q=G_\alpha(M(\lambda))\mu$. Let $M=M(\lambda)$. Since G_α is a bounded operator on L^2, $\int G_\alpha\mu d\mu<\infty$. Hence $G_\alpha\mu$ is in H_0^1 by Lemma 6.4. Similarly, using Lemma 6.1, $\int G_\alpha\mu(M,\alpha)d\mu(M,\alpha)<\infty$, so $G_\alpha\mu(M,\alpha)$ is in H_0^1 by Lemma 6.4. Hence q is in H_0^1. By Theorem 6.1, $\int q^2 d\lambda = \int_{W(\lambda)} q d\mu(M,\alpha) \le \int q d\mu(M,\alpha) \le \int G_\alpha\mu d\mu(M,\alpha) = \int G_\alpha\mu(M,\alpha)d\mu \le \int G_\alpha\mu d\mu < \infty$. Thus q is in $H_0^1 \cap L^2(\lambda)$. Let h be in $H_0^1 \cap L^2(\lambda)$ with compact support. $\int \nabla q \cdot \nabla h dm+\alpha\int qhdm+\int qhd\lambda = \int hd\mu - \int hd\mu(M,\alpha) + \int qhd\lambda = \int hd\mu$ by Lemma 6.4 and Theorem 6.1, and the theorem is proved.

Theorem 6.2 is proved in [5] by a more complicated argument.

Theorem 3.4 shows we can characterize Lebesgue-stable convergence of multiplicative functionals in terms of strong resolvent convergence. Since the operators are uniformly bounded, we need only consider dense subsets of $L^2(m)$. Since the variational Γ-convergence studied in [7],[8] is also characterized in terms of strong resolvent convergence (cf. [5]), Theorem 6.2 links the two forms of convergence.

7. Convergence of Stopped Diffusions

We will now discuss the problem of describing the behaviour of a diffusion in a medium containing many small absorbing bodies (see Problem A below). This problem was solved by Papanicolaou and Varadhan [15]. Working with N.C. Jain, the authors were able to prove a stronger result along the same lines [3], using the compactness of the stable topology. The compactness of the stable topology made it possible for the proof of convergence to be reduced to a convenient uniqueness question. In the present exposition, we will use the fact that the set of multiplicative functionals is closed, and the uniqueness result of Section 5, to make the proof shorter. In addition, the results of Section 3 give criteria for convergence of solutions of the diffusion equation at a point, once Lebesgue-stable convergence has been shown, so we need only consider Lebesgue-stable convergence.

Problem dealing with media containing many small bodies have been considered by many authors, beginning with Mark Kac [12]. The earlier papers considered time-independent problems such as the Dirichlet problem, which are somewhat easier to deal with than the time-dependent diffusion case. Recent papers on other time-dependent problems include [16].

We now give a more precise definition of the the basic diffusion problem, which we call

Problem A Fix $d \geq 1$. Let D_n be a closed set in R^d, for n=1,2,... . Let some finite initial distribution measure ν be given. For each n, let$\rho_n(t,x)$ denote the solution of the diffusion equation

(7.1a) $\partial\rho_n/\partial t = \Delta\rho_n$ on D_n^c,

with intial condition

(7.1b) $\rho_n(t, \cdot) \to \nu$ on D_n as $t \downarrow 0$,

and boundary condition

(7.1c) $\rho_n(t,x) = 0$ for x in ∂D_n.

Condition 7.1c represents the absorption or "killing" of the diffusing material on the boundary of D_n. When ν has a density f, Condition 7.1b means $\rho_n(t, \cdot) \to$ f() on D_n. Condition 7.1b is to be interpreted in the sense of generalized functions otherwise. We will shortly reexpress all of Problem A in a more natural way using Brownian motion.

We now suppose that the sets D_n are the union of many small bodies, which become more and more finely divided as $n \to \infty$. The first part of Problem A is to give conditions under which the solutions ρ_n converge to a nontrivial limit . The second part of Problem A is to identify the limit of the sequence ρ_n.

We will give the precise solution to this problem later. In order to describe the limit of the sequence ρ_n, it is necessary to consider a second problem, which we will call

Problem B Fix $d \geq 1$. Let λ be a nonnegative measure on R^d, which is not necessarily finite or even Radon (that is, even the measure of compact sets may be infinite), but gives measure 0 to all polar sets. Let some finite initial distribution measure ν be given. Problem B is to find the solutionρ of the diffusion equation with "killing measure" λ, namely

(7.2a) $\partial\rho/\partial t = \Delta\rho - \lambda\rho$,

with intial condition

(7.2b) $\rho(t, \cdot) \to \nu$ as $t \downarrow 0$.

There are two parts to Problem B. The first part of the problem is to explain what equation (7.2a) means, for a general measure λ. When λ has a density h, we can interpret (7.2a) pointwise as $\partial\rho/\partial t = \Delta\rho - h\rho$. In general we can consider (7.2a) as the shorthand for a variational problem which defines ρ. This approach, due to Dal Maso and Mosco [8], [9], is described in [5]. We will not approach Problem B is this way, but will instead use the standard Feynman-Kac construction. The fact that the variational and the probabilistic solutions are consistent is shown in [5].

Assuming for the moment that we can deal with Problem B in a satisfactory manner, we can now state the connection between Problems A and B, namely that the limit ρ of the sequence ρ_n of Problem A is the solution of the equation in Problem B, for an appropriate choice of killing measure λ. The killing measure λ is of course determined by the sequence (D_n). Not all sequences D_n determine a killing measure, but it will turn out (Theorem 7.2) that if a sequence (D_n) does have a killing measure λ, and λ is reasonably nice, then the sequence ρ_n must converge to the solution of Problem B, that is, no further conditions need be imposed to give convergence. We will refer to the measure λ as the "limiting capacity measure" for the sequence (D_n) (See Definition 7.1 below).

Let W be a compact set in R^d. If d=1 or 2, let $\alpha > 0$. Otherwise, let $\alpha \geq 0$. There is a unique measure $\psi_{W,\alpha}$ on W such $G_\alpha \Psi_{W,\alpha} = 1$ quasi-everywhere on W. We will refer to this measure as the α-equilibrium measure of W. We define the α-capacity of W, $c_\alpha(W)$, by

(7.3) $$c_\alpha(W) = \psi_{W,\alpha}(W).$$

Now let a sequence (D_n) of closed sets be given. For any compact set W, let

(7.4) $$\gamma_\alpha(W) = \limsup_{n\to\infty} c_\alpha(W \cap D_n).$$

It can be shown (see Lemma 7.1 below) that there is a unique minimal outer regular measure $\lambda(\gamma_\alpha)$ such that $\lambda(\gamma_\alpha) \geq \gamma_\alpha$. We shall call this measure $\lambda(\gamma_\alpha)$ the total α-capacity measure for the sequence (D_n). This measure is in fact independent of α (see Lemma 7.5), so that we may denote it simply by λ.

Definition 7.1 If any subsequence of the original sequence of sets (D_n) has the same total capacity measure λ, then we will say that λ is the limiting capacity measure for the sequence (D_n).

Naturally, for this definition to be useful, we need a verifiable criterion for a sequence (D_n) to have a limiting capacity measure. Such a criterion is given in [4] , (generalizing a result in [15]), and we extend this criterion slightly (with essentially the same proof as in [4]) as

Theorem 7.1 Let B(n,i) be a compact set in R^d, for i=1,...,k(n). Let $D_n = B(n,1) \cup \ldots \cup B(n,k(n))$. Fix $\beta \geq 0$ (if d=1 or d=2, $\beta > 0$). For each n and i, let $\psi(n,i)$ denote the β-equilibrium measure of B(n,i). Let $\lambda_n = \psi(n,1) + \ldots + \psi(n,k(n))$. Suppose λ_n converges weakly to a limit λ, and that the "uniformity condition" holds:

$$\langle\lambda_n,\lambda_n\rangle_\beta \to \langle\lambda,\lambda\rangle_\beta.$$

Then λ is the limiting capacity measure for the sequence (D_n).

We can now give a more precise statement of the limit result described above.

Theorem 7.2 Let (D_n) be a sequence of closed sets in R^d, with a limiting capacity measure λ, such that the restriction of λ to any bounded region is a measure in $\mathcal{M}$. Let ν be any finite measure which measure 0 to polar sets. Then the limit of the sequence ρ_n of Problem A is the solution of Problem B. The limit is in the sense of $L^1(m)$-convergence of $\rho_n(t,\cdot)$ on R^d for each t, where m denotes Lebesgue measure on R^d, and the convergence is uniform over t in bounded intervals in $[0,\infty)$.

We note that more can be proved, in particular one can say something about the pointwise convergence of the ρ_n (see [3],[4]). However, the results of Sections 2 and 3 show that these results, as well as Theorem 7.2, follow at once from the following:

Theorem 7.3 Let (D_n) be a sequence of closed sets in R^d, with a limiting capacity measure λ, such that the restriction of λ to any bounded region is a measure in $\mathcal{M}$. Let τ_n denote the first hitting time of D_n, and let T denote the randomized stopping time associated with $M(\lambda)$. Then τ_n converges Lebesgue-stably to T.

From now on we will simply concentrate on proving Theorem 7.3, since it gives everything else. We will follow the arguments in [4] but will be able to simplify the proof considerably by using our earlier results.

Definition 7.2 A set function γ from the collection of compact subsets of R^d to $[0,\infty]$ will be called a c-function if is subadditive and maps the empty set to 0.

Lemma 7.1 Let γ be a c-function. There is a unique minimal outer regular measure $\lambda\geq\gamma$. For G open,

(7.5) $$\lambda(G)=\sup\{\gamma(W_1)+...+\gamma(W_k)\},$$

where the sup is over all disjoint sequences $W_1,...,W_k$ of compact subsets of G.

This lemma is undoubtedly a well-known result. A proof is given in [4]. Several interesting properties of this type of construction are given in [1].

Definition 7.3 We will refer to the measure λ of Lemma 7.1 as $\lambda(\gamma)$.

Definition 7.4 A collection U of bounded open sets will be called a good base if it is a base for the usual topology on R^d and is closed under finite unions.

Lemma 7.2 Let γ be a c-function, U a good base. Then $\lambda(\gamma)$ is uniquely determined by the values of γ on the collection of closures of sets in U.

This result is immediate from (7.5).

Lemma 7.3 Let γ_1, γ_2 be c-functions, $\delta>0$. Suppose $\gamma_1(W)\leq\gamma_2(W)$ for all compact W with diameter<δ. Then $\lambda(\gamma_1)\leq \lambda(\gamma_2)$.

Proof $\lambda(\gamma_1)(G)\leq \lambda(\gamma_2)(G)$ for all open G with diameter<δ by (7.5). Hence $\lambda(\gamma_1)(A)\leq \lambda(\gamma_2)(A)$ for all Borel A with diameter<δ, and the lemma follows.

Lemma 7.4 For every $\alpha,\beta\geq0$ ($\alpha,\beta>0$ if d=1 or d=2), for every $\varepsilon>0$, there exists $\delta>0$ such that $c_\beta(W)\leq(1+\varepsilon)c_\alpha(W)$ for W compact with diameter<δ.

Proof It is easy to see that there exists $\delta>0$ such that

(7.6) $$g_\alpha(x)\leq(1+\varepsilon)g_\beta(x) \text{ if } |x|<\delta.$$

Let W be compact with diameter<δ. Then $G_\alpha\Psi_{W,\alpha}=1$ quasi everywhere on W, so $G_\beta(1+\varepsilon)\Psi_{W,\alpha}\geq G_\beta\Psi_{W,\beta}$ on W, and hence on all of R^d by the domination principle. The lemma follows.

Lemma 7.5 Let $\alpha\geq0$ ($\alpha>0$ if d=1 or d=2). Let γ_α defined by (7.4). Then $\lambda(\gamma_\alpha)$ is independent of α.

Proof Immediate by Lemmas 7.3 and 7.4.

Lemma 7.6 Let (D_n) be any sequence of closed subsets of R^d. Then there exists a subsequence $(D_{n(k)})$ with a limiting capacity measure.

Proof Let U be a countable good base. Fix $\alpha\geq0$ ($\alpha>0$ if d=1 or d=2). Choose a subsequence (n(k)) such that $\lim_{n\to\infty}c_\alpha(W\cap D_n)$ exists for W the closure of a set in U. Any subsequence of $(D_{n(k)})$ must give the same total capacity measure as $(D_{n(k)})$, by Lemma 7.2, so the lemma is proved.

Lemma 7.7 Let D be compact, and let D_n be a sequence of subsets of D. For each $\alpha\geq0$ ($\alpha>0$ if d=1 or d=2), let ν_α be a weak limit point of $\Psi_{D_n,\alpha}$. Suppose (D_n) has a limiting capacity measure λ. Then $\nu_\alpha\to\lambda$ in total variation norm as $\alpha\to\infty$.

Proof This argument is just as in [4], Theorem 4.1.

Proof of Theorem 7.3 (i) Case 1. We first assume that there is a compact set D with D_n contained in D for all n. Let T be a Lebesgue-stable limit point of τ_n. We must show the survival function of T is $M(\lambda)$. Let M denote a multiplicative functional which is the survival function of T.

Fix $\alpha>0$. Let $V(\alpha)$ be the randomized stopping time with survival function $(e^{-\alpha t})$. Since $G_\alpha\Psi_{D_n,\alpha}(B_t)$ is a martingale with respect to the Brownian motion with lifetime $V(\alpha)$, we find as usual that $G_\alpha\Psi_{D_n,\alpha}(x)=P^x(\tau_n<V(\alpha))$. Let μ be a probability measure on R^d which is absolutely

continuous with respect to m, and has a bounded density, so that $G_\alpha\mu$ is bounded and continuous. We have

(7.7) $$\int G_\alpha\mu d\Psi_{D_n,\alpha}=P^\mu(\tau_n<V(\alpha)).$$

Let us extend our sample space if necessary, and consider $V(\alpha)$ to be defined so that it is independent of T as well as G. Let λ_α be a weak limit point of $\Psi_{D_n,\alpha}$. Then by Lemma 2.1 and Lemma 3.1 we have

(7.8) $$\int G_\alpha\mu d\lambda_\alpha=P^\mu(T<V(\alpha)).$$

That is,

(7.9) $$E^\mu[A_{V(\alpha)}(\lambda_\alpha)]=E^\mu[1-M_{V(\alpha)}].$$

Thus for all x and all t>0, $E^x[A_{V(\alpha)}(\lambda_\alpha)\circ\theta_t]=E^x[1-M_{V(\alpha)}\circ\theta_t]$. Letting $t\to 0$, we have

(7.10) $$E^x[A_{V(\alpha)}(\lambda_\alpha)]=E^x[1-M_{V(\alpha)}].$$

In particular, for all $\alpha>0$,

(7.11) $$E^x[A_{V(\alpha)}(\lambda)]\geq E^x[1-M_{V(\alpha)}].$$

By Lemma 5.1,

(7.12) $$M_0=1 \text{ a.s.},$$

and

(7.13) $$P^x(T<\infty)\leq E^x[A_T(\lambda)].$$

For any $\alpha>0$, for $\beta>\alpha$ we have $E^x[A_{V(\beta)}(\lambda_\alpha)]\leq E^x[A_{V(\beta)}(\lambda_\beta)]=E^x[1-M_{V(\beta)}]$. Hence by Lemma 5.1,

(7.14) $$P^x(T<\infty)\geq E^x[A_T(\lambda_\alpha)].$$

Letting $\alpha\to\infty$, since a.s. $A_t(\lambda_\alpha)\to A_t(\lambda)$ uniformly in t, we see that $P^x(T<\infty)=E^x[A_T(\lambda)]$, and so by Theorem 5.1 $M(\lambda)$ is the survival function for T, so the theorem is proved in Case 1.

(ii) The general case. Let H(j) be a sequence of bounded open sets with $H(j)\uparrow R^d$. Let $D_n(j)$ be the intersection of D_n with the closure of H(j). Without loss of generality, by choosing a subsequence, we may assume that $(D_n(j))_{n\geq 1}$ has a limiting capacity measure λ_j for each j. If $\tau_n(j)$ denotes the first hitting time of $D_n(j)$, we have by Case 1 that $\tau_n(j)\to T(j)$ Lebesgue-stably as $n\to\infty$, for each j, where T(j) is the randomized stopping time with survival function $M(\lambda_j)$. Clearly $\lambda_j=\lambda$ on compact subsets of H(j). Let σ_j denote the first exit time of H(j). Clearly $T(j)\wedge\sigma_j=T\wedge\sigma_j$, for each j, where T is the randomized stoppping time with survival function $M(\lambda)$. By Corollary 2.1(i), $\tau_n\wedge\sigma_j=\tau_n(j)\wedge\sigma_j\to T(j)\wedge\sigma_j=T\wedge\sigma_j$ as $n\to\infty$, for each j. Hence, by Corollary 2.1(ii), $\tau_n\to T$, and the theorem is proved.

This work was supported in part by the National Science Foundation (U.S.A.) and the National Science and Engineering Research Council of Canada.

References

1. M. Balzano, A derivation theorem for countably subadditive set functions, SISSA Preprint, Trieste, 1987.

2. J.R. Baxter and R.V. Chacon, Compactness of stopping times, Z. Wahr. und Verw. Gebiete 40 (1977), 169-181.

3. J.R. Baxter, R.V. Chacon, N.C. Jain, Weak limits of stopped diffusions, Trans. Amer. Math. Soc.293(1986), 767-792.

4. J.R. Baxter and N.C. Jain, Asymptotic capacities for finely divided bodies and stopped diffusions, Illinois Jour. Math.31(1987), 469-495.

5. J.R. Baxter, U. Mosco, G. Dal Maso, Stopping times and Γ-convergence, Trans. Amer. Math. Soc.303(1987), 1-38.

6. R.M. Blumenthal and R.K. Getoor, Markov Processes and Potential Theory, Academic, New York 1968.

7. G. Dal Maso, Γ-convergence and μ-capacities, SISSA Preprint, Trieste, 1986.

8. G. Dal Maso and U. Mosco, Wiener's criterion and Γ-convergence, Applied Math. Optimization 15(1987), 15-63.

9. G. Dal Maso and U. Mosco, Wiener criteria and energy decay estimates for relaxed Dirichlet problems, Arch. Rational Mech. Anal. 95(1986), 345-387.

10. J.L.Doob, Classical Potential Theory and Its Probabilistic Counterpart, Springer-Verlag, New York 1984.

11. G.A. Edgar, A. Millet, L. Sucheston, On compactness and optimality of stopping times, pp. 36-61, in Proc. Conf. on Martingales in Harmonic Analysis and Banach Spaces, Lecture Notes in Mathematics 939, Springer-Verlag, New York 1982.

12. M. Kac, Probabilistic methods in some problems in scattering theory, Rocky Mountain Jour. Math.4 (1974), 511-538.

13. N.S. Landkof, Foundations of Modern Potential Theory, Springer-Verlag, New York 1972.

14. P.-A. Meyer, Convergence faible et compacite des temps d'arret d'apres Baxter at Chacon, pp. 411-423 in Seminaire de Probabilities XII, Lecture Notes in Mathematics 649, Springer-Verlag, New York 1978.

15. G.C. Papanicolaou and S.R. Varadhan, Diffusions in regions with many small holes, pp. 190-206 in Stochastic Differential Systems- Filtering and Control, ed. B. Grigelionis, Lecture Notes in Control and Information Sciences 25, Springer-Verlag, New York 1980.

16. A.S. Sznitman, Propagation of chaos for a systerm of annihilating Brownian spheres, Commun. Pure and Appl. Math.40(1987), 663-690.

17. J.B. Walsh, The perfection of multiplicative functionals, pp. 233-242 in Seminaire de Probabilities VI, Lecture Notes in Mathematics 258, Springer-Verlag, New York 1972.