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A PERTURBATION THEOREM FOR SEMIGROUPS OF LINEAR OPERATORS

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Let $\mathfrak{X}$ be a Banach space. Let T(t) and S(t) be strongly continuous semigroups of bounded linear operators on $\mathfrak{X}$ with infinitesimal generators A and B respectively. The main purpose of this note is to prove the following theorem :

THEOREM 1. Let $x\in\mathcal{D}(A)$. If $T(s)x\in\mathcal{D}(B)$ for $0<s<t$, and $\int_0^t \|BT(s)x\|ds < \infty$ (or equivalently $\int_0^t \|(B-A)T(s)x\|ds < \infty$), then we have

(1) $$S(t)x - T(t)x = \int_0^t S(t-s)(B-A)T(s)x\, ds$$

REMARK. If B-A is a bounded operator, the conditions are satisfied, and in this case formula (1) is due to R.S. Phillips [1] (see also [2], p. 77).

Before proving theorem 1 we give a useful corollary :

COROLLARY. Assume $\mathcal{D}(A)\subset\mathcal{D}(B)$ and B-A is A-bounded in the sense that

$$\|(B-A)y\| \leqq a\|y\| + b\|Ay\| \text{ for } y\in\mathcal{D}(A) ,$$

where $a\geqq 0$, $b\geqq 0$ are constants. Then (1) holds for $x\in\mathcal{D}(A)$.

Proof. One has $T(s)x\in\mathcal{D}(A)\subset\mathcal{D}(B)$, and

$$\|BT(s)x\| \leqq \|(B-A)T(s)x\| + \|AT(s)x\| \leqq a\|T(s)x\| + (b+1)\|T(s)Ax\| .$$

Since $\|T(s)\|\leqq Me^{\omega s}$ for some $M>0$ and $\omega\geqq 0$, we have $\int_0^t\|BT(s)x\|ds < \infty$, and therefore the conditions of theorem 1 are satisfied.

From now on we concentrate on the proof of theorem 1. Let

(2) $$G(t) = \int_0^t S(t-s)T(s)ds \quad .$$

It is easy to see that each G(t) is a bounded linear operator and G(.) is strongly continuous on $\mathfrak{X}$.

LEMMA 1. We have

(3) $$\lim_{h\downarrow 0} G(h)/h = I \text{ strongly on } \mathfrak{X}$$

(4) $$G(s+t) = S(s)G(t) + G(s)T(t)$$

Proof. Left to the reader.

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PROPOSITION 1. Let $x \in \mathcal{D}(A)$. Then $G(t)x \in \mathcal{D}(B)$, and we have

(5) $$S(t)x-T(t)x = BG(t)x - G(t)Ax \quad .$$

Proof. By (4) we have, for $h>0$

(6) $$\frac{1}{h}(G(t+h)-G(t))x = S(t)\frac{G(h)x}{h} + G(t)\frac{T(h)-I}{h}$$

Letting $h\downarrow 0$ we see that the right derivative exists and that

(7) $$G'(t)x = S(t)x + G(t)Ax \ .$$

On the other hand, the left side of (6) is also equal to

$$\frac{S(h)-I}{h}G(t)x + \frac{G(h)}{h}T(t)x$$

Thus $G(t)x \in \mathcal{D}(B)$, and we have

$$G'(t)x = BG(t)x + T(t)x \ .$$

Combining this with (7) gives (5).

LEMMA 2. Let $x \in \mathcal{D}(A)$. Assume that $T(s)x \in \mathcal{D}(B)$ for $0<s<t$ and

(8) $$\int_0^t \|BS(t-s)T(s)x\|ds < \infty$$

Then $s \mapsto BS(t-s)T(s)x$ is Bochner integrable on $[0,t]$ and we have

(9) $$BG(t)x = \int_0^t BS(t-s)T(s)x \ ds \ .$$

Proof. Let R_λ be the resolvent $\int_0^\infty e^{-\lambda u}S(u)du$, which is defined for λ large enough, and such that $Im(R_\lambda)=\mathcal{D}(B)$, BR_λ is a bounded operator, $BR_\lambda=R_\lambda B$ on $\mathcal{D}(B)$, $\lim_{\lambda\to\infty} \lambda R_\lambda = I$ in the strong sense.

Set $f(s) = BS(t-s)T(s)x$, $f_\lambda(s) = \lambda BR_\lambda S(t-s)T(s)x = \lambda R_\lambda f(s)$

We have $\lim_\lambda f_\lambda(s)=f(s)$ for $0<s<t$, and $f_\lambda(\cdot)$ is continuous since BR_λ is bounded. Hence $f(\cdot)$ is strongly measurable. So (8) is equivalent to the Bochner integrability of f (see [3], p.133). Therefore

$$\int_0^t f(s)ds = \lim_\lambda \lambda R_\lambda \int_0^t f(s)ds = \lim_\lambda \int_0^t \lambda R_\lambda f(s)ds$$
$$= \lim_\lambda \int_0^t \lambda R_\lambda BS(t-s)T(s)ds = \lim_\lambda \lambda R_\lambda BG(t)x = BG(t)x$$

which proves (9).

Now we are in a position to give the proof of theorem 1.

Proof of theorem 1. Let $x \in \mathcal{D}(A)$ such that $T(s)x \in \mathcal{D}(B)$ for $0<s<t$, so that $BS(t-s)T(s)x = S(t-s)BT(s)x$, and

$$\int_0^t \|BS(t-s)T(s)x\|ds = \int_0^t \|S(t-s)BT(s)x\|ds$$
$$\leqq \int_0^t \|S(t-s)\|\|BT(s)x\|ds < \infty \text{ by hypothesis} \ .$$

Thus according to (9), we have

(10) $$BG(t)x = \int_0^t BS(t-s)T(s)x \ ds = \int_0^t S(t-s)BT(s)x \ ds \ .$$

On the other hand, we have

$$(11) \qquad G(t)Ax = \int_0^t S(t-s)T(s)Ax\,ds = \int_0^t S(t-s)ATx\,ds$$

Combining (10) and (11) we get

$$BG(t)x-G(t)Ax = \int_0^t S(t-s)(B-A)T(s)\,ds$$

and we conclude using (5). $\square$

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