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A FORMULA FOR DENSITIES OF TRANSITION FUNCTIONS

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SUMMARY. Let $(P_{s,t})$ (resp. $(\hat{P}_{st})$) be a measurable forward (resp. backward) transition function on a measurable space $(E,\underline{\underline{E}})$ and let (μ_t) be a family of σ-finite measures on E. Under suitable assumptions of absolute continuity and duality, Wittmann [1] has constructed densities $P_{s,t}(x,dy)=p(s,t,x,y)\mu_t(dy)$, $\hat{P}_{s,t}(y,dx)=p(s,t,x,y)\mu_s(dx)$, satisfying identically the Chapman-Kolmogorov equation $p(s,t,x,y)=\int p(s,r,x,z)p(r,t,z,y)\mu_r(dz)$ for $s<r<t$. In the present paper we give a simpler proof of this result under weaker hypotheses. We give an explicit formula (cf. (4)) for the density. In the last section we construct a good density for any transition function, without duality hypotheses.

1. THE MAIN RESULT

Let Δ be $\{(s,t)\in\mathbb{R}\times\mathbb{R} : s<t\}$ and $(E,\underline{\underline{E}})$ be a measurable space. We consider two families of kernels on E ,

$$(P_{s,t}) \text{ and } (\hat{P}_{s,t}) \text{ indexed by } \Delta$$

such that for $s<r<t$

$$(1) \qquad P_{s,t} = P_{s,r}\circ P_{r,t}\ ,\ \hat{P}_{s,t} = \hat{P}_{r,t}\circ\hat{P}_{s,r}$$

$(P_{s,t})$, $(\hat{P}_{s,t})$ are respectively called a forward, resp. backward transition function. Let (μ_t) be a family of σ-finite measures on E. Throughout the sequel we assume that $P_{s,t}(x,.)$ is μ_t-absolutely continuous and $\hat{P}_{s,t}(x,.)$ μ_s-absolutely continuous for all $s<t$, $x\in E$, and that $(P_{s,t})$ and $(\hat{P}_{s,t})$ are in duality w.r.t. (μ_t), i.e.

$$(2) \qquad \int f\, P_{s,t}g\, d\mu_s = \int \hat{P}_{s,t}f\, g\, d\mu_t \qquad (s<t,\ f,g\in\underline{\underline{E}}_+)$$

where $\underline{\underline{E}}_+$ stands for the set of all non-negative measurable functions on E . We don't assume the transition functions to be submarkovian .

We choose preliminary versions of the transition densities as follows : $p_{st}(x,.)$ is a density of $P_{s,t}(x,\ )$ w.r.t. μ_t , and $\hat{p}_{st}(y,.)$ a density of $\hat{P}_{st}(y,.)$ w.r.t. μ_s. Our starting point is the following remark :

LEMMA 1. For fixed $x,y\in E$, $\int p_{sr}(x,z)\hat{p}_{rt}(y,z)\mu_r(dz)$ doesn't depend on r for $s<r<t$.

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Proof. Let v be such that $s<r<v<t$. Let us prove first that

(3) $$\int_E P_{r,v}(w,dz)\hat{p}_{vt}(y,z) = \hat{p}_{rt}(y,w) \qquad \mu_r(dw)\text{-a.e.}.$$

To see this, multiply both sides by f(w) ($f\in \underline{\underline{E}}_+$) and integrate w.r.t. $\mu_r(dw)$. The right side becomes $\hat{P}_{r,t}(y,f)$ by definition of $\hat{p}_{rt}$. The left side can be written (if we set $\phi=\hat{p}_{vt}(y,.)$)

$$\int f\, P_{r,v}\phi\, d\mu_r = \int \hat{P}_{r,v}f\, \phi\, d\mu_v \quad \text{(duality)}$$

By definition $\phi d\mu_v$ is the measure $\hat{P}_{vt}(y,.)$, and by (1) the right side is equal to $\hat{P}_{r,t}(y,f)$.

We now prove the lemma : multiply both sides by $p_{sr}(x,w)$ and integrate w.r.t. $\mu_r(dw)$. On the left we replace $p_{sr}(x,w)\mu_r(dw)$ by $P_{s,r}(x,dw)$ and

$$\int P_{s,r}(x,dw)P_{r,v}(w,dz)\hat{p}_{vt}(y,z) = \int P_{s,v}(x,dz)\hat{p}_{vt}(y,z)$$
$$= \int p_{sv}(x,z)\hat{p}_{vt}(y,z)\mu_v(dz)$$

On the right side we have $\int p_{sr}(x,w)\hat{p}_{rt}(y,w)\mu_v(dw)$, and the lemma is proved.

Given the lemma, we may set without ambiguity

(4) $$p(s,t,x,y) = \int p_{sr}(x,z)\hat{p}_{rt}(y,z)\mu_r(dz) \quad \text{for } s<r<t$$

Some comments about measurability are in order : if the σ-field $\underline{\underline{E}}$ is <u>separable</u> and all measures $P_{s,t}(x,.)$, $\hat{P}_{s,t}(x,.)$ are <u>bounded</u> (not necessarily of mass ≤ 1) it is well known that $p_{sr}(.,.)$ and $\hat{p}_{rt}(.,.)$ can be chosen to be $\underline{\underline{E}}\times\underline{\underline{E}}$-measurable, and by Fubini's theorem $p(s,t,.,.)$ <u>is automatically $\underline{\underline{E}}\times\underline{\underline{E}}$-measurable</u>. On the other hand, serious difficulties can arise if the transition functions consist just of σ-finite measures. The above hypotheses on $\underline{\underline{E}}$ and the transition functions will be assumed in the remainder of this note - but see however lemma 2 below, and the end of the paper .

Our main result is the following :

THEOREM 1. The function p defined by (4) satisfies the following properties, which characterize it uniquely : for $s<r<t$

(5) $$P_{s,t}(x,dy) = p(s,t,x,y)\mu_t(dy) \ , \ \hat{P}_{s,t}(x,dy) = p(s,t,y,x)\mu_s(dy)$$

(6) $$p(s,t,x,y) = \int p(s,r,x,z)p(r,t,z,y)\mu_r(dz) \qquad .$$

If furthermore $P_{s,t}(x,.)$ is measurable in $(s,x)\in]-\infty,t[\times E$ and $\hat{P}_{t,u}(x,.)$ measurable in $(u,x)\in]t,+\infty[\times E$ for all fixed t,x, the transition density p(s,t,x,y) is jointly measurable on $\Delta\times E\times E$.

Proof. Let p and p' be two functions satisfying the above properties. For $s<r<t$ and fixed x,y , we have μ_r-a.s.

$$p(s,r,x,.)=p'(s,r,x,.) \quad ; \quad p(r,t,.,y)=p'(r,t,.,y)$$

since these functions are the densities of the same measures w.r.t. μ_r. Using (6) we find that $p(s,t,x,y)=p'(s,t,x,y)$.

Uniqueness being established, we prove the first half of (5) (the second half is a consequence of (4) and (3). For $f\in\underline{\underline{E}}_+$ we have from (4)

$$\int p(s,t,x,y)f(y)\mu_t(dy) = < \hat{P}_{r,t}(\cdot,p_{sr}(x\cdot)), f >_{\mu_t} \text{ by } (2)$$
$$= < p_{sr}(x,\cdot),\ P_{r,t}f >_{\mu_r}$$
$$= P_{s,r}(x,P_{r,t}f) = P_{s,t}(x,f).$$

(5) being proved, note that the function p defined by (4) doesn't depend on the choice of the preliminary versions cf the densities. We may take now $p_{sr}(x,z)=p(s,r,x,z)$ and $\hat{p}_{rt}(y,z)=p(r,t,z,y)$, and (6) follows from (4).

It remains to prove the measurability. In formula (4) fix r rational, and choose the preliminary versions such that $p_{sr}(x,z)$ is measurable on $]-\infty,r[\times E\times E$ and $\hat{p}_{rt}(y,z)$ measurable on $]r,\infty[\times E\times E$. Then it follows that $p(s,t,x,y)$ is jointly measurable on $\Delta_r\times E\times E$, where Δ_r is the set of all (s,t) such that $s<r<t$. Taking an union over r gives the joint measurability over Δ .

REMARK. The additional hypotheses done by Wittmann in [1] were the following (mostly needed to use resolvent techniques)

- The family (μ_t) was assumed to be measurable, and "uniformly σ-finite", i.e. $\mu_t(f)$ was assumed to be locally bounded for some $f>0$.
- $P_{s,t}(x,\cdot)$ and $\hat{P}_{s,t}(x,\cdot)$ had to be jointly measurable and of mass $\leqq 1$.

We are going now to state the version of Theorem 1 which applies to the homogeneous case.

Let (P_t) and $(\hat{P}_t)$ be transition functions on E, which are in duality with respect to a σ-finite measure μ, and absolutely continuous with respect to it. We define

$$\mu_t=\mu \ , \ P_{s,t}=P_{t-s} \ , \ \hat{P}_{s,t}=\hat{P}_{t-s} \quad (s,t\in\mathbb{R} \ , \ s<t) \ .$$

Then we may apply theorem 1, and deduce from it a transition density $p(s,t,x,y)$ given by (4). In this formula, we may choose preliminary versions $p_{sr}(x,z)$ and $\hat{p}_{rt}(y,z)$ depending only on r-s, t-r, and then it is clear that $p(s,t,y,z)$ depends only on t-s. Then we have proved

THEOREM 2. Let (P_t) and $(\hat{P}_t)$ be transition functions in duality w.r.t. μ and absolutely continuous w.r.t. it . Then there exists a unique function $p(t,x,y)$ on $]0,\infty[\times E\times E$, measurable in (x,y) for fixed t, such that

$$P_t(x,dy) = p(t,x,y)\mu(dy) \ ; \ \hat{P}_t(x,dy) = p(t,y,x)\mu(dy) \tag{7}$$

(8) $$p(s+t,x,y) = \int p_s(x,z)p_t(z,y)\mu(dz) \, .$$

Furthermore, if the two transition functions are measurable (in (t,x)) the transition density is jointly measurable.

In particular, a symmetric transition function which is absolutely continuous w.r.t. μ has a symmetric transition density.

Dynkin has proved in [2], for the homogeneous case, a result which is partially stronger than Theorem 2. Namely, he doesn't assume the existence of the dual transition function $(\hat{P}_t)$, but only the absolute continuity of (P_t) w.r.t. μ ; then he constructs a transition density $p(t,x,y)$ satisfying (8) and deduces from this the existence of a dual transition function (neither $p(t,x,y)$ nor $\hat{P}_t$ are unique). Dynkin's proof is valid for a measurable space which isn't separable, provided one can show the existence of a preliminary version $q(t,x,y)$ of the density $P_t(x,dy)/\mu(dy)$ which is jointly measurable in (x,y) for fixed t . The following lemma (due to Ma [3]) shows that such a version always exists. See the end of the paper for the non-homogeneous case.

LEMMA 2. Let (P_t) be a transition function on an arbitrary measurable space $(E,\underline{\underline{E}})$ which is absolutely continuous w.r.t. a σ-finite measure μ on E. Then there exists a version $q(t,x,y)$ of $P_t(x,dy)/\mu(dy)$ such that for any $t>0$, $q(t,\cdot,\cdot)$ is measurable on $E\times E$. If furthermore (P_t) is measurable, $p(t,x,y)$ can be chosen to be measurable on $]0,\infty[\times E\times E$.

One may note that this lemma also frees theorem 2 from separability assumptions.

Proof. For $n=1,2,\ldots$ define on $E\times E$ a measure

$$\nu_n(dx,dy) = \mu(dx)P_{1/n}(x,dy)$$

This measure is absolutely continuous w.r.t. $\mu\times\mu$, and hence has a density $a_n(x,y)$ measurable on $E\times E$. One has for fixed $f\in\underline{\underline{E}}_+$

$$P_{1/n}(z,f) = \int a_n(z,y)f(y)\mu(dy) \qquad \mu(dz)\text{-a.e.}$$

If we set

$$q(t,x,y) = \int P_{t-1/n}(x,dz)a_n(z,y) \quad \text{for } \tfrac{1}{n}<t\leq \tfrac{1}{n-1}$$

it is easy to see that q fits our requirements.

2. AN APPLICATION TO FEYNMAN-KAC TRANSITION FUNCTIONS

Assume now that the transition functions $(P_{s,t})$ and $(\hat{P}_{s,t})$ are markovian (the submarkovian case can be reduced to the markovian case as usual). Let (X_t) be a Markov process on a probability space $(\Omega,\underline{\underline{F}},P)$ with these forward and backward transition functions. This means

$$P[X_t \in B \mid X_u, u \leqq s] = P_{s,t}(X_s, B) \quad \text{a.s. } P \quad (B \in \underline{\underline{E}})$$

$$P[X_s \in B \mid X_u, u\ t] = \hat{P}_{s,t}(X_t, B) \quad \text{a.s. } P$$

Put $\mathcal{F}_{\leqq t} = \sigma(X_u, u \leqq t)$ and $\mathcal{F}_{\geqq s} = \sigma(X_u, u \geqq s)$. Then there exist probability measures $P_{s,x}$ on $\mathcal{F}_{\geqq s}$ and $\hat{P}_{t,x}$ on $\mathcal{F}_{\leqq t}$ such that

$$P_{s,x}(X_t \in B) = P_{s,t}(x,B) \ , \ \hat{P}_{t,x}(X_s \in B) = \hat{P}_{s,t}(x,B) \ .$$

We assume that the transition functions $(P_{s,t})$ and $(\hat{P}_{s,t})$ satisfy the hypotheses of section 1 with respect to a family (μ_t) of σ-finite measures on E, and denote by $p(s,t,x,y)$ the corresponding transition density.

Now let $V(s,x)$ be a bounded Borel function on $\mathbb{R} \times E$. Put

(9) $$e_{s,t} = \exp \int_s^t V(u, X_u)\, du$$

(10) $$Q_{s,t} f(x) = E_{s,x}(f(X_t) e_{s,t}) \ , \ \hat{Q}_{s,t} f(x) = \hat{E}_{t,x}(f(X_s) e_{s,t})$$

(Some measurability is necessary to define (9), and some regularity on the space $(\Omega, \mathcal{F})$ is needed to guarantee the existence of the measures. We don't want to insist on technical details of this kind). It is well known that $(Q_{s,t})$ and $(\hat{Q}_{s,t})$ are forward (resp. backward) transition functions : we call them the <u>Feynman-Kac transition functions associated with</u> $V(s,x)$.

LEMMA 3. For any $f \in \underline{\underline{E}}_+$, we have

(11) $$Q_{s,t} f(x) = P_{s,t} f(x) + \int_s^t Q_{s,u}[V(u,\cdot) P_{u,t} f](x)\, du$$
$$= P_{s,t} f(x) + \int_s^t P_{s,u}[V(u,\cdot) Q_{u,t} f](x)\, du$$

(12) $$\hat{Q}_{s,t} f(x) = \hat{P}_{s,t} f(x) + \int_s^t \hat{Q}_{u,t}[V(u,\cdot) \hat{P}_{s,u} f](x)\, du$$
$$= \hat{P}_{s,t} f(x) + \int_s^t \hat{P}_{u,t}[V(u,\cdot) \hat{Q}_{s,u} f](x)\, du \ .$$

<u>Proof</u>. We have

(13) $$e_{s,t} = 1 + \int_s^t V(u, X_u) e_{s,u}\, du = 1 + \int_s^t V(u, X_u) e_{u,t}\, du \ .$$

The first (resp. second) relation (11) follows from (10) and the first (second) relation (13). The proof of (12) is analogous.

It is well known that the duality of transition functions implies a time reversal property for the process itself, namely

LEMMA 4. Let Z be a non-negative r.v., measurable w.r.t. $\sigma(X_u, s \leqq u \leqq t)$, and let $f, g \in \underline{\underline{E}}_+$. Then we have

(14) $$\int g(x) E_{s,x}(f(X_t) Z)\, \mu_s(dx) = \int \hat{E}_{t,y}(g(X_s) Z)\, f(y)\, \mu_t(dy) \ .$$

Taking $Z=e_{s,t}$ we have that $(Q_{s,t})$ and $(\hat{Q}_{s,t})$ are in duality w.r.t. (μ_t). Since absolute continuity is obvious, we can apply theorem 1, and construct a transition density $q(s,t,x,y)$.

The main result of this part is the following theorem.

THEOREM 3. The transition density $q(s,t,x,y)$ satisfies the following equations

$$(15)\quad q(s,t,x,y) = p(s,t,x,y) + \int_s^t du \int q(s,u,x,z)V(u,z)p(u,t,z,y)\mu_u(dz)$$

$$(16)\quad q(s,t,x,y) = p(s,t,x,y) + \int_s^t du \int p(s,u,x,z)V(u,z)q(u,t,z,y)\mu_u(dz).$$

Proof. We are going to prove (15), the proof of (16) being analogous. We denote by $\tilde{q}(s,t,x,y)$ the right side of (15). According to (11) and (12) we have

$$Q_{s,t}(x,dy) = \tilde{q}(s,t,x,y)\mu_t(dy)\ ,\ \hat{Q}_{s,t}(x,dy) = \tilde{q}(s,t,y,x)\mu_s(dy).$$

Then according to theorem 1 we have, for $s<r<u\leq t$

$$(17)\qquad q(s,u,x,z) = \int_E \tilde{q}(s,r,x,w)q(r,u,z,w)\mu_r(dw) = \hat{Q}_{r,u}(z,\tilde{q}(s,r,x,.))$$

Therefore

$$\int_s^t du \int_E q(s,u,x,z)V(u,z)p(u,t,z,y)\mu_u(dz) = \int_s^r + \int_r^t =$$

$$= \int_s^r du \int_E q(s,u,x,z)V(u,z)[\ \int_E p(u,r,z,w)p(r,t,w,y)\mu_r(dw)\ ]\ \mu_u(dz) +$$
$$+ \int_r^t du \int_E \hat{Q}_{r,u}(z,\tilde{q}(s,r,x,.)V(u,z)\hat{P}_{u,t}(y,dz)\quad (\text{ from } (17)).$$

On the 2^{nd} line we replace $\int_s^r du \int q(s,u,x,z)V(u,z)p(u,r,z,w)\mu_u(dz)$ by $\tilde{q}(s,r,x,w)-p(s,r,x,w)$ by definition of $\tilde{q}$ (remembering that $s<u<r$ in the first integral). The line has become

$$(18)\qquad \begin{aligned} &\int_E \{\tilde{q}(s,r,x,w)-p(s,r,x,w)\}p(r,t,w,y)\mu_r(dw) \\ = &\ \hat{P}_{r,t}(y,\tilde{q}(s,r,x,.))- p(s,t,x,z) \end{aligned}$$

On the other hand the last line can be written

$$(19)\qquad \int_r^t du \int \hat{P}_{u,t}(y,\ \hat{Q}_{r,u}\ [\ \tilde{q}(s,r,x,.)V(u,.)])$$

$$= \hat{Q}_{r,t}(y,\ \tilde{q}(s,r,x,.)) - \hat{P}_{r,t}(y,\ \tilde{q}(s,r,x,.)) \text{ from } (12)$$

The first term on the right is $\int\tilde{q}(s,r,x,z)q(r,t,z,y)\mu_r(dz) = q(s,t,x,y)$ since $\tilde{q}(s,r,x,.)$ is a density for $Q_{s,r}(x,dz)$. Adding then (18) and (19) we get simply $q(s,t,x,y)-p(s,t,x,y)$ and the theorem is proved.

Please see also the additions on next page.

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NOTE ADDED IN PROOF

The following improvements to the text couldn't be added in time to the paper. We apologize to the reader for the inconvenience.

1. In theorem 1, we are going to show that the measurability of the transition density $p_{s,t}(\cdot,\cdot)$ on $E\times E$ is true without any separability assumption, as well as the joint measurability of $p_{\cdot,\cdot}(\cdot,\cdot)$ if the additional properties of measurability in time hold as stated in thm. 1.

Four $u<v$ we define a measure on $E\times E$ by

$$\nu_{u,v}(dx,dy) = P_{u,v}(x,dy)\mu_u(dx)$$

and let $\beta(u,v,x,y)$ be a density of $\nu_{u,v}$ w.r.t. $\mu_u\times\mu_v$. Then for any $f\in\underline{\underline{E}}_+$ we have

$$\int\beta(u,v,z,w)f(w)\mu_v(dw) = P_{u,v}f(z) \quad \mu_u(dz)\text{-a.e.}$$

which implies that for $s<u<v<t$

$$\int P_{s,u}(x,dz)\beta(u,v,z,w)f(w)\mu_v(dw) = \int P_{s,u}(x,dz)P_{u,v}f(z) = P_{s,v}f(x)$$

since $P_{s,u}(x,\cdot)\ll\mu_u$. This means

$$\int P_{s,u}(x,dz)\beta(u,v,z,w) = p_{s,v}(x,w) \quad \mu_v(dw)\text{-a.e.}$$

from which and from (4) it follows that

$$(\ddagger) \qquad p(s,t,x,y) = \int P_{s,u}(x,dz)\beta(u,v,z,w)\hat{P}_{v,t}(y,dw) .$$

This implies that $p(s,t,\cdot,\cdot)$ is $\underline{\underline{E}}\times\underline{\underline{E}}$-measurable.

Now assume that $(P_{s,t})$ and $(\hat{P}_{s,t})$ satisfy the additional measurability condition stated in the theorem. Then from ($\ddagger$) we see that $p(s,t,x,y)$ is measurable on $\Delta_{u,v}\times E\times E$ for $u<v$, where $\Delta_{u,v} =]-\infty,u[\times]v,\infty[$. Taking an union over rational $(u,v)\in\Delta$ gives the joint measurability of $p(s,t,x,y)$ on $\Delta\times E\times E$.

2. In this section we are going to show that without assuming the existence of a dual transition function, any transition function $(P_{s,t})$ always has a transition density $p(s,t,x,y)$ which is measurable in (x,y) and satisfies the Chapman-Kolmogorov equation. This extends the result given in §2 for the homogeneous case.

We would like to state our result in the most general case. For $t\in\mathbb{R}$ let $(E_t,\underline{\underline{E}}_t)$ be a measurable space and let μ_t be a σ-finite measure on it. Assume that $(P_{s,t})_{(s,t)\in\Delta}$ is a transition function such that $P_{s,t}(x,dy)\ll\mu_t(dy)$ for all (s,t), $x\in E_s$. For $u<v$ define a measure on $E_u\times E_v$

$$\nu_{u,v}(dx,dy) = P_{u,v}(x,dy)\mu_u(dx)$$

and let $\beta(u,v,x,y)$ be a density of $\nu_{u,v}$ w.r.t. $\mu_u\times\mu_v$.

LEMMA 5. Set for $s<u<t$

(20) $$p_{s,t}(x,y) = \int_{E_u} \beta(u,t,z,y)P_{s,u}(x,dz)$$

Then $p_{s,t}(\cdot,\cdot)$ is measurable and $P_{s,t}(x,dy)=p_{s,t}(x,y)\mu_t(dy)$.

Proof. For any $f\in(\underline{\underline{E}}_t)_+$ we have

$$\int_{E_t} \beta(u,t,z,y)f(y)\mu_t(dy) = P_{u,t}f(z) \quad \mu_t(dz)\text{-a.e.}$$

Integrating both sides on E_u w.r.t. $P_{s,u}(x,dz)$ we get

$$\int_{E_t} p_{s,t}(x,y)f(y)\mu_t(dy) = P_{s,t}f(x) .$$

The measurability is obvious on (20).

THEOREM 4. There exists a function $p_{s,t}(x,y)$ on $\Delta\times E\times E$, measurable in (x,y) for fixed (s,t), such that for $s<r<t$

(21) $$P_{s,t}(x,dy) = p(s,t,x,y)\mu_t(dy)$$

(22) $$p(s,t,x,y) = \int p(s,r,x,z)p(r,t,z,y)\mu_r(dz)$$

Moreover, if we set

$$\hat{P}_{s,t}(x,dy) = p(s,t,y,x)\mu_s(dy)$$

then $(\hat{P}_{s,t})$ is a backward transition function in duality with $(P_{s,t})$ w.r.t. (μ_t).

<u>Proof</u>. The following proof is inspired by Dynkin [2]. For $(s,t)\in\Delta$ we define $p_{s,t}(x,y)$ by (20) with $u=(s+t)/2$ for example. Set for $u<r<v$

(23) $$p^r_{u,v}(x,z) = \int P_{u,r}(x,dy)p_{r,v}(y,z) \quad .$$

Since we have for $u<s<v$

$$\int \beta(s,v,w,z)P_{u,s}(x,dw) = \beta(u,v,x,z) \quad \mu_u\times\mu_v(dx,dy)\text{-a.e.}$$

(verification left to the reader), one sees easily that for $B_1\in\underline{\underline{E}}_u$ $B_2\in\underline{\underline{E}}_v$

(24) $$\int_{B_1\times B_2} \mu_u(dx)p^r_{u,v}(x,z)\mu_v(dz) = \int_{B_1\times B_2} \mu_u(dx)p_{u,v}(x,z)\mu_v(dz)$$

Therefore, if we set

(25) $$N_v = \{z\in E_v : p^r_{u,v}(x,z)=p_{u,v}(x,z) \ \mu_u(dx)\text{-a.e. } \forall u,r \text{ rationals, } u<r<v\}$$

then $N_v\in\underline{\underline{E}}_v$ and $\mu_v(E_v\setminus N_v)=0$.

Now let $(u,v)\in\Delta$ and let r and s be rationals such that $u<r<s<v$. Then for all $z\in N_v$ and $x\in E_u$ we have by (25) and (23)

$$p^r_{u,v}(x,z) = \int P_{u,r}(x,dy)p_{r,v}(y,z) = \int P_{u,r}(x,dy)p^s_{r,v}(y,z) =$$
$$= \int P_{u,r}(x,dy)P_{r,s}(y,dw)p_{s,v}(w,z) = \int P_{u,s}(x,dw)p_{s,v}(w,z) =$$
$$= p^s_{u,v}(x,z) .$$

Thus, for $z\in N_v$, $x\in E_u$ $p^r_{u,v}(x,z)$ doesn't depend on the rational $r\in]u,v[$, and we denote it by $p(u,v,x,z)$. For every $v\in\mathbb{R}$ we choose arbitrarily a $z_v\in N_v$ and put, if $x\in E_u$ and $z\notin N_v$

$$p(u,v,x,z) = p(u,v,x,z_v) \ .$$

Then $p(u,v,x,z)$ is well defined on $\Delta\times E_u\times E_v$. It is easy to see that all our requirements are fulfilled.