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THE PARTIAL MALLIAVIN CALCULUS

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1. Introduction

The notions of the partial Malliavin calculus were first introduced by Kusuoka and Stroock for the constant case (i.e. projections are taken on a fixed Hilbert subspace) and applied by them to prove regularity results in non-linear filtering theory. The theory of the partial Malliavin calculus has been developed in a different framework by Ikeda, Shigekawa and Taniguchi [4], in order to complete in detail the proof of some results of Malliavin (cf. [8]) on the long time asymptotics of stochastic oscillatory integrals.

The purpose of this paper is two fold, the first is to present an exposition of the approach of Ikeda, Shigekawa and Taniguchi [4] including some extensions and generalizations and the second is to derive conditions for the existence and smoothness of a conditional density under conditions which are more general than those considered previously.

In the next section we introduce the partial operators $D_{\mathcal{H}}$, $\delta_{\mathcal{H}}$ and $L_{\mathcal{H}}$ and following [4], we derive some properties of these operators. These operators are associated with a projection on a (possibly random) Hilbert subspace $\mathcal{H}$. In section 3 the subspace $\mathcal{H}$ is assumed to be the orthogonal complement of the Hilbert space induced by DG_i , $i \geq 1$ where $\{ G_i , i \geq 1 \}$ is a sequence of smooth random variables. The conditional integration by parts formula of [4] in this setup is derived in section 3. Sections 4 and 5 include new results (theorems 4.2 and 5.1) on the existence of a conditional density under relatively weak assumptions, these results were motivated by the results of Bouleau and Hirsch [3]. Conditions assuring the smoothness of the conditional density are discussed in section 5 (theorem 5.7), these results are based on the approach of [4] and [5]. The paper is concluded with an example related to the conditional law in the nonlinear filtering problem illustrating the direct applicability of the results of the earlier sections to this problem. This result states, very roughly, that for the one-dimensional nonlinear filtering and smoothing problem the existence of a non-conditional density implies the existence of a conditional density. It is also pointed out that the previously known results of Bismut and Michel [2] and of Kusuoka and Stroock [6] follow from the general approach presented here.

The rest of this section is devoted to establishing notation and to summarizing some basic results related to the Malliavin calculus. For a more detailed exposition of this subject c.f., e.g., Watanabe [11], Ikeda-Watanabe [5] or Zakai [12].

Let H be a real separable Hilbert space. Suppose that $W = \{ w(h), h \in H \}$ is a Gaussian process with zero mean and covariance function given by $E(w(h)w(g)) = \langle h, g \rangle$, defined in some probability space $(\Omega, \mathcal{F}, P)$. Here $\langle h, g \rangle$ denotes the scalar product in H. We also assume that $\mathcal{F}$ is generated by W.

Let E be another real separable Hilbert space. An E-valued random variable $F : \Omega \to E$ will be called *smooth* if

$$F = \sum_{i=1}^{M} f_i(w(h_1), \ldots, w(h_n))v_i \quad ,$$

where $f_i \in C_b^\infty(\mathbb{R}^n)$, $h_1, \ldots, h_n \in H$, and $v_1, \ldots, v_M \in E$.

Here $C_b^\infty(\mathbb{R}^n)$ denotes the set of C^∞ functions $f : \mathbb{R}^n \to \mathbb{R}$ which are bounded, together with all their derivatives.

The operator D is defined on E-valued smooth random variables as follows

$$DF = \sum_{i=1}^{M} \sum_{j=1}^{n} (\partial_j f_i)(w(h_1), \ldots, w(h_n))h_j \otimes v_i \quad .$$

That means, DF can be considered as an element of $L^2(\Omega ; H \otimes E)$. By iteration we introduce the N-th derivative of F, $D^N F$, which will be an element of $L^2(\Omega ; H^{\otimes N} \otimes E)$.

We also define the operator L by

$$LF = \sum_{i=1}^{M} \sum_{j=1}^{n} (\partial_j f_i)(w(h_1), \ldots, w(h_n))w(h_j)v_i - \sum_{i=1}^{M} \sum_{j,k=1}^{n} (\partial_j \partial_k f_i)(w(h_1), \ldots, w(h_n))$$
$$\cdot < h_j, h_k > v_i \quad .$$

In terms of the Wiener-Chaos decomposition, L coincides with the multiplication operator by the factor n.

Let F be a E-valued smooth random variable. For any $p > 1$ and $k \in \mathbb{R}$ we set

$$\|F\|_{p,k} = \|(I+L)^{k/2}F\|_p \quad , \tag{1.1}$$

and we denote by $\mathbb{D}_{p,k}(E)$ the Banach space which is the completion of the set of smooth functionals with respect to the norm (1.1). Set $\mathbb{D}_\infty(E) = \bigcap_{p,k} \mathbb{D}_{p,k}(E)$ and $\mathbb{D}_{-\infty}(E) = \bigcup_{p,k} \mathbb{D}_{p,k}(E)$. $\mathbb{D}_\infty(E)$ is the Fréchet space of tests random variables and $\mathbb{D}_{-\infty}(E)$ is its dual. For $E = \mathbb{R}$ we will simply write $\mathbb{D}_{p,k}$ for $\mathbb{D}_{p,k}(\mathbb{R})$.

The following equivalence of norms, proved by Meyer [9], provides a basic tool in studying the properties of the Sobolev spaces $\mathbb{D}_{p,k}$:

The Meyer inequalities: For any $p > 1$ and any positive integer k, there exists constants $a_{p,k}$, $A_{p,k} > 0$ such that

$$a_{p,k} \|D^k F\|_{L^p(\Omega; H^{\otimes k} \otimes E)} \le \|F\|_{p,k} \le A_{p,k} \left[\|F\|_{L^p(\Omega;E)} + \|D^k F\|_{L^p(\Omega;H^{\otimes k} \otimes E)} \right]$$

for any E-valued smooth functional F.

We introduce the operator δ, defined on H-valued smooth functionals $G = g(w(h_1), \ldots, w(h_n))h$ as follows

$$\delta(G) = g(w(h_1), \ldots, w(h_n))w(h) - \sum_{j=1}^{n} (\partial_j g)(w(h_1), \ldots, w(h_n)) < h_j, h > \quad . \tag{1.2}$$

Notice that $\delta(G)$ is a real valued random variable.

Finally we recall the following basic properties of these operators:

(1) The chain rule: If $F = (F_1, \ldots, F_m) \in \mathbb{D}_{2,1}(\mathbb{R}^m)$ and $\phi : \mathbb{R}^m \to \mathbb{R}$ is a C^1 function with bounded partial derivatives then

$$D\phi(F) = \sum_{i=1}^{m} (\partial_i \phi)(F) DF_i \quad . \tag{1.3}$$

(2) The integration by parts formula: If G and F are smooth random variables taking values in H and $\mathbb{R}$, respectively, then:

$$E(<G, DF>) = E(\delta(G)F) \quad . \tag{1.4}$$

This means that δ is the dual of D. If we denote by $Dom\,\delta \subset L^2(\Omega;H)$ the domain of the operator δ considered as the dual of the unbounded operator D on $L^2(\Omega)$ (with domain $\mathbb{D}_{2,1}$), then formula (1.4) holds for any $F \in \mathbb{D}_{2,1}$ and $G \in Dom\,\delta$.

(3) $LF = \delta DF$, for any F in the domain of L as an operator on $L^2(\Omega)$.

2. The operators $D_{\mathcal{H}}$, $\delta_{\mathcal{H}}$ and $L_{\mathcal{H}}$ associated with the projection on $\mathcal{H}$.

Let H be a real separable Hilbert space and $W = \{ w(h), h \in H \}$ a Gaussian process as defined in the previous section. Consider a (possible random) collection $\mathcal{H} = \{ K(\omega), \omega \in \Omega \}$ of closed subspaces $K(\omega)$ of H parameterized by ω, with a *measurable projection.* That means, we suppose that for any $h \in H$ the projection $\Pi_{K(\omega)}h$ is a measurable function of ω taking values in H. Namely, for every g in H, the scalar product of g with the projection of h on $K(\omega)$ is a real valued random variable. Notice that this implies that for any H-valued random variable $F : \Omega \to H$, $\Pi_{K(\omega)}(F(\omega)) : \Omega \to H$ is measurable. In fact, if $\{ e_i, \ i \geq 1 \}$ is a C.O.N.S. on H, we have $F = \sum_i <G, e_i> e_i$, and $\Pi_{K(\omega)}F = \sum_i <F, e_i> \Pi_{K(\omega)}e_i$. We will denote the random variable $\Pi_{K(\omega)}(F(\omega))$ by $\Pi_{\mathcal{H}}F$.

Definition 2.1. We define the *partial derivative operator* $D_{\mathcal{H}} : \mathbb{D}_{2,1} \to L^2(\Omega;H)$ as the projection of D on $\mathcal{H}$, namely, for any $F \in \mathbb{D}_{2,1}$,

$$D_{\mathcal{H}}F = \Pi_{\mathcal{H}}(DF) = \Pi_{K(\omega)}(DF)(\omega) \quad .$$

Some properties of this derivative:

(1) Let $F = f(w(h_1), \ldots, w(h_k))$ be a *smooth functional.* Then

$$DF = \sum_{i=1}^{k} (\partial_i f)(w(h_1), \ldots, w(h_k))h_i \quad \text{, and}$$

$$D_{\mathcal{H}}F = \sum_{i=1}^{k} (\partial_i f)(w(h_1), \ldots, w(h_k))\Pi_K h_i \quad .$$

Note that for any $h \in H$ we have

$$< D_{\mathcal{H}}F, h > \; = \; < DF, \Pi_K h > \; = \frac{d}{d\varepsilon}\Bigg|_{\varepsilon=0} F(\omega + \varepsilon \Pi_{K(\omega)}(h)) \; .$$

(2) The chain rule. Let $F_1, \ldots, F_m \in \mathbb{D}_{2,1}$ and let $\phi : \mathbb{R}^m \to \mathbb{R}$ be a continuously differentiable function with bounded first derivatives. Then $\phi(F) \in \mathbb{D}_{2,1}$, and

$$D_{\mathcal{H}}\phi(F) = \sum_{i=1}^{m} (\partial_i \phi)(F) D_{\mathcal{H}} F_i \; .$$

In fact, it suffices to project on $K(\omega)$ the ordinary chain rule for the derivative operator.

It is well known that D is a closed operator on $\mathbb{D}_{2,1}$. The assumptions of [4] assure that this remains true for $D_{\mathcal{H}}$ however we do not know whether this holds under the assumption of this paper. A convenient sufficient condition for this is the following lemma.

Lemma 2.2. If $\Pi_{\mathcal{H}} h \in Dom\, \delta$ for all $h \in H$, then $D_{\mathcal{H}}$ is a closed operator on $\mathbb{D}_{2,1}$.

Proof: For any $F \in \mathbb{D}_{2,1}$, we can write using integration by parts

$$E(< h, D_{\mathcal{H}} F >) = E(< \Pi_{\mathcal{H}} h, DF >) = E(\delta(\Pi_{\mathcal{H}} h) F) \; .$$

More generally, for any smooth H-valued random variable $G : \Omega \to H$ like $G = \sum_{i=1}^{m} \xi_i(\omega) h_i$, we have $\Pi_{\mathcal{H}} G \in Dom\, \delta$ (since $\Pi_{\mathcal{H}} h_i$ were assumed to be in $Dom\, \delta$, and the ξ_i are smooth), and

$$E(< G, D_{\mathcal{H}} F >) = E(< \Pi_{\mathcal{H}} G, DF >) = E(\delta(\Pi_{\mathcal{H}} G) F) \; . \tag{2.1}$$

This implies that $D_{\mathcal{H}}$ is closed since

$$\left.\begin{array}{ll} F_n \xrightarrow{L^2(\Omega)} 0 \; , & F_n \in \mathbb{D}_{2,1} \\[2em] D_{\mathcal{H}} F_n \xrightarrow{L^2(\Omega; H)} \eta & \end{array}\right\} \Rightarrow \eta = 0 \; .$$

In fact, setting $F = F_n$ in (2.1) and letting $n \to \infty$ yields the result. □

The converse of Lemma 2.2 is not true. Set, $\mathcal{H} = \{ K(\omega), \; \omega \in \Omega \}$, where $K(\omega) = \begin{cases} H & if \;\; w(e) \le 0 \; , \\ <e>^{\perp} & if \;\; w(e) > 0 \; , \end{cases}$ and where e is an element of H of norm one. For any $h \in H$ the H-valued random variable $\Pi_{\mathcal{H}} h = h - < h, e > \mathbf{1}_{\{ w(e) > 0 \}} e$ does not belong to $Dom\, \delta$, if $< h, e > \neq 0$; to see this, note that if $\mathbf{1}_{\{ w(e) > 0 \}} e \in Dom\, \delta$ then, if F is any smooth functional which vanishes in the set $\{ |w(e)| \le \varepsilon \}$ and $F = 1$ on $\{ |w(e)| > 2\varepsilon \}$ then $F \mathbf{1}_{\{ w(e) > 0 \}} \in \mathbb{D}_{2,1}$ and $D(F \mathbf{1}_{\{ w(e) > 0 \}}) = DF \cdot \mathbf{1}_{\{ w(e) > 0 \}}$. Consequently, we have by integration by parts (equation 1.4) that

$$E[F\delta(\mathbf{1}_{\{w(e)>0\}}e)] = E[\mathbf{1}_{\{w(e)>0\}}D_eF]$$
$$= E[D_e(F\,\mathbf{1}_{\{w(e)>0\}})] = E[F\,\mathbf{1}_{\{w(e)>0\}}w(e)]\ ,$$

which for ε small enough yields $E\ \delta(\mathbf{1}_{\{w(e)>0\}}e) > 0$ which is absurd since $E\ \delta = 0$. (cf. [10]). On the other hand, $D_{\mathcal{H}}$ is a closed operator in this case. In fact, suppose that $\{F_n\ ,\ n \geq 1\}$, is a sequence of functionals of $\mathbb{D}_{2,1}$, converging to zero in $L^2(\Omega)$, and such that

$$D_{\mathcal{H}}F_n = DF_n - < DF_n, e > \mathbf{1}_{\{w(e)>0\}}e \xrightarrow{L^2(\Omega;H)} \eta\ .$$

Suppose G is a H-valued smooth functional and let $\psi_\varepsilon : \mathbf{R} \to \mathbf{R}$ be a C^∞ function such that $\psi_\varepsilon(x) = 0$ if $x \leq 0$, $\psi_\varepsilon(x) = 1$ if $x \geq \varepsilon > 0$. Then using integration by parts and the above limit we deduce that $E(< G, \eta > \psi_\varepsilon(w(e))) = 0$ and $E(< G, \eta > \psi_\varepsilon(-w(e))) = 0$, which implies $\eta = 0$.

Definition 2.3. Set $Dom\,\delta_{\mathcal{H}} = \{u \in L^2(\Omega;H) : \Pi_{\mathcal{H}}u \in Dom\,\delta\}$, and for any $u \in Dom\,\delta_{\mathcal{H}}$, set $\delta_{\mathcal{H}}u = \delta\Pi_{\mathcal{H}}u$.

With this definition we have the following *integration by parts formula:*

$$\begin{aligned} E(F\delta_{\mathcal{H}}u) &= E(F\delta\Pi_{\mathcal{H}}u) \\ &= E(< DF, \Pi_{\mathcal{H}}u >) \\ &= (< D_{\mathcal{H}}F, u >)\ , \end{aligned} \tag{2.2}$$

for any $u \in Dom\,\delta_{\mathcal{H}}$ and $F \in \mathbb{D}_{2,1}$.

Notice that the condition in Lemma 2.2 implies that the H-valued smooth random variables belong to $Dom\,\delta_{\mathcal{H}}$. So, $Dom\,\delta_{\mathcal{H}}$ is a dense subset of $L^2(\Omega;H)$.

Some properties of the operator $\delta_{\mathcal{H}}$:

(1) Let $u \in Dom\,\delta_{\mathcal{H}}$, then it is clear from the definition that $\Pi_{\mathcal{H}}u \in Dom\,\delta_{\mathcal{H}}$, and $\delta_{\mathcal{H}}\Pi_{\mathcal{H}}u = \delta\Pi_{\mathcal{H}}u = \delta_{\mathcal{H}}u$.

(2) Let $u \in Dom\,\delta_{\mathcal{H}}$ and $F \in \mathbb{D}_{2,1}$. Then $Fu \in Dom\,\delta_{\mathcal{H}}$ and

$$\delta_{\mathcal{H}}(Fu) = F\delta_{\mathcal{H}}u - < u, D_{\mathcal{H}}F >\ , \tag{2.3}$$

provided that the right hand side is square integrable.

The proof is a direct consequence of the same result without $\mathcal{H}$ (see [10]).

Definition 2.4. Set

$$DomL_{\mathcal{H}} = \{F \in \mathbb{D}_{2,1} : D_{\mathcal{H}}F \in Dom\,\delta\} = \{F \in \mathbb{D}_{2,1} : DF \in Dom\,\delta_{\mathcal{H}}\}\ ,$$

and for any $F \in DomL_{\mathcal{H}}$ we define

$$L_{\mathcal{H}}F = \delta_{\mathcal{H}}D_{\mathcal{H}}F = \delta D_{\mathcal{H}}F\ .$$

Properties of the operator $L_{\mathcal{H}}$:

(1) It follows from property (2) of $\delta_{\mathcal{H}}$ that

$$L_{\mathcal{H}}\psi(F_1, \ldots, F_m) = \delta\Pi_{\mathcal{H}}(\sum_{i=1}^{m}(\partial_i\psi)(F)DF_i)$$

$$= L(\sum_{i=1}^{m}(\partial_i\psi)(F)D_{\mathcal{H}}F_i)$$

$$= \sum_{i=1}^{m}(\partial_i\psi)(F)L_{\mathcal{H}}F_i - \sum_{i,j=1}^{m}(\partial_i\partial_j\psi)(F) < D_{\mathcal{H}}F_i, D_{\mathcal{H}}F_j > \quad ,$$

provided that the components of $F = (F_1, \ldots, F_m)$ belong to $DomL_{\mathcal{H}}$, ψ is a smooth bounded function with bounded first and second partial derivatives, and $E(||DF_i||^4) < \infty$, $i = 1, \ldots, m$.

(2) Under the condition of Lemma 2.2, smooth functionals of the form $f(w(h_1), \ldots, w(h_k))$ belong to $DomL_{\mathcal{H}}$, and, therefore, $DomL_{\mathcal{H}}$ is dense in $L^2(\Omega)$.

3. The conditional integration by parts formula

Definition 3.1. A sub-σ-field $\mathcal{G}$ of $\mathcal{F}$ is said to be *countably smoothly generated* if $\mathcal{G}$ is generated by some sequence of random variables $\{ G_i \ , \ i \geq 1 \}$, such that $G_i \in \mathbb{D}_{2,1}$ for all $i \geq 1$.

Note that by taking $\tilde{G}_i = arctan \ G_i$, we may assume that the random variables generating the σ-field are bounded.

We can associate to $\mathcal{G}$ the family of subspaces defined by the orthogonal complement to the subspace generated by $\{ DG_i(\omega) \ , \ i \geq 1 \}$, i.e.,

$$K(\omega) = \ < DG_i(\omega) \ , \ i \geq 1 >^{\perp} \ . \tag{3.1}$$

It is clear that this family $\mathcal{H} = \{ K(\omega) \ , \ \omega \in \Omega \}$ has a measurable projection. This follows from the fact that for any $h \in H$ and $\omega \in \Omega$ we have

$$\Pi_{K(\omega)}h = \lim_n \{ h - \Pi_{< DG_1(\omega), \ldots, DG_n(\omega) >}(h) \} \ . \tag{3.2}$$

The next result gives a sufficient condition under which $\mathcal{H}$ is independent of the particular sequence of generators $\{ G_i \ , \ i \geq 1 \}$ of $\mathcal{G}$.

Proposition 3.2. Suppose that $\{ F_i \ , \ i \geq 1 \}$ and $\{ G_i \ , \ i \geq 1 \}$ generate the same σ-field $\mathcal{G}$, and $F_i, G_i \in \mathbb{D}_{2,1}$ for any $i \geq 1$. Assume that the families $\mathcal{H}_F = \{ \ < DF_i \ , \ i \geq 1 >^{\perp} \}$ and $\mathcal{H}_G = \{ \ < DG_i \ , \ i \geq 1 >^{\perp} \}$ are such that $D_{\mathcal{H}_F}$ and $D_{\mathcal{H}_G}$ are *closed operators.* Then $\mathcal{H}_F = \mathcal{H}_G$.

Proof: It suffices to show that $DF \in \ < DG_i \ , \ i \geq 1 >$ for any $\mathcal{G}$-measurable $F \in \mathbb{D}_{2,1}$. There exists a sequence $\psi_n(G_1, \ldots, G_n) \to F$ as $n \to \infty$, in $L^2(\Omega)$ and a.s. We may assume that the functions ψ_n are in $C_b^{\infty}(\mathbb{R}^n)$. Clearly $D_{\mathcal{H}_G}[\psi_n(G_1, \ldots, G_n)] = 0$, since the projection is on the orthogonal to $< DG_i \ , \ i \geq 1 >$. So $D_{\mathcal{H}_G}F = 0$ a.s., because $D_{\mathcal{H}_G}$ is closed, and this implies that $DF \in \mathcal{H}_G^{\perp} = \ < DG_i \ , \ i \geq 1 >$. □

Throughout this section we assume that $\mathcal{G} = \sigma\{ G_i \ , \ i \geq 1 \}$ is countably smoothly generated and $\mathcal{H} = \mathcal{H}_G$.

Lemma 3.3. Let $u \in Dom\ \delta_{\mathcal{H}}$ and let R be a random variable such that $R\,\delta_{\mathcal{H}}u$ is a square integrable random variable. If either

(a) $R = \psi(G_1, \dots, G_m)$ where ψ is a C^1 bounded function with bounded first derivatives, or

(b) $D_{\mathcal{H}}$ is closed and $R \in \mathbb{D}_{2,1}$ is $\mathcal{G}$-measurable and square integrable,

then $Ru \in Dom\ \delta_{\mathcal{H}}$ and

$$\delta_{\mathcal{H}}(Ru) = R\,\delta_{\mathcal{H}}u \quad . \tag{3.3}$$

The proof follows directly from the fact that in this case $D_{\mathcal{H}}R = 0$ and from equation (2.3).

Remark: As pointed out earlier, for $\mathcal{H}{=}\mathcal{H}_G$, $D_{\mathcal{H}}\psi(G_1, \dots, G_n) = 0$. This result and (3.3) indicate that, very roughly speaking, as far as $\delta_{\mathcal{H}}$ and $D_{\mathcal{H}}$ are concerned, $\mathcal{G}$ measurable random variables play the role of "frozen parameters" in the partial Malliavin calculus. This is also suggested by the following proposition:

Proposition 3.4.

(a) Conditional integration by parts formula: For any $F \in \mathbb{D}_{2,1}$ and $u \in Dom\ \delta_{\mathcal{H}}$, we have

$$E(< u, D_{\mathcal{H}}F > \mid \mathcal{G}) = E(F\,\delta_{\mathcal{H}}u \mid \mathcal{G})$$

(b) $L_{\mathcal{H}}$ is "conditionally self-adjoint": For any F, Q in the domain of $L_{\mathcal{H}}$,

$$E(QL_{\mathcal{H}}F \mid \mathcal{G}) = E(FL_{\mathcal{H}}Q \mid \mathcal{G}) \ .$$

Proof: Let $\psi : \mathbb{R}^m \to \mathbb{R}$ be a C^1-function bounded and with bounded derivatives. Set $R = \psi(G_1, \dots, G_m)$. Then by (2.2)

$$\begin{aligned} E(FR\,\delta_{\mathcal{H}}u) &= E(< D_{\mathcal{H}}(FR), u >) \\ &= E(< FD_{\mathcal{H}}R, u > + < RD_{\mathcal{H}}F, u >) \\ &= E(R < D_{\mathcal{H}}F, u >) \ , \end{aligned}$$

which proves the first part. The second part follows since

$$\begin{aligned} &E(QL_{\mathcal{H}}F \mid \mathcal{G}) = E(Q\,\delta_{\mathcal{H}}D_{\mathcal{H}}F \mid \mathcal{G}) \\ &= E(< D_{\mathcal{H}}Q, D_{\mathcal{H}}F > \mid \mathcal{G}) = E(FL_{\mathcal{H}}G \mid \mathcal{G}) \ . \qquad \square \end{aligned}$$

Corollary 3.5. $L_{\mathcal{H}}$ is "conditionally non-negative" in the sense that if $F \in DomL_{\mathcal{H}}$ then

$$E(FL_{\mathcal{H}}F \mid \mathcal{G}) \geq 0 \qquad a.s.$$

This follows directly from the conditional integration by parts formula.

Remark: Let $p(\omega, A)$ be a regular version of the conditional probability given $\mathcal{G}$. That means, $p : \Omega \times \mathcal{F} \to [0,1]$ is a stochastic kernel such that $p(\cdot, A)$ is $\mathcal{G}$-measurable, and

$$P(A \cap B) = \int_B p(\omega, A)\, dP(\omega) \ , \qquad \text{for } all \quad B \in \mathcal{G} \ .$$

Then, the second part of Proposition 3.4 means that, for almost every ω, the operator $L_{\mathcal{H}}$ is symmetric with respect to the probability $p(\omega, \cdot\,)$.

An important special case is the case where $\mathcal{G}$ is finitely smoothly generated, namely, $\mathcal{G} = \sigma\{ G_1, \dots, G_m \}$, where $G_i \in \mathbb{D}_{2,1}$, and moreover the Malliavin matrix $\gamma = \ < DG_i, DG_j >$ is assumed to be a.s. invertible. Then, for any $h \in H$,

$$\begin{aligned} \Pi_{\mathcal{H}} h &= h - \Pi_{< DG_1, \dots, DG_m >} h \\ &= h - \sum_{i,j=1}^{m} (\gamma^{-1})_{ij} \ < h, DG_j > DG_i \ . \end{aligned} \tag{3.4}$$

In this case, if we further assume that $(\gamma^{-1})_{ij} < h, DG_j > DG_i \in Dom\, \delta$ for any $h \in H$ then by Lemma 2.2 $D_{\mathcal{H}}$ is a closed operator on $\mathbb{D}_{2,1}$. This condition is satisfied if, for instance, $G_i \in \mathbb{D}_{p,2}$ and $E(\, | (\gamma_{ij}^{-1}) |^{\,p}\,) < \infty$ for $p \geq 8$, $\gamma_{ij}^{-1} \in \mathbb{D}_{2,1}$ and $\| D(\gamma_{ij}^{-1}) \| \in L^8$, since

$$\begin{aligned} \delta((\gamma^{-1})_{ij} < h, DG_j > DG_i) &= (\gamma^{-1})_{ij} < h, DG_j > \delta(DG_i) \\ &\quad -(\gamma^{-1})_{ij} < D^2 G_j, DG_i \otimes h >_{H \otimes H} - < h, DG_j > \ < D((\gamma^{-1})_{ij}), DG_i > \ . \end{aligned} \tag{3.5}$$

Therefore since for any $CONS \ \{ h_q \ , \ q \geq 1 \}$, $F \in \mathbb{D}_{2,1}$

$$D_{\mathcal{H}} F = \sum_{q=1}^{\infty} < DF, h_q > \Pi_{\mathcal{H}} h_q$$

it follows that

$$D_{\mathcal{H}} F = DF - \sum_{i,j=1}^{m} (\gamma^{-1})_{ij} < DF, DG_j > DG_i \tag{3.6}$$

In particular, for $m = 1$ we have

$$D_{\mathcal{H}} F = DF - \frac{< DF, DG >}{\| DG_1 \|} \mathbf{1}_{\{ \|DG_1\|_H \neq 0 \}} \cdot DG_1$$

Turning to $\delta_{\mathcal{H}} h$, it follows from (3.4) and (2.3) that:

$$\begin{aligned} \delta_{\mathcal{H}} h &= \delta h - \sum_{i,j=1}^{m} \delta[(\gamma^{-1})_{ij} < h, DG_j > DG_i] \\ &= \delta h - \sum_{i,j}^{m} (\gamma^{-1})_{ij} < h, DG_j > \delta DG_i + \\ &\quad + \sum_{i,j}^{m} < h, DG_j > \ < D(\gamma^{-1})_{ij}, DG_i > \\ &\quad + \sum_{i,j}^{m} (\gamma^{-1})_{ij} < D^2 G_j, h \otimes DG_i >_{H \otimes H} \ . \end{aligned} \tag{3.7}$$

4. The existence of a conditional density

In this section we derive two results regarding the existence of conditional densities. These results hold under relatively weak assumptions on the Malliavin derivatives but are restricted in other directions. For the first result the conditioning σ-field is restricted to be finitely smoothly generated. For the second result the last restriction is dropped, however it is assumed that the random variable for which the conditional density is obtained is one-dimensional (and not a finite dimensional vector as in the previous case). Both results are motivated by the work of Bouleau-Hirsch [3]. In the next section we consider stronger assumptions on the partial Malliavin matrix, without the restriction described above. Also, conditions for the smoothness of the density will be considered in the next section.

In this and the following section we assume that H is a real separable Hilbert space and $W = \{ w(h), h \in H \}$ is a Gaussian process defined as in section 1.

Theorem 4.1: Let $G_1, \ldots, G_n$ be elements of $\mathbb{D}_{2,1}$ satisfying $\det < DG_i , DG_j > \; > 0$ a.s. Set $\mathcal{H} = \{ K(\omega), \omega \in \Omega \}$ with $K(\omega) = < DG_i(\omega), \; i = 1, \ldots, n >^{\perp}$. Let $F = (F_1, \ldots, F_m)$, $F_i \in \mathbb{D}_{2,1}$ and assume that

$$\det < D_{\mathcal{H}} F_i \, , \, D_{\mathcal{H}} F_j > \; > 0 \qquad a.s.$$

Then, there exists a conditional density for the law of F given the σ-field $\sigma\{ G_1, \ldots, G_n \}$.

Proof: Consider the augmented vector

$$(G_1, \ldots, G_n \, , \, F_1, \ldots, F_m) \; .$$

Note that in order to prove the theorem it suffices to show that the augmented vector possesses a joint density. The determinant of the Malliavin matrix of the augmented vector is given by:

$$Q = \det \begin{bmatrix} < DG_i \, , DG_j > & < DG_i \, , DF_j > \\ < DG_i \, , DF_j >^T & < DF_i \, , DF_j > \end{bmatrix} \; . \tag{4.1}$$

The result of Bouleau and Hirsch is that if the above determinant is a.s. non zero then the augmented vector has a probability density.

On the other hand, it was shown by Ikeda, Shigekawa and Taniguchi (equation 3.29 of [4]) that

$$Q = \det[< DG_i \, , \; DG_j >] \cdot \det[< D_{\mathcal{H}} F_i \, , \; D_{\mathcal{H}\neq} F_j >] \tag{4.2}$$

where Q is as defined by (4.1). By our assumptions this expression is positive and this completes the proof. □

Theorem 4.2: Let $F \in \mathbb{D}_{2,1}$ be a real valued random variable, and $\underline{G} = (G_i \, , \; i \geq 1)$, $G_i \in \mathbb{D}_{2,1}$. Assume that $D_{\mathcal{H}}$ is a closed operator where $\mathcal{H}$ is induced by $\underline{G}$, that means, $\mathcal{H} = \{ K(\omega), \omega \in \Omega \}$ and $K(\omega) = < DG_i(\omega), i \geq 1 >^{\perp}$ (cf. Lemma 2.2). If $< D_{\mathcal{H}} F \, , D_{\mathcal{H}} F > \; > 0$ a.s., then F has a conditional density with respect to the sub-σ-field generated by $\underline{G}$.

Proof: Without any loss of generality we may assume that F is bounded, namely $| F | < 1$. Denote by $P_{\underline{G}}$ the probability law induced by $\underline{G}$ on $\mathbb{R}^{\infty}$. Then it suffices to show that the probability law induced by the vector $(F, \underline{G})$ on $(-1,1) \times \mathbb{R}^{\infty}$, denoted by $P_{(F,\underline{G})}$, is absolutely continuous with respect to the product measure $d\alpha dP_{\underline{G}}(\underline{x})$. In that case the Radon-Nikodym derivative

$$f(\alpha,\underline{x}) = \frac{dP_{(F,\underline{G})}(\alpha,\underline{x})}{d\alpha dP_{\underline{G}}(\underline{x})} \qquad \textbf{(4.3)}$$

will provide a version for the conditional density of F given $\underline{G} = \underline{x}$.

We have, therefore, to show that for any measurable function $g : (-1,1)\times\mathbb{R}^\infty \to [0,1]$ such that $\int g(\alpha,\underline{x})d\alpha dP_{\underline{G}}(\underline{x}) = 0$ we have $E[g(F,\underline{G})] = 0$. If g is such a function we have

$$\int g(\alpha,\underline{x})d\alpha = 0 \qquad (4.4)$$

for almost all $\underline{x}$ with respect to the law of $\underline{G}$. Consequently, there exists a sequence of continuously differentiable functions with bounded derivatives $g^n : (-1,1)\times\mathbb{R}^n \to [0,1]$ such that $g^n(\alpha,x_1,\ldots,x_n)$ converges to $g(\alpha,\underline{x})$ for almost all $(\alpha,\underline{x})$ with respect to the measure $dP_{(F,\underline{G})}(\alpha,\underline{x}) + d\alpha dP_{\underline{G}}(\underline{x})$. Take

$$\psi^n(y,x_1,\ldots,x_n) = \int_{-1}^{y} g^n(\alpha,x_1,\ldots,x_n)d\alpha$$

and

$$\psi(y,\underline{x}) = \int_{-1}^{y} g(\alpha,\underline{x})d\alpha \ .$$

Then $\psi^n(F,G_1,\ldots,G_n) \in \mathbb{D}_{2,1}$ and

$$D[\psi^n(F,G_1,\ldots,G_n)] = g^n(F,G_1,\ldots,G_n)DF + \sum_{i=1}^{n} \frac{\partial\psi^n}{\partial x_i}(F,G_1,\ldots,G_n)DG_i. \qquad (4.5)$$

We have

$$\psi^n(f,G_1,\ldots,G_n) \to \psi(F,\underline{G})$$

a.s., as $n\to\infty$, and in $L^2(\Omega)$ by dominated convergence. Because of (4.4) with $g(\alpha,\underline{x})$ nonnegative, it holds that $\psi(F,\underline{G}) = 0$ a.s. Now from (4.5)

$$D_{\mathcal{H}}[\psi^n(F,G_1,\ldots,G_n)] = g^n(F,G_1,\ldots,G_n)D_{\mathcal{H}}F \ , \qquad (4.6)$$

which converges a.s. to $g(F,\underline{G})D_{\mathcal{H}}F$. Thus $g(F,\underline{G})D_{\mathcal{H}}F = 0$ because $D_{\mathcal{H}}$ was assumed to be a closed operator, and, therefore, $g(F,\underline{G}) = 0$ a.s., because $< D_{\mathcal{H}}F\ , D_{\mathcal{H}}F >\ > 0$ a.s., which completes the proof of the theorem. □

Remark: The technique used in the proof of Theorem 4.2 can be applied, in a similar way, to obtain a very simple proof of the absolute continuity criterion of Bouleau and Hirsch, in dimension one.

5. Another condition for the existence of a conditional density and a condition for its smoothness.

In this section we consider first the existence of a conditional density under conditions which are different from those of the previous section. After this we consider conditions for smoothness of the density. Our approach will follow that of Watanabe (cf. [11]) and we will construct the conditional expectation of some generalized functionals obtained by pull-back.

Recall that the σ-algebra $\mathcal{G}$ is assumed to be smoothly countably generated by $\{ G_i \, , \, i \geq 1 \}$, and $\mathcal{H} = \{ K(\omega) \, , \, \omega \in \Omega \}$ with $K = \langle DG_i \, , \, i \geq 1 \rangle^{\perp}$.

5.1 A result on the existence of a conditional density:

Theorem 5.1: Let $F = (F_1, \ldots, F_m)$ be a k-dimensional random vector verifying the following conditions:

(i) $F_i \in \mathbb{D}_{2,1}$, $D_{\mathcal{H}} F_i \in Dom\, \delta$ and

$\langle D_{\mathcal{H}} F_i \, , D_{\mathcal{H}} F_j \rangle \in \mathbb{D}_{2,1}$ for any $i\, j = 1, \ldots, k$.

(ii) The partial Malliavin matrix $\gamma_{\mathcal{H}}^{ij} = \langle D_{\mathcal{H}} F_i \, , D_{\mathcal{H}} F_j \rangle$ is invertible a.s.

Then there exists a conditional density for the law of F given the σ-algebra $\mathcal{G}$.

Proof: For any integer $N \geq 1$ we consider a function $\psi_N \in C_0^{\infty}(\mathbb{R}^m \otimes \mathbb{R}^m)$ (C^{∞} and with compact support) such that

(a) $\psi_N(\sigma) = 1$ if $\sigma \in K_N$,

(b) $\psi_N(\sigma) = 0$ if $\sigma \notin K_{N+1}$, where

$K_N = \{ \sigma \in \mathbb{R}^m \otimes \mathbb{R}^m : |\sigma^{ij}| \leq N$ for any $i\, j$ and $|\det \sigma| \geq \frac{1}{N} \}$, i.e. K_N is a compact subset of $GL(m) \subset \mathbb{R}^m \otimes \mathbb{R}^m$.

We fix a function $\phi \in C_b^{\infty}(\mathbb{R}^m)$. Using the differentiation rules of the partial Malliavin calculus we deduce $\phi(F) \in \mathbb{D}_{2,1}$, and

$$D_{\mathcal{H}} \phi(F) = \sum_{i=1}^{m} (\partial_i \phi)(F) D_{\mathcal{H}} F_i \quad .$$

Hence,

$$\langle D_{\mathcal{H}} \phi(F) \, , D_{\mathcal{H}} F_j \rangle = \sum_{i=1}^{m} (\partial_i \phi)(F) \gamma_{\mathcal{H}}^{ij} \quad ,$$

where $\gamma_{\mathcal{H}}^{ij}$ is as defined above in the statement of the theorem. Then, we have

$$E[\psi_N(\gamma_{\mathcal{H}})(\partial_i\phi)(F)\mid\mathcal{G}]$$

$$=\sum_{j=1}^{m}E[\psi_N(\gamma_{\mathcal{H}})<D_{\mathcal{H}}\phi(F),D_{\mathcal{H}}F_j>(\gamma_{\mathcal{H}}^{-1})^{ij}\mid\mathcal{G}]$$

$$=\sum_{j=1}^{m}E[<D_{\mathcal{H}}(\phi(F)(\gamma_{\mathcal{H}}^{-1})^{ij}\psi_N(\gamma_{\mathcal{H}})),\ D_{\mathcal{H}}F_j>$$

$$-\phi(F)<D_{\mathcal{H}}((\gamma_{\mathcal{H}}^{-1})^{ij}\psi_N(\gamma_{\mathcal{H}})),\ D_{\mathcal{H}}F_j>\mid\mathcal{G}]$$

$$=E\{\ \phi(F)\sum_{j=1}^{m}[(\gamma_{\mathcal{H}}^{-1})^{ij}\psi_N(\gamma_{\mathcal{H}})\delta D_{\mathcal{H}}F_j-<D_{\mathcal{H}}((\gamma_{\mathcal{H}}^{-1})^{ij}\psi_N(\gamma_{\mathcal{H}}),D_{\mathcal{H}}F_j>]\mid\mathcal{G}\ \}$$

$$=E(\phi(F)A_N\mid\mathcal{G})\ ,$$

where A_N is some integrable random variable.

Denote by $p_N(\omega,B)$, $B\in\mathcal{B}(\mathbb{R}^m)$, a regular version of the conditional distribution of the random vector F (with respect to the measure $\psi_N(\gamma_{\mathcal{H}})dP$) given the σ-field $\mathcal{G}$. The above relations imply that for any i,

$$\left|\int_{\mathbb{R}^m}(\partial_i\phi)(x)p_N(\omega,dx)\right|\ \le\ \|\phi\|_\infty E(\,|A_N|\mid\mathcal{G})(\omega)\ ,\quad a.s.\ . \tag{5.1}$$

There exists a countable subset $\mathcal{S}$ in $C_b^\infty(\mathbb{R}^m)$ such that for any finite measure ν on $\mathbb{R}^m$, the property

$$\left|\int_{\mathbb{R}^m}(\partial_i\phi)(x)\nu(dx)\right|\ \le K_\nu\|\phi\|_\infty,\ \forall\ \phi\in\mathcal{S}\ ,\ \forall\ i=1,\ldots,m$$

implies the same inequality for any function ϕ in $C_b^\infty(\mathbb{R}^m)$. As a consequence we may assume that (5.1) holds for any function $\phi\in C_b^\infty(\mathbb{R}^m)$ and any $\omega\notin N$ with $P(N)=0$. By Malliavin's lemma (cf. [7]), for any $\omega\notin N$, the measure $p_N(\omega,dx)$ is absolutely continuous with respect to the Lebesgue measure on $\mathbb{R}^m$ and it has a density $f_N(\omega,x)$ which is $\mathcal{G}\otimes\mathcal{B}(\mathbb{R}^m)$-measurable. Consider the measures ν_N on $\mathcal{G}\otimes\mathcal{B}(\mathbb{R}^m)$ defined by

$$\nu_N(A\times B)=\int_{A\cap\{F\in B\}}\psi_N(\gamma_{\mathcal{H}})dP=\int_{A\times B}f_N(\omega,x)P(d\omega)dx\ ,$$

where $A\in\mathcal{G}$ and $B\in\mathcal{B}(\mathbb{R}^m)$.

The sequence ν_N is increasing, and $\nu=\sup_N\nu_N$ is a finite measure verifying $\nu(A\times B)=P[A\cap\{F\in B\}]$ due to condition (ii). Besides, ν is absolutely continuous with respect to $dPdx$ because so are the measures ν_N. Therefore, the Radon-Nikodym derivative of ν with respect to $dPdx$ on $\mathcal{G}\otimes\mathcal{B}(\mathbb{R}^m)$ will be a version of the desired conditional density. □

5.2 The conditional pull-back of Schwartz distributions, and the regularity of conditional laws

Assume that $F=(F_1,\ldots,F_m)$ is a random vector such that $F_i\in\mathbb{D}_\infty$ for any $i=1,\ldots,m$.

Let $\mathcal{G}=\sigma\{G_i\ ,\ i\ge1\}$ be a countably smoothly generated σ-algebra such that the following condition holds:

$$(C) \quad \eta \in \mathbb{D}_\infty(H) \quad implies \quad \Pi_{\mathcal{H}}\eta \in \mathbb{D}_\infty(H) \ .$$

This condition holds, for example, if the number of generators is finite, say $G_1, \ldots, G_n$, $(det < DG_i , DG_j >)^{-1} \in \bigcap_{p>1} L^p(\Omega)$, and $G_i \in \mathbb{D}_\infty$, $i = 1, \ldots, n$.

Consider the partial Malliavin matrix of F, defined as before by

$$\gamma_{\mathcal{H}}^{ij} = < D_{\mathcal{H}}F_i , D_{\mathcal{H}}F_j > \ .$$

Lemma 5.2. If $(\gamma_{\mathcal{H}}^{-1})^{ij} \in L^p(\Omega)$ for some $p > k+1$, then $(\gamma_{\mathcal{H}}^{-1})^{ij} \in \mathbb{D}_{r,k}$ for all $1 \le r < p/k+1$, and any integer $k \ge 1$.

The proof of this lemma is the same as given on page 18 of Ikeda Watanabe ([5]).

Lemma 5.3. Suppose that $(\gamma_{\mathcal{H}}^{-1})^{ij} \in L^p(\Omega)$ for some $p > 2k$, $R \in \mathbb{D}_{q,k}$ with $q > 1$, and $\frac{1}{q} + \frac{2k}{p} < 1$. Then there exists random variables $A_{i_1}, \ldots, A_{i_k}$ depending linearly on R, such that:

(i) For any $\phi \in C_b^\infty(\mathbb{R}^m)$

$$E[(\partial_{i_1}\partial_{i_2} \cdots \partial_{i_k}\phi)(F)\, R \mid \mathcal{G}] = E[\phi(F) A_{i_k}(A_{i_{k-1}}(\cdots (A_{i_1}(R)) \cdots)) \mid \mathcal{G}] \ .$$

(ii) $\sup_{R \in \mathbb{D}_{q,k},\ \|R\|_{q,k} \le 1} \|A_{i_k}(\cdots (A_{i_1}(R)) \cdots)\|_r < \infty$ for any $r \ge 1$ such that $\frac{1}{r} > \frac{1}{q} + \frac{2k}{p}$.

Proof: We fix a function $\phi \in C_b^\infty(\mathbb{R}^m)$. Suppose first that $k = 1$. We know that

$$D_{\mathcal{H}}\phi(F) = \sum_{i=1}^m (\partial_i \phi)(F) D_{\mathcal{H}}F_i \ ,$$

and

$$(\partial_i \phi)(F) = \sum_{j=1}^m (\gamma_{\mathcal{H}}^{-1})^{ij} < D_{\mathcal{H}}\phi(F), D_{\mathcal{H}}F_j > \ .$$

Hence, if $R \in \mathbb{D}_{q,1}$ for some $q > 1$ such that $\frac{2}{p} + \frac{1}{q} < 1$, we obtain as on pages 18-19 of [5] that

$$\begin{aligned}
& E[(\partial_i \phi)(F) R \mid \mathcal{G}] \\
&= \sum_{j=1}^m E[< D_{\mathcal{H}}\phi(F), (\gamma_{\mathcal{H}}^{-1})^{ij} R D_{\mathcal{H}}F_j > \mid \mathcal{G}] \\
&= \sum_{j=1}^m E[\phi(F)\delta((\gamma_{\mathcal{H}}^{-1})^{ij} R D_{\mathcal{H}}F_j) \mid \mathcal{G}] \\
&= E[\phi(F) A_i(R) \mid \mathcal{G}] \ ,
\end{aligned}$$

where

$$A_i(R) = \sum_{j=1}^{m} \delta((\gamma_{\mathcal{H}}^{-1})^{ij} R D_{\mathcal{H}} F_j)$$
$$= \sum_{j=1}^{m} \{ (\gamma_{\mathcal{H}}^{-1})^{ij} R L_{\mathcal{H}} F_j - R < D(\gamma_{\mathcal{H}}^{-1})^{ij}, D_{\mathcal{H}} F_j > -(\gamma_{\mathcal{H}}^{-1})^{ij} < D_{\mathcal{H}} R, DF_j > \} \ .$$

We assume $p > 2$. Then $D_{\mathcal{H}} R \in L^q(\Omega;H)$, $D_{\mathcal{H}}(\gamma_{\mathcal{H}}^{-1})^{ij} \in L^r$ (because $1 \le r < \frac{p}{2}$), $(\gamma_{\mathcal{H}}^{-1})^{ij} \in L^p$ and $R \in L^2$. Therefore, condition (ii) holds for any $r \le 1$ such that $\frac{1}{r} > \frac{1}{q} + \frac{2}{p}$.

Repeating the above arguments, the result can be proved for an arbitrary $k \ge 1$. □

In a similar way, if we assume $p > 4k$ and $r \in \mathbb{D}_{q,2k}$ for $q > 1$ with $\frac{1}{q} + \frac{4k}{p} < 1$, then we have

$$E[(\{1 + |x|^2 - \Delta\}^k \phi)(F) R \mid \mathcal{G}] = E[\phi(F) B_{2k}(R) \mid \mathcal{G}] \ ,$$

where Δ is the Laplacian and $B_{2k}(R)$ is a random variable depending linearly on R and satisfying

$$\sup_{R \in \mathbb{D}_{q,2k},\ \|R\|_{q,2k} \le 1} \|B_{2k}(R)\|_r < \infty \ ,$$

for any $r \ge 1$ such that $\frac{1}{r} > \frac{1}{q} + \frac{4k}{p}$.

In the sequel we will assume that Ω is a Polish space and denote by $p(\omega, B)$ (B Borel subset of Ω) a regular version of the probability P conditioned by $\mathcal{G}$.

Define the following random seminorm on $\mathbb{D}_\infty$:

$$\|F\|_{p_o,-2k,\omega} = \sup_{R \in \mathbb{D}_{q,2k},\ \|R\|_{q,2k} \le 1} \left| \int_\Omega (FR)(y) p(\omega, dy) \right|$$
$$= \sup_{R \in \mathbb{D}_{q,2k},\ \|R\|_{q,2k} \le 1} |E(FR / \mathcal{G})| \ ,$$

where $\frac{1}{p_o} + \frac{1}{q} = 1$, $k \ge 1$.

Notice that the following inequality holds true

$$\|F\|_{p_o,-2k} \le E(\|F\|_{p_o,-2k,\omega}) \ .$$

Denote by $\mathcal{S}(\mathbb{R}^m)$ the Schwartz space of rapidly decreasing C^∞ functions on $\mathbb{R}^m$. For $\phi \in \mathcal{S}(\mathbb{R}^m)$ and $k \in \mathbb{Z}$ set

$$\|\phi\|_{2k} = \|(1 + |x|^2 - \Delta)^k \phi\|_\infty \ ,$$

and let ξ_{2k} be the completion of $\mathcal{S}(\mathbb{R}^m)$ by the norm $\|\cdot\|_{2k}$. Then $\mathcal{S}'(\mathbb{R}^m) = \bigcup_{k>0} \xi_{-2k}$ is the Schwartz space of tempered distributions on $\mathbb{R}^m$.

Proposition 5.4. Let k be a positive integer. If $(\gamma_{\mathcal{H}}^{-1})^{ij} \in L^p$ for some $p > 4k$ and if we take $q > 1$ satisfying $\frac{1}{q} + \frac{4k}{p} < 1$, then the mapping

$$\mathcal{S}(\mathbb{R}^m) \ni \phi \mapsto \phi(F) \in \mathbb{D}_\infty$$

is continuous with respect to the norm $\|\cdot\|_{-2k}$ on $\mathcal{S}(\mathbb{R}^m)$, and the norm $\|\cdot\|_{p_o,-2k,\omega}$ on $\mathbb{D}_\infty$, for almost all ω, where $\frac{1}{p_o} + \frac{1}{q} = 1$.

Proof: For $\phi \in \mathcal{S}(\mathbb{R}^m)$ and $R \in \mathbb{D}_{q,2k}$, $\|R\|_{q,2k} \le 1$, we have, using Lemma 6.3,

$$\begin{aligned}
&| \int_\Omega \phi(F)(y) R(y) p(\omega,dy) | \\
&= | \int_\Omega (\{(1+|x|^2-\Delta)^k (1+|x|^2-\Delta)^{-k}\phi\})(F)(y) R(y) p(\omega,dy) | \\
&= | \int_\Omega ((1+|x|^2-\Delta)^{-k}\phi)(F)(y) B_{2k}(R)(y) p(\omega,dy) | \\
&\le \|(1+|x|^2-\Delta)^{-k}\phi\|_\infty E(|B_{2k}(R)| \mid \mathcal{G}) \\
&\le \|\phi\|_{-2k} E(|B_{2k}(R)| \mid \mathcal{G}) ,
\end{aligned}$$

for almost all ω.

Taking countable and dense subsets of $\mathcal{S}(\mathbb{R}^m)$ and $\mathbb{D}_{q,2k}$, we may assume that the above inequality holds for all ϕ and R, a.s., and this concludes the proof. □

As a consequence we deduce the following results on the conditional pull-back of Schwartz distributions:

Proposition 5.5. Under the assumptions of Proposition 5.4, the mapping

$$\mathcal{S}(\mathbb{R}^m) \ni \phi \mapsto \phi(F) \in \mathbb{D}_\infty$$

extends a.s. to a unique continuous linear mapping

$$\xi_{-2k} \ni T \mapsto T(F) \in \mathbb{D}_{p_o,-2k,\omega} .$$

Here $T(F)$ is a generalized random variable in the sense that ω-a.s., the "conditional expectation" $E[T(F)R \mid \mathcal{G}]$ exists for all R in $\mathbb{D}_{q,2k}$.

Proposition 5.6. If F is such that $(\gamma_{\mathcal{H}}^{-1})^{ij} \in \bigcap_{p>1} L^p$, then, ω-a.s., the mapping

$$\xi_{-2k} \ni T \mapsto T(F) \in \mathbb{D}_{p_o,2k,\omega}$$

is continuous for every $k \ge 1$ and $p_o > 1$. In particular,

$$\mathcal{S}'(\mathbb{R}^m) \ni T \mapsto T(F) \in \mathbb{D}_{-\infty,\omega} := \bigcup_{k\ge 1} (\bigcup_{p>1} \mathbb{D}_{p,-2k,\omega})$$

is well defined.

These results can be applied to derive the existence of a smooth conditional density of F given $\mathcal{G}$ as follows:

Theorem 5.7. Take $m_o = [\frac{m}{2}]+1$. Assume $(\gamma_{\mathcal{H}}^{-1})^{ij} \in L^p$ for all $i,j = 1, \ldots, m$, and for some $p > 4(m_o+k)$. Then there exists a version of the conditional density $f(\omega,x) : \Omega\times\mathbb{R}^m \to \mathbb{R}_+$ of F

given $\mathcal{G}$ such that for all ω, $f(\omega,\cdot)$ is of class C^{2k}.

Sketch of the proof: We introduce the Dirac δ-function δ_x which belongs to ξ_{-2k} if $k > \frac{m}{2}$. Furthermore, $\mathbb{R}^m \ni x \mapsto \delta_x \in \xi_{-2m_o-2k}$ is $2k$-times continuously differentiable for any $k \geq 0$. Therefore, if $1 < p_o < p/4(m_o+k)$ then $\delta_x(F) \in \mathbb{D}_{p_o,-2m_o,\omega}$ for all $x \in \mathbb{R}^m$, a.s., and the mapping

$$\mathbb{R}^m \ni x \mapsto \delta_x(F) \in \mathbb{D}_{p_o,-2m_o-2k,\omega}$$

is $2k$-times continuously differentiable.

As a consequence, the function $E[\delta_x(F) \mid \mathcal{G}]$ is $2k$-times continuously differentiable, a.s., and it can be seen as in Watanabe [11] that this function provides a version of the conditional density of F given $\mathcal{G}$. $\square$

6. Applications to regularity of conditional laws in filtering problems

In this section we will discuss two different applications of the results obtained in the previous sections. First we present a criterion for the existence of a density which is based on Theorem 5.2. We show this criterion in a setup that can be considered as a very general formulation of the filtering problem without feedback.

Let $W = \{W(A),\ A \in \Theta\}$ be a zero mean Gaussian measure on the finite atomless measure space (T,Θ,μ), on a separable σ-field Θ and $EW(A)W(B) = \mu(A \cap B)$. Fix a measurable subset A of T and set $H_o = \{h \in H : h$ vanishes on $A^c\}$. Also set $\mathcal{F}_A = \sigma\{W(B), B \in \Theta, B \subset A\}$. Suppose we are given a random variable $F \in \mathbb{D}_{2,1}$ and a real valued process $u = \{u_t,\ t \in T\}$ belonging to $L^2(T \times \Omega)$ such that they are both $\mathcal{F}_A$-measurable.

Consider the stochastic process $Y = \{Y(B), B \in \Theta, B \subset A^c\}$ indexed by measurable subsets of A^c, defined by

$$Y(B) = \int_B u_t \mu(dt) + W(B)\ . \tag{6.1}$$

In order to point out the relevance of (6.1) to the nonlinear filtering problem, let $T = [0,2)$, let Θ be the Borel σ-field on T and μ the Lebesgue measure. Set $A = [0,1)$, then (6.1) can be rewritten as

$$Y([1,1+s)) = \int_0^s u_{1+\theta} d\theta + W(1+s) - W(1), \quad s \in [0,1)$$

Setting $Z(s) = Y([1,1+s))$, $u_{1+\theta} = v_\theta$, $W(1+s) - W(1) = \nu(s)$ yields

$$Z(s) = \int_0^s v_\theta d\theta + \nu(s)$$

where v_s, $s \in [0,1)$ is independent of $\{\nu(\theta),\ \theta \in [0,1)\}$. Note that $\mathcal{F}_{[0,1)}$ is independent of $\sigma\{\nu(\theta),\ \theta \in [0,1)\}$, v_s is assumed to be adapted to $\mathcal{F}_{[0,1)}$ and is not restricted to be adapted to $\mathcal{F}_{[0,s)}$ (and to be a solution to a stochastic differential equation) as in the classical setup.

Theorem 6.1. Under the above assumptions, the law of F conditioned by the σ-field

$\mathcal{G} = \sigma\{ Y(B), B \in \Theta, B \subset A^c \}$ has a density, provided that the following conditions are satisfied:

(a) $u \in \mathbb{D}_{2,2}(L^2(T))$,

(b) $< DF, DF > \; > 0 \quad a.s.$

Namely: under the regularity hypothesis $u \in \mathbb{D}_{2,2}(L^2(T))$, the condition for the existence of a density for F, i.e. $< DF, DF > \; > 0$ a.s., *also* implies the existence of a conditional density.

Proof: First note that the σ-field $\mathcal{G}$ is countably smoothly generated because we can take $\mathcal{G} = \sigma\{ < Y, e_i > , \; i \ge 1 \}$, where $\{ e_i , \; i \ge 1 \}$ is a C.O.N.S. for $L^2(A^c)$, and $< Y, e_i > \; = \; < u, e_i > + W(e_i)$. Set $\mathcal{H} = \{ K(\omega), \; \omega \in \Omega \}$, where

$$K(\omega) = \; < D(< Y, e_i >), \quad i \ge 1 >^{\perp} .$$

The proof will be carried out in several steps:

(i) We claim that

$$K^{\perp} = \{ g \in L^2(T) : g(t) = \int_{A^c} g(s) D_t u_s \, \mu(ds), \; for \; any \; t \in A , \quad \mu\text{-}a.e. \; \} .$$

That means, the values of every function $g \in K^{\perp}$ on the set A depend linearly on its values on A^c, and there is no restriction on the values of g on A^c. This property is an easy consequence of the following formula

$$D_t(< Y, e_i >) = \begin{cases} e_i(t), \; if \; t \in A^c \text{ , } because \; u \; is \; \mathcal{F}_A\text{-}measurable \; . \\ \int_{A^c} e_i(s) D_t u_s \, \mu(ds), \; if \; t \in A \text{ , } because \;\; e_i(t) = 0 \; . \end{cases}$$

(ii) The operator $D_{\mathcal{H}}$ is closed.

In fact, suppose that $F_n \xrightarrow{L^2} 0$, $F_n \in \mathbb{D}_{2,1}$, and $D_{\mathcal{H}} F_n \xrightarrow{L^2} \eta$. Then property (i) implies that

$$(D_{\mathcal{H}} F_n)_t = D_t F_n - (D_{\mathcal{H}^{\perp}} F_n)_t = D_t F_n - \int_{A^c} (D_{\mathcal{H}^{\perp}} F_n)_s D_t u_s \, \mu(ds) \; , \tag{6.2}$$

for any $t \in A$. We know that $\eta(\omega) \in K(\omega)$ for almost all ω. Then it suffices to check that

$$\eta_t = \int_{A^c} \eta_s D_t u_s \, \mu(ds) \quad , \quad for \; any \quad t \in A \; , \tag{6.3}$$

because in view of (i) (6.3) implies that $\eta \in K^{\perp}$ and, consequently, $\eta = 0$. Let R be a smooth random variable, and $h \in L^2(A)$. Using (6.2) we have

$$\begin{aligned} E[R \int_A \eta_t h_t \, \mu(dt)] &= E(< \eta, Rh >) = \lim_n E(< D_{\mathcal{H}} F_n, Rh >) \\ &= \lim_n E[\int_A (D_t F_n - \int_{A^c} (D_{\mathcal{H}^{\perp}} F_n)_s D_t u_s \, \mu(ds)) R h_t \, \mu(dt)] \\ &= \lim_n E[F_n \delta(R h_t \mathbf{1}_A(t))] - \lim_n E[\int_{A^c} (D_{\mathcal{H}^{\perp}} F_n)_s (\int_A D_t u_s R h_t \, \mu(dt)) \mu(ds)] \; . \end{aligned} \tag{6.4}$$

The first limit is equal to zero. For the second one we write $D_{\mathcal{H}^\perp} F_n = DF_n - D_{\mathcal{H}} F_n$. Integrating by parts and using the fact that the process $\{ \mathbf{1}_{A^c}(s)(\int_A D_t u_s R h_t \mu(dt)),\ s \in T \}$ belongs to the domain of δ (due to assumption (a)) we obtain that (6.4) is equal to

$$- \lim_n E[F_n \delta\{ \mathbf{1}_{A^c}(s)(\int_A D_t u_s R h_t \mu(dt)) \}] + E[\int_{A^c} \eta_s (\int_A D_t u_s R h_t \mu(dt)) \mu(ds)] =$$

$$= E[R \int_A h_t (\int_{A^c} \eta_s D_t u_s \mu(ds)) \mu(dt)] ,$$

which completes the proof of (6.3).

(iii) We will show now that assumption (b) implies that $< D_{\mathcal{H}} F , D_{\mathcal{H}} F > \ > 0$ a.s. This inequality will follow from the inclusion

$$\{ \omega : < D_{\mathcal{H}} F , D_{\mathcal{H}} F > = 0 \} \subset \{ \omega : < DF , DF > = 0 \} \quad a.s. \tag{6.5}$$

In order to prove (6.5), assume that $< D_{\mathcal{H}} F , D_{\mathcal{H}} F > = 0$ for a fixed value of ω. We have

$$(D_{\mathcal{H}} F)_t = \begin{cases} -(D_{\mathcal{H}^\perp} F)_t , & if \ \ t \in A^c , \\ D_t F - \int_{A^c} (D_{\mathcal{H}^\perp} F)_s D_t u_s \mu(ds) , & if \ \ t \in A . \end{cases}$$

Therefore $(D_{\mathcal{H}^\perp} F)_t = 0$ for $t \in A^c$, which implies $D_t F = 0$ for $t \in A$ and, consequently, $< DF , DF > = 0$.

Finally the result follows form Theorem 4.2 and properties (ii) and (iii). □

We conclude this section by pointing out that the methodology developed in the previous sections can be used to derive the regularity of conditional laws in a filtering problem with feedback.

Denote by (x_t , z_t) the solution of the following stochastic system

$$dx_t = X_o(x_t , z_t) dt + X_i(x_t , z_t) dw_t^i + \tilde{X}_i(x_t , z_t) dz_t^i$$
$$dz_t = l(x_t , z_t) dt + d\tilde{w}_t ,$$

where $x_t \in \mathbb{R}^n$, $z_t \in \mathbb{R}^p$. The processes $\{ w_t^i ,\ t \geq 0 ,\ i = 1, \dots, m \}$ and $\{ \tilde{w}_t^i ,\ t \geq 0 ,\ i = 1, \dots, p \}$ are independent Brownian motions. We assume that the coefficients X_i^j, $\tilde{X}_i^j$, l^j are smooth functions which are bounded together with their derivatives. The stochastic integrals are taken here in the Stratonovich sense.

Consider the vector fields on $\mathbb{R}^n \times \mathbb{R}^p$ defined by

$$X_i(x , z) = X_i^j (x , z) \frac{\partial}{\partial x_j} , \qquad 0 \leq i \leq m ,$$

$$\tilde{Y}_i(x , z) = \tilde{X}_i^j (x , z) \frac{\partial}{\partial x_j} + \frac{\partial}{\partial z_i} , \qquad 1 \leq i \leq p .$$

Then we have:

Theorem 6.2. Assume the following Hörmander-type condition, (H): The Lie algebra spanned by

$X_1, \ldots, X_m$ and the brackets of $X_0, X_1, \ldots, X_m, \tilde{Y}_1, \ldots, \tilde{Y}_p$ where there is at least one X_i, $1 \le i \le m$, at point (x_o, z_o) has dimension n. Then, the law of x_t $(t > 0)$ given $\{z_s, \ 0 \le s \le t\}$ has a C^∞ density $p_t(x)$.

A theorem of this type (including smoothness and integrability properties of the density as a function of (t, x)) has been proved using Malliavin calculus by Bismut-Michel [2] and Kusuoka-Stroock [6]. A rough sketch of the proof in the context of the results of the paper would be as follows. The first step in proving Theorem 6.2 is to use a Girsanov transformation in such a way that z_t becomes a Brownian motion. Denote by P_o the new probability measure and by Λ_t the Radon-Nikodym derivative dP / dP_o at time t. Now we can estimate a conditional expectation like

$$E_o\left[\frac{\partial f}{\partial x_i}(x_t)\Lambda_t \mid z_s, \quad 0 \le s \le t\right]$$

by

$$E_o\left[\det \langle D_{\mathcal{H}}x_t, D_{\mathcal{H}}x_t\rangle^{-1} \, |R| \;\Big|\; z_s, \quad 0 \le s \le t\right] \|f\|_\infty ,$$

where R is a random variable in $\mathbb{D}_\infty$, as it has been done in Section 5 (see the proofs of Theorem 5.1 and Lemma 5.3). Under the probability P_o, the family of Hilbert spaces $\mathcal{H}$ is constant, because z_t is a Brownian motion independent of w_t. So we can compute $\langle D_{\mathcal{H}}x_t, D_{\mathcal{H}}x_t\rangle$ as in the classical case, and the Hörmander's condition (H) in Theorem 6.2 implies that $E_o(\det \langle D_{\mathcal{H}}x_t, D_{\mathcal{H}}x_t\rangle^{-p}) < \infty$ for every $p \ge 2$. This property allows to conclude the proof as in the nonconditional case.

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