SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

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Séminaire de probabilités (Strasbourg), tome 22 (1988), p. 542-599
<http://www.numdam.org/item?id=SPS_1988__22__542_0>

Pathwise approximations of processes

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ABSTRACT

A stochastic approximation technique is developed so that an $\mathbf{R}^d$-valued random process X in discrete or continuous time can be viewed as a limit of "simple" quantities $(\xi^{(n)})$ $(n \geq 0)$, subsequently called "skeletons". The important feature of the approximation is that convergence is understood in the strongest possible sense, not only for the processes (where pathwise convergence is required), but also for the underlying filtrations (for which a natural convergence concept is introduced): the simple processes $\xi^{(n)}$ take finitely many different values at any point in time, change values at a finite number of times and possess an information structure (filtration) that is discrete and finite. The constructions provide a tool to explicitly deal with the fine structure of a given filtration and can be used to gain insight into the dynamic nature of the underlying process. The sequences of skeletons that are constructed (i) approximate a given random process X pathwise and (ii) preserve structural properties of X (for example, (sub-, super-) martingale property, martingale representation property, Markov property) by means of convergence of information. This ability to "deal with information" is illustrated with a variety of examples, including pathwise approximations of standard Brownian motion in one and higher dimensions.

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* Research supported by the National Science Foundation grant ECS-8696-090 at Boston University.

1. Introduction

There exist several different kinds of probabilistic convergence that can be applied when approximating a given stochastic process. The concept of *weak convergence* (for which Billingsley (1968) is the standard reference) is certainly the most popular and widely used concept. It considers two processes identical if they have the same function space distribution. One of the main uses of weak convergence is in studying the distributions of functionals of sample paths. However, there exist interesting objects that cannot be represented as a functional of sample paths; for example, the optimally-stopped value, $g(X) \equiv \sup\{E[X_T] : T \text{ a stopping time }\}$ of a process X (see Aldous (1981)).

The concept of *extended weak convergence,* developed by Aldous (1981), is capable of approximating optimally-stopped values such as $g(X)$. Extended weak convergence was partly introduced in response to the modern French general theory of processes (à la Dellacherie and Meyer (1978, 1982)). Whereas the general theory stresses sample path aspects of stochastic processes and emphasises the importance of the underlying filtrations, ordinary weak convergence is concerned with function space distributions and cannot distinguish between certain processes with dramatically different dynamic structures (see Aldous (1981)). Extended weak convergence connects these two theories by considering two processes "essentially" the same if their corresponding *prediction processes* (see Aldous (1981)) have the same function space distribution. Consequently, a sequence of processes (with corresponding filtrations) converges in the extended weak sense if the associated sequence of prediction processes converges weakly. One of the attractive features of the notion of extended weak convergence is a series of "almost structure-preserving" results that take the following form.

(*) If a sequence of processes approximates a given process X in the extended weak sense and X has a certain structural property (for example, (sub-, super-) martingale property) then this property "almost" holds for each process "far enough out" in the approximating sequence.

There are, however, (limiting) processes X with structural properties which cannot be even "almost" preserved along sequences that approximate X in the extended weak sense. One such example concerns d-dimensional Brownian motion ($1 \le d < \infty$) and its property of *completeness*, also called the *martingale representation property*. (Generally attributed to Itô (1951), different proofs of this property appear in Kunita and Watanabe (1967), Clark (1970), Dellacherie (1975).) The completeness property is important, for instance, in the context of the theory of continuous trading where one would like the processes approximating d-dimensional Brownian motion to also be complete (see Harrison and Pliska (1981, 1983), Kreps (1982), Duffie and Huang (1985)).

The purpose of this paper is to develop an approximation scheme for stochastic processes that emphasizes the stronger "structure-preserving" condition (**).

(**) If a sequence of processes approximates a given process X and X has a certain structural property, then this property holds *as well* for each process along the approximating sequence.

In a subsequent paper (Willinger and Taqqu (1987)) we use the approximation scheme to show that the martingale representation property can be preserved along approximating sequences and to construct stochastic integrals sample path by sample path. This allows us to provide a solution to an open problem stated in Kopp (1986, p. 169).

We concentrate here on processes $X = (X_t : 0 \leq t \leq T)$ with continuous sample paths and postpone a discussion of processes with jumps to a later paper (see Section 5.4, however, for a process with jumps at fixed points in time). Strictly maintaining a structural property of X for approximating processes requires explicit manipulation of the fine structure of the filtration $\mathbf{F}$ associated with X. We construct simple approximating processes $\xi^{(n)}$ and filtrations $\mathbf{F}^{(n)}$ $(n \geq 0)$ such that the pairs $(\mathbf{F}^{(n)}, \xi^{(n)})$, called skeletons, converge to $(\mathbf{F}, X)$ in the following two ways:

(i) *pathwise convergence*, that is,

$$P[\{\omega \in \Omega : \lim_{n \to \infty} \sup_{0 \leq t \leq T} |X_t(\omega) - \xi_t^{(n)}(\omega)| = 0\}] = 1 \quad ,$$

and

(ii) *convergence of information*, that is, for each $0 \leq t \leq T$ and for each $n \geq 0$,

$$\mathcal{F}_t = \sigma(\bigcup_{n \geq 0} \mathcal{F}_t^{(k)}) \supseteq \cdots \supseteq \mathcal{F}_t^{(n+1)} \supseteq \mathcal{F}_t^{(n)} \supseteq \cdots \supseteq \mathcal{F}_t^{(0)} \quad (\text{up } P\text{-null sets})$$

This strong dual form of convergence guarantees condition (**) for the approximations we construct.

The rest of the paper is organized as follows. In Section 2, we derive our basic building block; namely, a method for simultaneously (i) approximating almost surely an arbitrary random vector X by a particular sequence of simple random vectors, and (ii) approximating, $\sigma(X)$, the σ-algebra generated by X, by certain finitely generated σ-algebras. This approximation scheme is first applied to a simple change of measure problem and then used in Section 3 to construct skeletons of discrete-time stochastic processes $X = (X_t : t = 0,1,...,T)$. Examples are given which demonstrate the structure-preserving nature of the skeletons. Section 4 generalizes the skeleton-technique to the case of continuous-time stochastic processes $X = (X_t : 0 \leq t \leq T)$ with continuous sample paths. We present explicit constructions of skeletons which again maintain a given structural property of X. The ability of the skeleton-technique to explicitly control the flow of information is illustrated in Section 5 with several examples of skeleton-approximations of well-known stochastic processes with quite different dynamic natures.

2. An almost-sure approximation scheme for random vectors

In this section, we develop an almost-sure approximation technique for random vectors X and their associated σ-algebras $\sigma(X)$ which serves as the foundation for our subsequent constructions. As an application, we solve a simple change of measure problem and illustrate why the information contained in the filtrations along the approximation suffices to solve problems about X.

2.1 Notations, definitions, and a construction

Consider an arbitrary d-dimensional ($d \geq 1$), half-open rectangle R in $\mathbf{R}^d$ and let $y = (y^1, y^2,..., y^d)$ denote an interior point of R. Then y gives rise to a natural partition of R into $D = 2^d$ half-open, d-dimensional subrectangles $R(1), R(2),..., R(D)$, where for $i \in \{1,2,...,D\}$,

$$R(i) = \{x \in R : x\Delta(i)y\} .$$

Here, $x\Delta(i)y$ is shorthand for $x^j > y^j$ or $x^j \leq y^j$ depending on whether the j^{th} digit in the binary expansion of $i-1$ is 0 or 1 ($j = 1,2,...,d$). The use of $\Delta(\cdot)$ is convenient for describing in a unique manner the position of subrectangle $R(\cdot)$ relative to the point $y \in R$. For a simple illustration, see Figure 2.1.1.

Next, we introduce a particular indexing scheme for systematically enumerating all the pieces obtained by partitioning a rectangle in $\mathbf{R}^d$ successively into smaller and smaller subrectangles. For $n \geq 0$, let $M(n)$ denote the set of all D-adic words of length n (i.e. all words of length n over the alphabet $\{1,2,...,D = 2^d\}$). We define $M(n)$ recursively as follows:

(0) For $n = 0$, set $M(0) = \{\varnothing\}$.

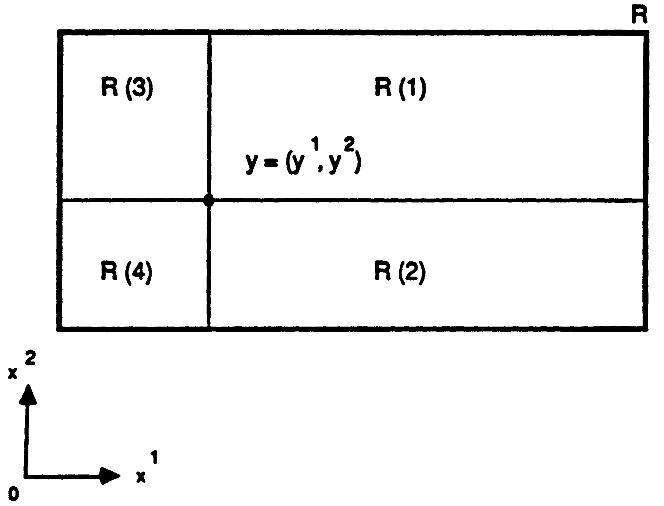


Figure 2.1.1 Partition of a rectangle R ($d = 2, D = 2^2 = 4$).

(1) For $n > 0$, assume $M(n-1)$ is defined and set

$$M(n) = \{m : m = m'i, m' \in M(n-1) \text{ and } i \in \{1, 2, ..., D\}\} \ .$$

(Here, for $m_1 \in M(n_1)$ and $m_2 \in M(n_2)$, $m_1 m_2$ denotes the word of length $n_1 + n_2$ in $M(n_1 + n_2)$ which is the concatanation of the words m_1 and m_2. Note that $\emptyset m = m\emptyset = m$.) For example, in the case $d = 2$, we have $M(0) = \{\emptyset\}, M(1) = \{1,2,3,4\}, M(2) = \{11, 12, 13, 14, 21, ..., 43, 44\}$, etc.

Lastly, fix an arbitrary probability space $(\Omega, \mathcal{F}, P)$and consider an integrable random variable X with values in $\mathbf{R}^d$ $(1 \leq d < \infty)$; that is, assume that $E_P[\ |X\ |\ ] < \infty$ where $E_P[\cdot]$ denotes the mathematical expectation with respect to the probability measure P. We call a random variable *simple (or elementary)* it it takes only finitely many values.

Our goal is to construct a sequence $(X_n)_{n\geq 0}$ of simple random variables X_n with values in $\mathbf{R}^d$ and a sequence $(\mathcal{A}_n)_{n\geq 0}$ of finitely generated, nondecreasing σ-algebras $\mathcal{A}_n$

such that

(i) $X_n \to X$ P-a.s. as $n \to \infty$,

and

(ii) $\sigma(\bigcup_{n\geq 0} \mathcal{A}_n) = \sigma(X)$ (up to P-null sets).

The idea of the construction is to subdivide $\mathbf{R}^d$ (and consequently Ω) successively into smaller and smaller "pieces". This "size" of each piece will be determined by the random variable X to be approximated and the underlying probability measure P.

The following recursive procedures uses the notation introduced above and completely describes our approximation scheme.

Explicit construction of the approximation scheme for X

(0) For $n = 0$, set

$$R(\varnothing) = \mathbf{R}^d \quad ,$$

$$A(\varnothing) = \Omega \quad ,$$

and define

$$\mathcal{A}_0 = \{\varnothing, \Omega\} \quad ,$$

$$X_0(\omega) = E_P[X] \text{ for all } \omega \in \Omega \quad .$$

(1) For $n > 0$, assume that $\{R(m): m \in M(n-1)\}$, $\{A(m): m \in M(n-1)\}$, $\mathcal{A}_{n-1}$, and X_{n-1} are already defined. For all $m \in M(n-1)$ and for $i = 1, 2, ..., D$ set

$$R(mi) = \begin{cases} \{x \in R(m): x\Delta(i)x(m)\}, & \text{if } P[A(m)] > 0 \\ R(m), & \text{if } P[A(m)] = 0 \end{cases}$$

(where $x(m)$ denotes the value of X_{n-1} on $A(m)$),

$$A(mi) = X^{-1}(R(mi)) = \{\omega \in \Omega : X(\omega) \in R(mi)\} \quad ,$$

and define

$$\mathcal{A}_n = \sigma(1_{A(m)} : m \in M(n)) ,$$

$$X_n(\omega) = \begin{cases} (1/P[A(m)])\, E_P[X; A(m)], & \text{if } \omega \in A(m) \text{ for some} \\ & m \in M(n) \text{ with } P[A(m)] > 0, \\ X_{n-1}(\omega) , & \text{otherwise} . \end{cases}$$

Here, $E_P[X;A(m)]$ means $\int_{A(m)} X(\omega)\, dP(\omega)$ so $(1/P[A(m)])E_P[X;A(m)] = E_P[X \mid A(m)]$. Therefore, $X_n = E_P[X \mid \mathcal{A}_n]$ P-a.s. The definition of X_n given above specifies its values even when ω belongs to a set of P-measure zero. Figure 2.2.2 provides an illustration in the case $d = 2$. Observe that the recursive procedure always yields elements $R(m)$ $(m \in M(n),\ n \geq 0)$ which are subrectangles in $\mathbf{R}^d$. Moreover, the size of each $R(m)$ is uniquely determined by X and the underlying probability measure P.

2.2 Probabilistic and geometric properties of the approximation scheme

Theorem 2.2.1 below summarizes the probabilistic properties of this approximation scheme. In particular, it shows that the sequence $(X_n)_{n\geq 0}$ converges to X almost surely and that $\sigma(\cup_{n\geq 0} \mathcal{A}_n)$ and $\sigma(X)$ contain the same information.

Theorem 2.2.1

(1) For each $n \geq 0$, X_n is a simple random variable with values in $\mathbf{R}^d$.

(2) For each $n \geq 0$, $\mathcal{A}_n = \sigma(X_n) = \sigma(\mathcal{P}_n)$, where $\mathcal{P}_n$ denotes the partition of Ω generated by the elements of the set $\{A(m) : m \in M(n)\}$.

(3) The sequence $(\mathcal{A}_n)_{n\geq 0}$ satisfies $\sigma(X) = \sigma(\cup_{n\geq 0} \mathcal{A}_n) \supseteq \dots \mathcal{A}_n \supseteq \mathcal{A}_{n-1} \supseteq \dots \supseteq \mathcal{A}_0 = \{\emptyset,\Omega\}$ (up to P-null sets).

(4) $(X_n)_{n\geq 0}$ is a uniformly integrable $\mathbf{R}^d$-valued $(\mathcal{A}_n, P)$-martingale and therefore, $X_n \to X$ P-a.s. and in $L^1(\Omega, \mathcal{F}, P)$ as in $n \to \infty$.

Proof: Properties (1), (2) and the fact that the $\mathcal{A}_n$'s are nondecreasing are immediate consequences of the construction of the sequences $(X_n)_{n\geq 0}$ and $(\mathcal{A}_n)_{n\geq 0}$. Moreover, we obviously have $\sigma(X) \supseteq \sigma(\cup_{n\geq 0} \mathcal{A}_n)$ and to complete the proof of (3) it therefore suffices to show that $\sigma(\cup_{n\geq 0} \mathcal{A}_n) \supseteq \sigma(X)$ (up to P-null sets). However, for every pair $a, b \in \mathbf{R}^d$ with $a < b$ (i.e. $a^j < b^j$, $j = 1, 2, ..., d$), one can find a sequence $(A_k)_{k\geq 0}$ of pairwise disjoint sets in $\cup_{n\geq 0} \mathcal{P}_n$ with $X^{-1}((a, b]) = \{\omega \in \Omega : a < X(\omega) \leq b\} = \cup_{k\geq 0} A_k$. By definition, $\sigma(X) = \sigma(X^{-1}((a, b]) : a < b;\, a, b \in \mathbf{R}^d)$ which proves the result. Property (4) is standard martingale theory (for example, see Dellacherie and Meyer (1982, p. 26)) after observing that $X_n = E[X \mid \mathcal{A}_n]$ P-a.s. and $X \in L^1(\Omega, \mathcal{F}, P)$. □

Remarks 2.2.1 1) In addition to properties (1)-(4), it is easy to see that the sequence $(X_n)_{n\geq 0}$ also has the Markov property, that is

$$P[X_{n+1} = x_{n+1} \mid X_0 = x_0, X_1 = x_1, ..., X_n = x_n] = P[X_{n+1} = x_{n+1} \mid X_n = x_n] \quad ,$$

provided the conditioning events have positive probability. Indeed, $X_n = x_n$ indicates the partition cell of $\mathcal{P}_n$ in which $X(\omega)$ falls, and this specifies $X_0, ..., X_{n-1}$.

2) The approximation scheme can also be used to approximate an arbitrary probability measure μ on $(\mathbf{R}^d, \mathcal{B}(\mathbf{R}^d))$ with $\int x\mu(dx) < \infty$ by a particular sequence $(\mu_n)_{n\geq 0}$ of probability measures. Simply, set $\Omega = \mathbf{R}^d$, $\mathcal{F} = \mathcal{B}(\mathbf{R}^d)$, $P = \mu$, let X denote the identity mapping and for $n \geq 0$ define $\mu_n = \mu \circ X_n^{-1}$, the image of μ under X_n. Then for each $n \geq 0$, μ_n is a discrete probability measure with finite support in $\mathbf{R}^d$ and moreover, μ_n converges *weakly* to μ as $n \to \infty$. This is the context in which Dubins (1968) originally introduced this approximation scheme in the one-dimensional case. Dubins used this construction as a main tool for his new proof of the Skorohod-embedding of martingales which works only when $d = 1$. A beautiful presentation of Dubin's construction (as it became known) can be found in Meyer (1971) (see also Billingsley (1979, p. 459)).

3) The approximation scheme provides an almost sure convergent sequence $(X_n)_{n\geq 0}$ of simple, $\mathbf{R}^d$-valued random variable even when X is *not* assumed to be integrable. Simply, set

$$Y = \arctan X \text{ (componentwise)}$$

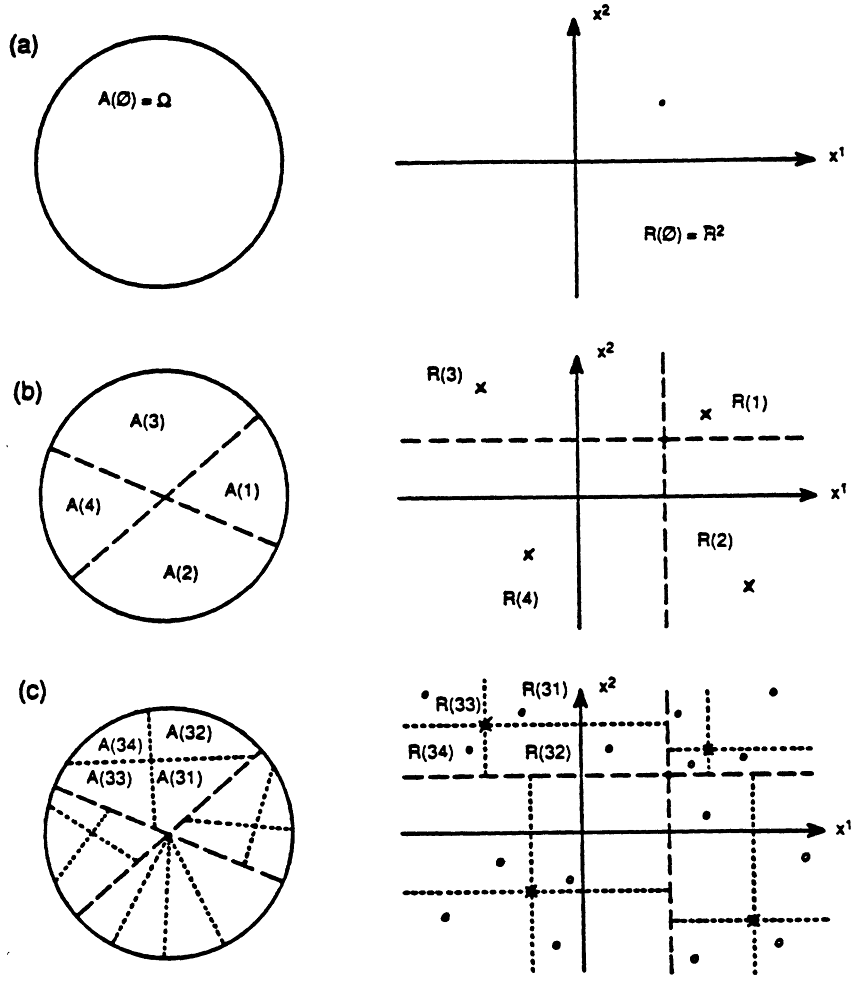


Figure 2.1.2 Approximation scheme for $d = 2$.

$(a)\ n = 0$: $*$ value of X_0
$(b)\ n = 1$: x values of X_1
$(c)\ n = 2$: $\circ$ values of X_2

and note that since arctan is one-to-one, $\sigma(X) = \sigma(Y)$. In addition, Y is a bounded random vector so that our approximation scheme applied to Y and P yields sequences $(Y_n)_{n \geq 0}$ and $(\mathcal{B}_n)_{n \geq 0}$ with

(i) $Y_n = E[Y \mid \mathcal{B}_n] \to Y$ P-a.s. as $n \to \infty$, and

(ii) $\sigma(\bigcup_{n \geq 0} \mathcal{B}) = \sigma(Y)$ $(= \sigma(X)$, up the P-null sets).

Therefore, the sequence $(X_n)_{n \geq 0}$ with $X_n = \tan Y_n$ P-a.s. $(n \geq 0)$ satisfies $P[\{\omega \in \Omega : X_n(\omega) \to X(\omega)\}] = 1$.

The construction of the approximating sequences $(X_n)_{n \geq 0}$ and $(\mathcal{A}_n)_{n \geq 0}$ together with the results of Theorem 2.2.1 suggest a geometric interpretation of the martingale nature of the approximation scheme. The corresponding geometry is provided by Theorem 2.2.2 below. It uses some concepts from convex analysis which we now introduce

Consider a simple, $\mathbf{R}^d$-valued random variable Y taking values $y_1, y_2, \ldots, y_n$. The *convex hull* of the set $\{Y(\omega) : \omega \in \Omega\}$ is

$$\text{conv}(\{Y(\omega) : \omega \in \Omega\}) = \{y \in \mathbf{R}^d : y = \sum_{k=1}^{n} \lambda_k y_k \, , \sum_{k=1}^{n} \lambda_k = 1, \lambda_k \geq 0 \; \forall \, k\}$$

and its *affine hull* is

$$\text{aff}(\{Y(\omega) : \omega \in \Omega\}) = \{y \in \mathbf{R}^d : \; y \; = \; \sum_{k=1}^{n} \lambda_k y_k \, , \; \sum_{k=1}^{n} \lambda_k = 1\} \quad .$$

When dealing with convex sets C in $\mathbf{R}^d$, the concept of interior is replaced by the more convenient one of relative interior, reflecting the fact that C, regarded as subset of $\mathbf{R}^d$, does not have an interior when $\dim(C) < d$. The *relative interior* of C, denoted $ri(C)$, is the interior of C when C is viewed as a subset of $\text{aff}(C)$. Formally

$$ri(C) = \{y \in \text{aff}(C) : \exists \, \varepsilon > 0 \text{ such that } C \supset (y + \varepsilon B) \cap \text{aff}(C)\}$$

where $B = \{y \in \mathbf{R}^d : \; |y| \leq 1\}$ denotes the Euclidian unit ball in $\mathbf{R}^d$.

Theorem 2.2.2. For each $n \geq 0$, set $C_n = \text{conv}(\{X_n(\omega) : \omega \in \Omega\})$. Then we have:

(1) For each $n \geq 0$, C_n is a bounded polyhedron, that is, a convex polytope, and hence closed.

(2) For each $n \geq 0$, $C_{n+1} \supseteq C_n$, and moreover, $ri(C_{n+1}) \supseteq ri(C_n)$.

(3) $ri(\bigcup_{n\geq 0} C_n) = \bigcup_{n\geq 0} ri(C_n)$.

Proof. (1) follows directly from Theorem 2.2.1(1) and the definition of C_n. To prove (2), note that in this case, the martingale property of the sequence $(X_n)_{n\geq 0}$ (Theorem 2.2.1(4)), namely the fact that for each $n \geq 0$,

$$X_n = E_P[X_{n+1} \mid \mathcal{A}_n] \quad P\text{-a.s.} \ ,$$

can be paraphrased as follows: each value of X_n is a convex combination of certain values of the random vector X_{n+1}. Therefore, $C_{n+1} \supseteq C_n$ for each $n \geq 0$. To show that $ri(C_{n+1}) \supseteq ri(C_n)$, note that the martingale property of $(X_n)_{n\geq 0}$ prevents C_n from being entirely contained in the relative boundary $C_{n+1} \backslash ri(C_{n+1})$ of C_{n+1}; i.e. for each $n \geq 0$, $C_{n+1} \backslash ri(C_{n+1}) \not\supseteq C_n$. The conclusion $ri(C_{n+1}) \supseteq ri(C_n)$, $n \geq 0$, follows from a straightforward application of Rockafellar (1970, Corollary 6.5.2).

Finally, we prove property (3). In order to show that $\bigcup_{n\geq 0} ri(C_n) \supseteq ri(\bigcup_{n\geq 0} C_n)$, we can assume without loss of generality that for all $n \geq 1$, $\dim(C_n) = d' \leq d$ (otherwise, neglect the first few C_n's with $\dim(C_n) < d'$). In fact, we can assume that $d' = d$ (otherwise, the problem can be reduced to one in $\mathbf{R}^{d'}$). Now let $x \in ri(\bigcup_{n\geq 0} C_n)$. By definition there exists $\varepsilon > 0$ such that

$$\bigcup_{n\geq 0} C_n \supseteq (x + \varepsilon B) \ , \tag{2.2.1}$$

where B denotes the Euclidean unit ball in $\mathbf{R}^d$. Let $x_1, x_2, \ldots, x_{d+1}$ denote the $d+1$ vertices of a d-dimensional simplex $S_x{}^d$ with center x inscribed in the ε-ball with center x. Since for each n, C_n is convex and $C_{n+1} \supseteq C_n$, $\bigcup_{n\geq 0} C_n$ is convex and hence, there exists $0 < n^* < \infty$ with $x_i \in C_{n^*}$ $(i = 1, 2, \ldots, d+1)$. Thus, $C_{n^*} \supseteq S_x{}^d$, that is, there exists $\varepsilon > \varepsilon' > 0$ with

$$C_{n^*} \supseteq x + \varepsilon' B \ , \tag{2.2.2}$$

i.e. $x \in ri(C_{n^*})$ and consequently $x \in \bigcup_{n\geq 0} ri(C_n)$. On the other hand, for each $m \geq 0$, $\bigcup_{n\geq 0} C_n \supseteq C_m$ and in order to conclude that $ri(\bigcup_{n\geq 0} C_n) \supseteq ri(C_m)$, we only have to show again that C_m is not entirely contained in the relative boundary of $\bigcup_{n\geq 0} C_n$, i.e.

$(\cup_{n\geq 0} C_n) \setminus ri(\cup_{n\geq 0} C_n) \not\supseteq C_m$. This is done as in the proof of (2). □

2.3 A simple application: changing probability measures

In this subsection, the approximation scheme is applied to a simple problem involving transformations of the underlying probability measure P. First, we introduce some notation. A probability measure Q on $(\Omega,\mathcal{F})$ is said to be *absolutely continuous with respect to P* $(Q \ll P)$ if $P[A]=0$ implies $Q[A]=0$ $(A \in \mathcal{F})$. Q is said to be *equivalent to P* $(Q \sim P)$ if P and Q are mutually absolutely continuous (i.e. $Q \ll P$ and $P \ll Q$).

The change of measure problem under consideration can now be formulated as follows. For an integrable, $\mathbf{R}^d$-valued random variable X, let $m \in \mathbf{R}^d$ denote its mean vector and assume that $m \neq 0$. What are necessary and sufficient conditions (on X and $\mathcal{F}$) for the existence of a new probability measure Q on $(\Omega, \mathcal{F})$, equivalent to P and such that X has zero mean under Q? Thus, we are looking for condition which allow us to redistribute probability mass among the "possible" events in $\mathcal{F}$ so as to "center" X around the origin.

The solution to this change of measure problem is given in Theorem 2.3.1 below and requires an application of our approximation scheme to the random variable X and the probability measure P. The resulting sequences $(X_n)_{n\geq 0}$ and $(\mathcal{A}_n)_{n\geq 0}$ are shown to contain all the relevant information needed for such an equivalent measure transformation. The main idea of the proof can be illustrated as follows.

As stated in Theorem 2.2.1(2), the elements of $\mathcal{P}_n = \{A(m): m \in M(n)\}$ form the cells of the minimal partition of Ω generating the σ-algebra $\mathcal{A}_n$ $(n \geq 0)$. By invoking the law of total probability, for any $B \in \mathcal{F}$ and $n \geq 0$ we can write

$$P[B] = \sum_{A(m)\in M(n)} P[B \cap A(m)]$$

$$= \sum_{A(m)\in M(n)} P[B \mid A(m)] P_n[A(m)] \; .$$

Here, P_n denotes the restriction $P\,|_{\mathcal{A}_n}$ of the probability measure P on the σ-algebra $\mathcal{A}_n$, and each conditional probability $P[\cdot \mid A(m)]$ is defined in an elementary way (with the convention $0/0=1$). This decomposition of the measure P into an "initial distribution" P_n and "transition probabilities" $P[\cdot \mid A(m)]$ contains the fundamental property of our change of measure problem. In order to find $Q \sim P$ such that $E_Q[X]=0$, it suffices to be able to change P_n in an equivalent way so that X_n becomes "centered". Whereas the original problem typically involves a random variable which takes more than countably many different values in $\mathbf{R}^d$, the latter is concerned with a *simple* random variable (Theorem 2.2.1(1)) and hence constitutes a finite-dimensional problem. However, such a finite-dimensional change of measure problem has already been studied in Taqqu and Willinger (1987).

Theorem 2.3.1. The following statements are equivalent.

(I) $0 \in ri(\cup_{n\geq 0} C_n) = \cup_{n\geq 0} ri(C_n)$, where $C_n = \operatorname{conv}(\{X_n(\omega): \omega \in \Omega\})$ $(n \geq 0)$.

(II) There exists no (nonrandom) vector $\alpha \in \mathbf{R}^d$ such that $\alpha \cdot X \geq 0$ P-a.s. and $P[\{\omega \in \Omega : \alpha \cdot X(\omega) \neq 0\}] > 0$.

(III) There exists a probability measure Q on $(\Omega, \mathcal{F})$ with $Q \sim P$ and such that $E_Q[X] = 0$.

Proof. 1) First, we prove (I) $\Rightarrow$ (III). Statement (I) implies that there exists $0 < n^* < \infty$ with $0 \in ri(C_{n^*})$. We then focus on the n^*-th step in our approximation scheme, namely on the random variable X_{n^*}, the partition $\mathcal{P}_{n^*}$ of Ω, and the probability measure $P_{n^*} = P \;|_{\mathcal{A}_{n^*}}$ on $(\Omega, \mathcal{A}_{n^*})$. The change of measure problem in such a finite setup (involving $(\Omega, \mathcal{A}_{n^*}, \mathcal{P}_{n^*})$ and X_{n^*}) has been solved in Taqqu and Wilinger (1987) and by applying their Proposition 3.1, we conclude: there exists a new probability measure Q_{n^*} on $(\Omega, \mathcal{A}_{n^*})$ with $Q_{n^*} \sim P_{n^*}$ such that $E_{Q_{n^*}}[X_{n^*}] = 0$. Next, for each $B \in \mathcal{F}$, we set

$$Q[B] = \sum_{A(m)\in P_{n^*}} P[B \;|A(m)] Q_{n^*}[A(m)] \;.$$

(Interpret $Q_{n^*}[\cdot]$ as "new initial distribution", $P[\cdot \;|A(m)]$ as "transition probabilities".) Clearly, Q defines a probability measure on $(\Omega, \mathcal{F})$ and moreover, $Q \sim P$ with (setting $(0/0 = 1)$

$$(dQ/dP)(\cdot) = \sum_{A(m)\in P_{n^*}} (Q_{n^*}[A(m)]/P_{n^*}[A(m)]) \, 1_{A(m)}(\cdot) \qquad P\text{-a.s.}$$

Thus we have $X \in L^1(\Omega, \mathcal{F}, Q)$ and

$$E_Q[X] = E_Q[E_Q[X \mid \mathcal{A}_{n*}]]$$

$$= E_Q[E_P[X \mid \mathcal{A}_{n*}]] \qquad (Q[\cdot \mid \mathcal{A}_{n*}] = P[\cdot \mid \mathcal{A}_{n*}] \;\; P\text{–a.s.})$$

$$= E_Q[X_{n*}] \qquad ((X_n)_{n\geq 0} \text{ is a } (\mathcal{A}_n\ P)\text{–martingale with } X_n = E_P[X \mid \mathcal{A}_n] \;\; P\text{–a.s.})$$

$$= E_{Q_{n*}}[X_{n*}] \qquad (Q_{n*} = Q\ |_{\mathcal{A}_{n*}})$$

$$= 0 \quad ,$$

that is, Q has the desired properties.

2) Next we show (III) => (II). Let $M = dQ/dP$ denote the Radon-Nikodym derivative of Q with respect to P. Since $Q \sim P$, the Radon-Nikodym theorem states, among other things, that $0 < M < \infty$ P-a.s. and hence Q-a.s. Moreover, $MX \in L^1(\Omega, \mathcal{F}, P)$ iff $X \in L^1(\Omega, \mathcal{F}, Q)$ and consequently,

$$E_P[MX] = E_Q[X] = 0 \quad ,$$

i.e. for any $\alpha \in \mathbf{R}^d$, we have

$$E_P[(\alpha \cdot X)M] = \alpha \cdot E_P[MX] = 0 \quad .$$

Thus, there cannot exist a vector $\alpha \in \mathbf{R}^d$ with

$$\alpha \cdot X \geq 0 \quad P \text{ a.s. and } P[\alpha \cdot X \neq 0] > 0 \quad .$$

3) Finally, we prove (II) => (I) by contradiction. Assuming that 0 $ri(\cup_{n\geq 0} C_n)$, we will show that (II) cannot hold. First we note that $\cup_{n\geq 0} C_n$ is a convex set in $\mathbf{R}^d$ (use Theorem 2.2.2(2)). Therefore, our assumption provides us with two non-empty convex sets, namely $\{0\}$ and $\cup_{n\geq 0} C_n$, whose respective relative interior have no points in common, i.e. $ri(\{0\}) \cap ri(\cup_{n\geq 0} C_n) = \varnothing$. However, this condition guarantees (see Rockafellar (1970, p.97)) the existence of a hyperplane separating $\{0\}$ and $\cup_{n\geq 0} C_n$ properly.

That is, there exists $\alpha_0 \in \mathbf{R}^d$ (independent of n) such that for each $n \geq 0$, we have

(2.3.1) $$\alpha_0 \cdot X_n \geq 0 \text{ and } \alpha_0 \cdot X_n \not\equiv 0 \quad .$$

The martingale nature of the sequence $(X_n)_{n \geq 0}$ (Theorem 2.2.1(4)) together with (2.3.1) will allow us to show that α_0 also satisfies

(2.3.2) $$\alpha_0 \cdot X \geq 0 \quad P\text{-a.s. and } P[\alpha_0 \cdot X \neq 0] > 0$$

Indeed, on the one hand, almost-sure convergence implies

(2.3.3) $$\alpha_0 \cdot X = \alpha_0 \cdot (\lim_{n \to \infty} X_n) = \lim_{n \to \infty} \alpha_0 \cdot X_n \geq 0 \quad P\text{-a.s.}$$

In the other hand, L^1-convergence yields for each $n \geq 0$,

(2.3.4) $$E_P[\alpha_0 \cdot X] = E_P[\alpha_0 \cdot X_n] > 0 \quad .$$

Together, (2.3.3) and (2.3.4) show that α_0 satisfies (2.3.2) which proves the desired contradiction.

The equality $ri(\cup_{n \geq 0} Cn) = \cup_{n \geq 0} ri(C_n)$ results from Theorem 2.2.2(3). □

The results of Theorem 2.3.1 do not depend on the particular (i.e., rectangular) shape of the elements $R(m) \subseteq \mathbf{R}^d$ ($m \in M(n)$, $n \geq 0$) constructed in Step 1 of the approximation scheme. In fact, any partition of $\mathbf{R}^d$ works as long as the corresponding partitions of Ω become finer along the approximation. More precisely, we have the following result the proof of which is identical to the one given above.

Corollary 2.3.1 Let $(\mathcal{P}_n)_{n \geq 0}$ be a non-decreasing sequence of finite partitions of Ω such that each cell of $\mathcal{P}_n$ ($n \geq 0$) has positive probability, and set $\mathcal{B}_n = \sigma(\mathcal{P}_n)$ ($n \geq 0$). Assume that $\sigma(\bigcup_{n \geq 0} \mathcal{B}_n) = \sigma(X)$ (up to P-null sets) for some random vector $X \in L^1(\Omega, \mathcal{F}, P)$. If the sequences $(X_n)_{n \geq 0}$ and $(C_n)_{n \geq 0}$ are defined by

$$X_n = E_P[X \mid \mathcal{B}_n] \; P\text{-a.s.}, n \geq 0 \text{ , and}$$

$$C_n = \operatorname{conv}(\{X_n(\omega) : \omega \in \Omega\}),\ n \geq 0, \text{resp.},$$

then the results of Theorem 2.3.1 hold.

The next result follows directly from the proof of Theorem 2.3.1 and states explicitly the most appealing feature of the change of measure problem under consideration, namely its finite-dimensional nature. Note that this reduction to finite dimensions is a consequence of the approximation scheme and can be viewed as a first example of a "structure-preserving" result (see property (**) in Section 1).

Corollary 2.3.2 There exists a probability measure Q on $(\Omega, \mathcal{F})$ with $Q \sim P$ and such that $E_Q[X] = 0$ if and only if for some $0 < n < \infty$ there exists a probability measure Q_n on $(\Omega, \mathcal{A}_n)$ with $Q_n \sim P\ |_{\mathcal{A}_n}$ and such that $E_{Q_n}[X_n] = 0$.

Finally we observe that the solution to the change of measure problem stated in Theorem 2.3.1 does not impose any conditions on the underlying information structure, mathematically modelled by the σ-algebra $\mathcal{F}$. In fact, the proof of Theorem 2.3.1 makes it clear that all the information relevant for finding $Q \sim P$ with $E_Q[X] = 0$ is contained in $\sigma(X)$, an intuitively obvious result. However, when one focuses on the *uniqueness* of such a Q, the situation changes and $\mathcal{F}$ plays a crucial role. In order to see this, we first observe that if Q is to be unique, the approximating sequence $(X_n)_{n \geq 0}$ has to terminate, that is, all the X_n must be equal for n greater than some finite n^*. Indeed, it is easily seen (recall the proof of Theorem 2.3.1 ((I) => (III))) that in the case of a non-terminating sequence $(X_n)_{n \geq 0}$ all choices of $n \geq n^*$ are suitable (where n^* is such that $0 \in ri(C_{n^*})$), and each choice yields a different Q. Thus, for Q to be unique, it is necessary that X be a *simple* random vector (up to P-null sets). In that case the finite sample space results of Taqqu and Wilinger (1987, Theorem 4.1 and Corollary 4.1) are applicable (see also Remark 2.3.1), and they provide the corresponding sufficient conditions. We thus obtain the following uniqueness result for the change of measure problem.

Corollary 2.3.3 There exists a unique probability measure Q on $(\Omega, \mathcal{F})$ with $Q \sim P$ and such that $E_Q[X] = 0$ if and only if there exists $n > 0$ with the following properties:

(i) $X = X_n$ P-a.s.

(ii) $0 \in ri(C_n)$

(iii) $\dim \{span\ \{X_n(\omega) : \omega \in \Omega\}\} =$ cardinality $(A \in \mathcal{P}_n : P[A] > 0) - 1$

(iv) $\mathcal{F} = \sigma(X)$ (up to P-null sets).

Observe first that because of the definition of X_n on a partition set A with $P[A] = 0$, we can write

$$\text{cardinality } (A \in \mathcal{P}_n : P[A] > 0) \leq \text{ number of distinct values of } X_n .$$

But X_n can take at most 2^{nd} different values and therefore,

$$\text{cardinality } (A \in \mathcal{P}_n : P[A] > 0) \leq 2^{nd}.$$

However, since X_n takes values in $\mathbf{R}^d$, condition (iii) imposes the more stringent restriction

$$\text{cardinality } (A \in \mathcal{P}_n : P[A] > 0) \leq d + 1 \quad .$$

Furthermore, together with (ii), condition (iii) states that there is a unique convex combination of the values of X_n representing zero; that is, the different values of X_n are *affine independent*. Thus, condition (iii) relates the values of X_n with the fine structure of the $\mathcal{F}$.

Remarks 2.3.1 1) Taqqu and Willinger (1987) work in a strictly *finite* "world", i.e. they assume $|\Omega| < \infty$. Therefore, each reference to that paper implicitly assumes an identification of the probabilistic setting presently under consideration with its finite-dimensional "counterpart". For example, when proving Theorem 2.3.1 ((I) => (III)), the probabilistic setting at the time we apply Proposition 3.1 of Taqqu and Willinger (1987) consists of the triplet $(\Omega, \mathcal{A}_{n*}, P_{n*})$ and X_{n*}. Although X_{n*} is simple, Ω is arbitrary. But one can identify $(\Omega, \mathcal{A}_{n*}, P_{n*})$ and X_{n*} with the corresponding finite "version" $(\Omega', \mathcal{A}', P')$ and X', respectively, as follows.

$$\Omega' \equiv \{\omega' : \omega' \hat{=} A(m), m \in M(n^*), P_{n^*}[A(m)] > 0\} \quad ,$$

$$\mathcal{A}' \equiv 2^{\Omega'} = \text{set of all subsets of } \Omega' \quad ,$$

$$P'[\{\omega'\}] \equiv P_{n^*}[A(m)] \text{ where } \omega' \hat{=} A(m) \quad ,$$

$$X'(\omega') \equiv X_{n^*}(\omega) \text{ if } \omega \in A(m) \text{ and } \omega' \hat{=} A(m)$$

($\hat{=}$ means "corresponds to"). This identification yields a framework to which the results of Taqqu and Willinger (1987) are directly applicable. It also allows a re-interpretation of these results in the original setting consisting of $(\Omega, \mathcal{A}_{n^*}, P_{n^*})$ and X_{n^*}.

2) Condition (I) (or equivalently, condition (II)) of Theorem 2.3.1 is also sufficient for the existence of an absolutely continuous probability measure Q on $(\Omega, \mathcal{F})$ $(Q \ll P)$ with $E_Q[X] = 0$. However, it is easy to construct simple examples for which necessity may fail. A common feature of all these examples is that zero is contained in the relative boundary rel $\delta(\bigcup_{n \geq 0} C_n)$ of $\bigcup_{n \geq 0} C_n$ rather than in the relative interior and that the relative boundary, has positive probability. Thus it should be clear that our approximation scheme can also be used to find necessary and sufficient conditions for the existence of $Q \ll P$ with $E_Q[X] = 0$ (see Willinger (1987)).

3. Pathwise approximation of stochastic processes: the discrete-time case

This section defines and provides explicit constructions of skeleton-approximations for discrete-time stochastic processes. Skeletons take only finitely many values at any point in time, and they have finitely generated filtrations. The dynamic structure of a skeleton can be uniquely described by a tree-structure: the nodes represent possible values the process can take, and the branches depict the flow of information over time. The skeleton-approach is illustrated with examples and then used to solve a change of measure problem.

3.1 Definition of a skeleton

We fix a *stochastic base* $(\Omega, \mathcal{F}, P, \mathbf{F})$ where $(\Omega, \mathbf{F}, P)$ is a an arbitrary probability space and $\mathbf{F} = (\mathcal{F}_t : t = 0,1,...,T < \infty)$ is a given *filtration,* that is, a nondecreasing sequence of sub-σ-algebras of $\mathcal{F}$. Without loss of generality we shall assume that $\mathcal{F}_0 = \{\varnothing, \Omega\}$ and $\mathcal{F}_T = \mathcal{F}$. Next, let $X = (X_t : t = 0,1,...,T)$ denote an $\mathbf{R}^d$ -valued stochastic process defined on $(\Omega, \mathcal{F}, P)$ with component processes $X^1, \ldots, X^d$ $(1 \le d < \infty)$. We require each component X^k to be integrable and $\mathbf{F}$-adapted, i.e. X_t^k is measurable with respect to $\mathcal{F}_t$ (we write $X_t^k \in \mathcal{F}_t$) for each k and t.

The process X gives rise to the so-called *minimal* (or *natural*) filtration $\mathbf{F}^X = (\mathcal{F}_t^X : t = 0,1,...,T)$, where $\mathcal{F}_t^X$ denotes the σ -algebra generated by $X_0, X_1, \ldots, X_t$. Whereas the minimal filtration is based on knowledge of past and present values of X only, an arbitrary filtration $\mathbf{F}$ is capable of modelling additional information such as knowledge of certain future "outcomes".

A skeleton of the stochastic process X and the underlying filtration $\mathbf{F}$ is defined as follows.

Definition 3.1.1. A *skeleton of the pair* $(\mathbf{F}, X)$ is a pair $(\mathbf{F}^\xi, \xi)$ consisting of a filtration $\mathbf{F}^\xi = (\mathcal{F}_t^\xi : t = 0,1,...,T)$ (the *skeleton-filtration)* and an $\mathbf{R}^d$-valued stochastic process $\xi = (\xi_t : t = 0,1,...,T)$ (the *skeleton- process)* such that

(3.1.1a) for each t, $\mathcal{F}_t^\xi$ is finitely generated, i.e.,there exists a minimal partition $\mathcal{P}_t^\xi$ of Ω such that $\mathcal{F}_t^\xi = \sigma(\mathcal{P}_t^\xi)$,

(3.1.1b) for each t, $\mathcal{F}_t^\xi \subseteq \mathcal{F}_t$ (in particular, $\mathcal{F}_0^\xi = \{\emptyset,\Omega\}$), and

(3.1.1c) for each t, $\xi_t \in \mathcal{F}_t^\xi$.

For example, if we set $\xi_t = E_P[X_t \mid \mathcal{F}_t^\xi]$ P–$a.s.$ $(t = 0,1,...,T)$, then (3.1.1c) obviously holds and the skeleton-process ξ is simply a projection (in the sense of a conditional expectation operator) of X onto a finitely generated filtration and therefore automatically $\mathbf{F}^\xi$-adapted and *simple*. The latter means that for each t, ξ_t is a simple random vector. In general, we require only that ξ_t be measurable with respect to the finitely generated σ-algebra $\mathcal{F}_t^\xi$ $(t = 0,1,...,T)$. As a result, skeletons are completely described by their corresponding tree-structure and can be dealt with using elementary probability theory.

It follows from Definition 3.1.1 that a skeleton-process provides a more or less accurate description of X depending on whether the skeleton-filtration is capable of capturing the essential features of $\mathbf{F}$. Consequently, a successful use of skeletons for the purpose of approximating X depends almost exclusively on our ability to explicitly construct "good" skeleton-filtrations. For this purpose, the approximation scheme of Section 2 turns out to be extremely useful as shown below.

3.2 A general construction of skeletons and skeleton-approximations

A general construction of sequences of skeletons of the pair $(\mathbf{F},X)$ based on the approximation scheme of Section 2 is given next.

Explicit construction of skeletons of $(\mathbf{F},X)$

Step 0. For each $t = 0,1,...,T$, choose an increasing sequence $(m(t,n))_{n\geq 0}$ of non-negative integers.

Step 1. For $t = 0,1,...,T$, apply the approximation scheme of Section 2 to $(\Omega,\sigma(X_t),P)$ and the random vector X_t. Consider the resulting sequences $(\mathcal{A}_n(t))_{n\geq 0} (t = 0,1,...,T)$.

Step 2. For each $n \geq 0$, set

$$\mathcal{F}_t^{(n)} = \sigma(\bigcup_{s=0}^{t} \mathcal{A}_{m(s,n)}(s)) \quad (t = 0,1,...,T),$$

$$\xi_t^{(n)} = E_P[X_t \mid \mathcal{F}_t^{(n)}] \ P\text{-}a.s. \quad (t = 0,1,...,T),$$

and consider the pair $(\mathbf{F}^{(n)},\xi^{(n)})$ where

$$\mathbf{F}^{(n)} = (\mathcal{F}_t^{(n)}: t = 0,1,...,T), \text{ and}$$

$$\xi^{(n)} = (\xi_t^{(n)}: t = 0,1,...,T).$$

When the underlying filtration $\mathbf{F}$ is minimal (i.e. $\mathbf{F} = \mathbf{F}^X$) then the sequence $(\mathbf{F}^{(n)},\xi^{(n)})_{n\geq 0}$ resulting from the above construction will be called a *skeleton-approximation of the pair* $(\mathbf{F}^X, X)$. Such an approximation always exists. In general, it is not unique since it depends on the choice of the sequences $(m(t,\cdot))_{t=0,1,...,T}$ (set of "free parameters"). The properties of skeleton-approximations of $(\mathbf{F}^X, X)$ obtained this way are stated in the following Theorem.

Theorem 3.2.1. For each $t = 0,1,...,T$, let $(m(t,n))_{n\geq 0}$ denote an increasing sequence of non-negative integers and consider the resulting skeleton-approximation $(\mathbf{F}^{(n)},\xi^{(n)})_{n\geq 0}$ of the pair $(\mathbf{F}^X, X)$. Then the following properties hold.

(1) For each $n \geq 0$, $(\mathbf{F}^{(n)},\xi^{(n)})$ defines a skeleton of the pair $(\mathbf{F}^X, X)$.

(2) $\mathbf{F}^{(n)} \uparrow \mathbf{F}^X$ as $n \to \infty$, that is, for each $t = 0,1,...,T$,
$\mathcal{F}_t^X = \sigma(\cup_{k\geq 0} \mathcal{F}_t^{(k)}) \supseteq \cdots \supseteq \mathcal{F}_t^{(n)} \supseteq \mathcal{F}_t^{(n-1)} \supseteq \cdots \supseteq \mathcal{F}_t^{(0)}$ (up to P-null sets) ("convergence of information").

(3) $\xi^{(n)} \to X$ as $n \to \infty$ (uniformly in t) P-a.s., that is,
$P[\{\omega \in \Omega: \lim_{n\to\infty} \max_{0\leq t\leq T} |X_t(\omega) - \xi_t^{(n)}(\omega)| = 0\}] = 1$ ("pathwise approximation").

Proof. All the results follow directly from (i) the properties of our approximation scheme (see Theorem 2.2.1), and (ii) the particular construction of the sequence $(\mathbf{F}^{(n)}, \xi^{(n)})_{n\geq 0}$. Indeed, properties (1) and (2) rely on Theorem 2.2.1 ((2),(3)) and the definition of $\mathcal{F}_t^{(n)}$. In order to prove (3), it is enough to observe that for each $t = 0,1,2,...,T$, the sequence $(\xi_t^{(n)})_{n\geq 0}$ is a uniformly integrable $((\mathcal{F}_t^{(n)})_{n\geq 0}, P)$ -martingale with $\xi_t^{(n)} \to X_t$ P-a.s. as $n \to \infty$. □

Observe that the σ -algebras $\mathcal{F}_t^{(n)}$ $(t = 0,1,...,T;\ n \geq 0)$ obtained in Step 2 of the construction increase not only when n is fixed and t varies ($\mathbf{F}^{(n)}$ is a filtration) but also when t is fixed and n varies (i.e. for each $t = 0,1,...,T$, $\mathcal{F}_t^{(n)} \uparrow \sigma(\bigcup_{k\geq 0} \mathcal{F}_t^{(k)}) = \mathcal{F}_t^X$ (up to P-null sets, as $n \to \infty$). This kind of "convergence of information" is expressed as

$$\mathbf{F}^{(n)} \uparrow \mathbf{F}^X \quad \text{as } n \to \infty .$$

Suppose now that the minimal filtration $\mathbf{F}^X$ is replaced by an arbitrary filtration $\mathbf{F} = (\mathcal{F}_t : t = 0,1,...,T:\ \mathcal{F}_0 = \{\varnothing, \Omega\},\ \mathcal{F}_T = \mathcal{F})$ with respect to which X is adapted. Our construction yields sequences $(\mathbf{F}^{(n)}, \xi^{(n)})_{n\geq 0}$ of skeletons that in general no longer possess property (2) of Theorem 3.2.1. Instead they satisfy

$$\mathbf{F} \supseteq \mathbf{F}^X, \text{ i.e. } \mathcal{F}_t \supseteq \mathcal{F}_t^X = \sigma(\bigcup_{n\geq 0} \mathcal{F}_t^{(n)})\ (t = 0,1,...,T).$$

Except for very special cases (for example, when $\mathcal{F}_t \supseteq \mathcal{F}_t^X$ is generated by a countable partition of Ω; $t = 0,1,...,T$), we cannot "close the gap" between $\mathcal{F}_t^X$ and $\mathcal{F}_t$ by constructing additional skeletons of $(\mathbf{F}, X)$ based on the information in $\mathbf{F} \setminus \mathbf{F}^X$ alone and such that property (2) holds along the "extended" skeleton-approximation. Properties (1) and (3) of Theorem 3.2.1 are still valid, however, since they involve only knowledge about X reflected in $\mathbf{F}^X$.

Finally we note that when dealing with skeletons, the sample paths of X, $t \to X_t(\omega)$, are the fundamental objects of study and the natural mode of convergence is *almost-sure* or *pathwise* convergence. To fully appreciate the skeleton-approach for obtaining pathwise convergence results, we refer the reader to Sections 4 and 5 concerning continuous-time stochastic processes.

We conclude this subsection with examples that illustrate the "structure-preserving" character of our construction. In particular, we show that by appropriately choosing the "free parameters" or by slightly modifying the basic construction, a given structural property of X can be maintained along the entire skeleton sequence. The examples concern the martingale- and Markov-property of $(\mathbf{F}, X)$, but other properties can be considered as well. Note that the results below do not depend on whether the underlying filtration is minimal.

Example 3.2.1. (Pathwise approximation of a martingale). In the case where X is a $(\mathbf{F}, P)$ -martingale, any choice of sequences $(m(t,\cdot))_{t=0,1,\ldots,T}$ of increasing, non-negative integers (i.e., for all t, take $m(t,n) = n$) yields a sequence $(\mathbf{F}^{(n)}, \xi^{(n)})_{n\geq 0}$ of skeletons of $(\mathbf{F}, X)$, such that each skeleton $(\mathbf{F}^{(n)}, \xi^{(n)})$ along the sequence is a $(\mathbf{F}^{(n)}, P)$-martingale. Indeed, for $t = 0,1,\ldots,T-1$ and for each $n \geq 0$,

$$E_P[\xi_{t+1}^{(n)} \mid \mathcal{F}_t^{(n)}] = E_P[E_P[X_{t+1} \mid \mathcal{F}_{t+1}^{(n)}] \mid \mathcal{F}_t^{(n)}] \quad \text{(by Step 2 of the construction)}$$

$$= E_P[X_{t+1} \mid \mathcal{F}_t^{(n)}] \qquad (\mathcal{F}_{t+1}^{(n)} \supseteq \mathcal{F}_t^{(n)})$$

$$= E_P[E_P[X_{t+1} \mid \mathcal{F}_t] \mid \mathcal{F}_t^{(n)}] \qquad (\mathcal{F}_t \supseteq \mathcal{F}_t^{(n)})$$

$$= E_P[X_t \mid \mathcal{F}_t^{(n)}] \qquad (X \text{ is a } (\mathbf{F}, P)\text{–martingale})$$

$$= \xi_t^{(n)} \quad P\text{–}a.s.$$

Thus we obtain a pathwise approximation of the $(\mathbf{F}, P)$– martingale X by $(\mathbf{F}^{(n)}, P)$ skeleton-martingales $\xi^{(n)}$ $(n \geq 0)$; if moreover $\mathbf{F} = \mathbf{F}^X$ then we also have the convergence of information $\mathbf{F}^{(n)} \uparrow \mathbf{F}^X$ (i.e. a skeleton-approximation).

Example 3.2.2 (Skeleton-approximation of a Markov process with a stationary distribution). Let the pair $(\mathbf{F}^X, X)$ define a time-homogeneous Markov process with transition-kernel $P(\cdot,\cdot)$, with the stationary distribution μ as its initial distribution (typically $\mathcal{F}_0 = \sigma(X_0) \neq \{\varnothing, \Omega\}$), and with state space $\mathbf{R}^d$ $(1 \leq d < \infty)$. Let P_μ denote the unique probability measure on $(\Omega, \mathcal{F})$ for which

(i) $P_\mu[X_0 \in A] = \mu(A)$, $A \in \mathcal{B}(\mathbf{R}^d)$, and

(ii) $P_\mu[X_{t+1} \in A \mid \mathcal{F}_t](\cdot) = P(X_t(\cdot), A), \quad A \in \mathcal{B}(\mathbf{R}^d)$,

and recall that the stationary distribution μ satisfies

$$\mu P(A) = \mu(A), \quad A \in \mathcal{B}(\mathbf{R}^d), \quad \text{where}$$

$$\mu P(A) \equiv \int_{\mathbf{R}^d} P(x, A)\mu(dx).$$

Using the following construction based on μ, $P(\cdot,\cdot)$ and the approximation scheme of Section 2, we obtain a skeleton-approximation $(\mathbf{F}^{(n)}, \xi^{(n)})_{n\geq 0}$ of the Markov process X consisting of time-homogeneous finite Markov chains.

Step 0′. Choose an increasing sequence $(m(n))_{n\geq 0}$ of non-negative integers.

Step 1′. Apply the approximation scheme of Section 2 to $(\Omega, \sigma(X_0), P_\mu)$ and X_0.

Step 2′. Using the same notation as in Section 2.1, for each $n \geq 0$, we set

$$S^{(n)} = \{E_{P_\mu}[X_0 \mid A(i)]: \ P_\mu[A(i)] > 0, \ \ i \in M(m(n))\},$$

$$\xi_t^{(n)}(\omega) = \begin{cases} E_{P_\mu}[X_0 \mid A(i)] \ \text{if} \ P_\mu[A(i)] > 0 \ \text{and} \ X_t(\omega) \in R(i), i \in M(m(n)) \\ 0 \qquad \text{otherwise} \end{cases}$$

$$\mathcal{F}_t^{(n)} = \sigma(\xi_s^{(n)}: \ s = 0,1,\ldots,t) \quad (t = 0,1,2,\ldots),$$

$$P^{(n)} = (p_{ij}^{(n)})_{i,j \in M(m(n))}$$

with $p_{ij}^{(n)} = P_\mu[X_1 \in R(j) \mid X_0 \in R(i)] \quad (i,j \in M(m(n)))$,

$$\mu^{(n)} = (\mu_i^{(n)})_{i \in M(m(n))}$$

with $\mu_i^{(n)} = P_\mu[X_0 \in R(i)] \quad (i \in M(m(n)))$.

This construction yields a sequence $(\mathbf{F}^{(n)}, \xi^{(n)})_{n\geq 0}$ of skeletons satisfying properties (1)-(3) of Theorem 2.3.1 and such that each skeleton $\xi^{(n)}$ is a time-homogeneous finite Markov chain, defined on $(\Omega,\mathcal{F})$, with transition-matrix $P^{(n)}$, with the stationary distribution $\mu^{(n)}$ as its initial distribution and with $S^{(n)}$ as its state space. In particular, note that $\mu^{(n)}$ converges weakly to μ as $n \to \infty$.

3.3 A skeleton-characterization of the uniqueness problem of an equivalent martingale measure

The change of measure problem discussed in Section 2.3 has a natural analog in the context of (discrete-time) stochastic processes. We are interested in a new probability measure Q on $(\Omega,\mathcal{F})$, equivalent to P and such that the discrete-time stochastic process $X = (X_t : t = 0,1,\ldots,T)$ becomes a martingale under Q and with respect to the filtration $\mathbf{F}$; that is, X must satisfy

(i) $X_t \in L^1(\Omega, \mathcal{F}, Q)$ $(t = 0,1,\ldots,T)$, and

(ii) $E_Q[X_{t+1} \mid \mathcal{F}_t] = X_t$ Q*–a.s.* $(t = 0,1,\ldots,T-1)$.

Such a probability measure Q is called an *equivalent martingale measure* for $(\mathbf{F}, X)$.

In the simple case of a single period stochastic process $X = (X_0, X_1)$, the problem of finding an equivalent martingale measure for X and the change of measure problem considered in Section 2.3 are obviously identical: in both cases the goal is to obtain $Q \sim P$ with $E_Q[Y] = 0$ where $Y \equiv X_1 - X_0$. However, Corollary 2.3.2 states that this problem can be reduced to a finite-dimensional one involving skeletons of $(\sigma(Y), Y)$. This suggests that skeletons provide a natural setting for addressing the more general change of measure problem.

We consider here the uniqueness question for an equivalent martingale measure for the $\mathbf{R}^d$-valued, discrete-time stochastic process $X = (X_t : t = 0,1,\ldots,T)$ defined on the stochastic base $(\Omega, \mathcal{F}, P, \mathbf{F})$. Recalling Corollary 2.3.3, the uniqueness result for the single period case, and applying it to each time period, we observe that for Q to be unique, the pair $(\mathbf{F}, X)$ must have the structure of a skeleton. In fact, the conditions of Corollary 2.3.3 applied to $(\mathbf{F}, X)$ are both necessary and sufficient. Thus, we obtain the following result.

Theorem 3.3.1. There exists a *unique* equivalent martingale measure Q for the stochastic process $(\mathbf{F}, X)$ if and only if there exists a skeleton $(\mathbf{F}^{\xi}, \xi)$ of the pair $(\mathbf{F}, X)$ with the following properties:

(i) $(\mathbf{F},X)$ and $(\mathbf{F}^{\xi},\xi)$ are *indistinguishable* (i.e. $\mathbf{F}=\mathbf{F}^{\xi}$ up to P-null sets, and $P[\{\omega\in\Omega\colon X_t(\omega)=\xi_t(\omega);\ t=0,1,\ldots,T\}]=1$).

(ii) $(\mathbf{F}^{\xi},\xi)$ satisfies condition (C_{ξ}):

(C_{ξ}) For each $t=1,2,\ldots,T$, and for all $\mathbf{R}^d$-valued random variables $\alpha\in\mathcal{F}^{\xi}_{t-1}$,

$$P[\alpha\cdot(\xi_t-\xi_{t-1})\geq 0 \mid \mathcal{F}^{\xi}_t](\cdot)=1 \ \ P\text{-}a.s. \Rightarrow$$

$$P[\alpha\cdot(\xi_t-\xi_{t-1})=0 \mid \mathcal{F}^{\xi}_t](\cdot)=1 \ \ P\text{-}a.s.$$

(iii) $(\mathbf{F}^{\xi},\xi)$ satisfies condition (C_{ξ}'):

(C_{ξ}') For each $t=1,\ldots,T$, and for each $A\in\mathcal{P}^{\xi}_{t-1}$ with $P[A]>0$,

$$\dim(\mathrm{span}\{\xi_t(\omega)-\xi_{t-1}(\omega)\colon \omega\in A\setminus A^0(t,A)\})$$

$$=\text{cardinality }(A'\in\mathcal{P}^{\xi}_t\colon A\setminus A^0(t,A))\supseteq A')-1$$

$$(\text{where } A^0(t,A)\equiv\cup\{A''\in\mathcal{P}^{\xi}_t\colon A\supseteq A'', P[A'']=0\}).$$

Moreover, a skeleton with these properties is unique (up to P -indistinguishability).

Thus for an equivalent martingale measure Q to be unique, the stochastic process $(\mathbf{F},X)$ must not only have the structure of a skeleton (condition (i)) but must also satisfy the stringent requirement of condition (iii) of Theorem 3.3.1. That requirement involves the fine structure of the filtration, relating the flow of information between times $t-1$ and t to the possible changes of the value of X from $t-1$ to t along each sample path. It also shows that the minimal filtration is necessary for uniqueness. In the sequel, we consider continuous-time stochastic processes and show that one can continue to deal with the fine struture of the filtration through conditions of the type (C_{ξ}').

4. Continuous-time stochastic processes and the skeleton-approach

In this section, the skeleton-approach introduced in Section 3 is extended to the case of a continuous-time stochastaic process $X = (X_t: 0 \le t \le T)$ with continuous sample paths. We present explicit constructions and discuss their ability to explicitly deal with filtrations.

4.1 Basic assumptions

We consider a fixed probability space $(\Omega, \mathcal{F}, P)$ and a given *finite* time horizon $T (0 < T < \infty)$. We shall assume $(\Omega, \mathcal{F}, P)$ to be *complete:* that is, if there exists $A \supseteq B$ where $A \in \mathcal{F}$ is such that $P[A] = 0$, then B must necessarily belong to $\mathcal{F}$ (and clearly, $P[B] = 0$). We also specify a *filtration,* i.e. an increasing sequence $\mathbf{F} = (\mathcal{F}_t: 0 \le t \le T)$ of sub σ-algebras of $\mathcal{F}$, and we require that $\mathbf{F}$ satisfies the following, so-called *"usual conditions"* (Dellacherie and Meyer (1978, p. 115)):

(4.1.1)

$$(i)\ \mathcal{F}_0 \text{ contains all } P\text{-null sets (i.e. } \mathcal{F}_0 \text{ is complete)}.$$

$$(ii)\ \mathbf{F} \text{ is right-continuous (i.e. } \mathcal{F}_t = \mathcal{F}_{t+} \equiv \cap_{s>t} \mathcal{F}_s,\ 0 \le t \le T).$$

In fact, for convenience we require $\mathcal{F}_0$ to be almost trivial (i.e. $\mathcal{F}_0$ contains only sets of P-measure zero and one) and $\mathcal{F}_T = \mathcal{F}$.

Next we let $X = (X_t: 0 \le t \le T)$ denote a given $\mathbf{R}^d$-valued stochastic process, defined on $(\Omega, \mathcal{F}, P)$ and with component processes $X^1, \ldots, X^d$. Here each $X^k (1 \le k \le d)$ represents an $\mathbf{F}$-adapted integrable (real-valued) stochastic process with continuous sample paths. By requiring the sample paths of X to be continuous, we exclude stochastic processes with jumps. The development of an appropriate skeleton-approach for jump process is the subject of current research and will appear in a later paper. (See Section 5.4, however, for a process with jumps at fixed points in time.) We identify two $\mathbf{F}$-adapted stochastic processes $X = (X_t: 0 \le 1 \le T)$ and $Y = (Y_t: 0 \le t \le T)$ if they are *indistinguishable,* i.e. if $P[\{\omega \in \Omega: X_t(\omega) = Y_t(\omega) \text{ for all } t \in [0,T]\}) = 1$.

Finally, let $\mathbf{F}^X$ denote the *minimal* filtration satisfying the usual conditions. By this we mean that $\mathcal{F}_t^X$ defines the sub-σ-algebra of $\mathcal{F}$ generated by the set $\{X_s: s \le t\}$ of random vectors X_s and all P-null sets. Thus $\mathcal{F}_t^X$ is the completion of $\mathcal{G}_t \equiv \sigma(X_s: s \le t)\ (0 \le t \le T)$ and right-continuity of $\mathbf{F}^X$ follows from the sample path continuity of X.

4.2 Skeletons and skeleton-approximations of continuous stochastic processes

We start by generalizing the idea of a skeleton to the case of a continuous sample path process. More precisely, we consider the given $\mathbf{R}^d$ -valued stochastic process X, together with the underlying filtration $\mathbf{F}$, and define a continuous-time skeleton of the pair $(\mathbf{F},X)$ to be a piecewise constant (and, of course, right-continuous) skeleton. Formally, we have the following:

Definition 4.2.1. A *continuous-time skeleton* of the pair $(\mathbf{F},X)$ is a triplet $(I^{\xi},\mathbf{F}^X,\xi)$, consisting of a deterministic index-set I^{ξ}, a filtration $\mathbf{F}^X = (\mathcal{F}_t^X : 0 \le t \le T)$ (the *skeleton-filtration*), and an $\mathbf{R}^d$ -valued stochastic proces $\xi = (\xi_t : 0 \le t \le T)$ (the *skeleton-process*) such that

(4.2.1a) • $I^{\xi} = \{t(\xi,0), t(\xi,1),...,t(\xi,N^{\xi})\}$ with $0 = t(\xi,0) < \cdots < t(\xi,N^{\xi}) = T$, $N^{\xi} < \infty$.

(4.2.1b) • for each $t \in I^{\xi}$, $\mathcal{F}_t^{\xi}$ is a finitely generated sub- σ-algebra of $\mathcal{F}_t$, i.e. there exists a minimal partition $\mathcal{P}_t^{\xi}$ of Ω such that $\mathcal{F}_t \supseteq \sigma(\mathcal{P}_t^{\xi}) = \mathcal{F}_t^{\xi}$.
• for $t \notin I^{\xi}$, we set
$\mathcal{F}_t^{\xi} = \mathcal{F}_{t(\xi,k)}^{\xi}$ if $t \in [t(\xi,k), t(\xi,k+1))$ for some $0 \le k < N^{\xi}$.

(4.2.1c) • for each $t \in I^{\xi}$, ξ_t is an $\mathbf{R}^d$ -valued, $\mathcal{F}_t^{\xi}$ -measurable random variable.
• for $t \notin I^{\xi}$, we set
$\xi_t = \xi_{t(\xi,k)}$ if $t \in [t(\xi,k), t(\xi,k+1))$ for some $0 \le k < N^{\xi}$.

Next we extend the notion of a skeleton-approximation to the continuous-time case.

Definition 4.2.2. A sequence $(I^{(n)},\mathbf{F}^{(n)},\xi^{(n)})_{n\ge 0}$ of continuous-time skeletons $(I^{(n)},\mathbf{F}^{(n)},\xi^{(n)})$ of the pair $(\mathbf{F},X)$ is called a *continuous-time skeleton-approximation of the pair* $(\mathbf{F},X)$ if the following three properties hold.

(4.2.2a) *Dense subset property:*

The sequence $(I^{(n)})_{n\ge 0}$ of the finite and deterministic index-sets satisfies

$$(i) \quad |\, I^{(n)} \,| \equiv \max_{1 \le k \le N^{(n)}} |\, t(\xi^{(n)},k) - t(\xi^{(n)},k-1) \,| \to 0 \text{ as } n \to \infty .$$

(ii) $I \equiv \bigcup_{n\geq 0} I^{(n)}$ is a dense subset of $[0,T]$.

(4.2.2b) *Convergence of information:*

$\mathbf{F}^{(n)} \uparrow \mathbf{F}$ as $n \to \infty$, that is, for each $0 \leq t \leq T$,

$$\mathcal{F}_t = \sigma(\bigcup_{k\geq 0} \mathcal{F}_t^{(k)}) \supseteq \cdots \supseteq \mathcal{F}_t^{(n+1)} \supseteq \mathcal{F}_t^{(n)} \supseteq \cdots \supseteq \mathcal{F}_t^{(0)} \text{ up to } P\text{-null sets } (n \geq 1)$$

(4.2.2c) *Pathwise approximation:*

$\xi^{(n)} \to X$ as $n \to \infty$ (uniformly in t) P-a.s.; that is,

$$P[\{\omega \in \Omega: \lim_{n\to\infty} \sup_{0\leq t\leq T} \mid X_t(\omega) - \xi_t^{(n)}(\omega) \mid = 0\}] = 1.$$

As in the discrete-time case, continuous-time skeleton-approximations of the pair $(\mathbf{F}, X)$ enable us to view (i) the stochastic process X as pathwise limits of continuous-time skeleton-processes and (ii) the underlying filtration as limits of skeleton-filtrations. Because of condition (4.2.2b), we shall restrict ourselves in the sequel to the case $\mathbf{F} = \mathbf{F}^X$. (A remark at the end of this section concerns the case $\mathbf{F} \supseteq \mathbf{F}^X$.) We now establish the existence of continuous-time skeleton-approximations of $(\mathbf{F}^X, X)$ through two explicit constructions.

4.3 A general construction of continuous-time skeleton-approximations

The following skeleton-approximation of $(\mathbf{F}^X, X)$ imitates the discrete-case version described in Section 3.2 and essentially ignores the continuity of the time parameter. The construction yields continuous-time skeleton-approximations of the pair $(\mathbf{F}^X, X)$ and gives a first indication of the role of the fine structure of the filtration $\mathbf{F}^X$. The "Special construction" presented in the next subsection will explore this role in greater detail.

For each $n \geq 0$, let $D_n = \{kT/2^n : k = 0,1,\ldots,2^n\}$ be the n^{th}*-dyadic partition of* $[0,T]$, and $D \equiv \bigcup_{n\geq 0} D_n$ the *set of all dyadic numbers* in $[0,T]$. Without loss of generality, we take D as our generic countable dense subset of $[0,T]$.

Construction of continuous-time skeletons of $(\mathbf{F}^X, X)$

For each $n = 0,1,2,\ldots$ do the following four steps ($D_{-1} = \varnothing$):

Step 0. For each $t \in D_n \setminus D_{n-1}$, choose an increasing sequence $(m(t,k))_{k\geq n}$ of non-negative integers.

Step 1. For each $t \in D_n \setminus D_{n-1}$, apply the approximation scheme of Section 2 to $(\Omega, \sigma(X_t), P)$ and to the random vector X_t. Consider the resulting sequences

$(\mathcal{A}_k(t))_{k\geq 0}$.

Step 2. For each $t \in D_n$, set

$$\mathcal{F}_t^{(n)} = \sigma(\bigcup_{\substack{s\leq t \\ s\in D_n}} \mathcal{A}_{m(s,n)}(s)), \quad \text{and}$$

$$\xi_t^{(n)} = E_P[X_t \mid \mathcal{F}_t^{(n)}] \ P\text{-a.s.}$$

Step 3. Define the triplet $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$ where

$$I^{(n)} = D_n ,$$

$$\mathbf{F}^{(n)} = (\mathcal{F}_t^{(n)}: 0 \leq t \leq T)$$

with $\mathcal{F}_t^{(n)} = \mathcal{F}_{t(\xi^{(n)},k)}^{(n)}$ if $t \in [t(\xi^{(n)},k), t(\xi^{(n)},k+1))$ $,0 \leq k < N^{\xi^{(n)}}$, and

$$\xi^{(n)} = (\xi_t^{(n)}: 0 \leq t \leq T)$$

with $\xi_t^{(n)} = \xi_{t(\xi^{(n)},k)}^{(n)}$ if $t \in [t(\xi^{(n)},k), t(\xi^{(n)},k+1))$, $0 \leq k < N^{\xi^{(n)}}$.

Terminology: The triplet $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$ obtained in Step 3 will be called the *continuous-time version* of the discrete-time skeleton $((\mathcal{F}_t^{(n)}: t \in I^{(n)}), (\xi_t^{(n)}: t \in I^{(n)}))$. Conversely, $((\mathcal{F}_t^{(n)}: t \in I^{(n)}), (\xi_t^{(n)}: t \in I^{(n)}))$ will be called the *discrete-time version* of the continuous-time triplet $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$.

An illustration of the construction ($T = 1$)

n = 0. Here $D_0 = \{0,1\}$. Now $X_0 =$ const. P-a.s. and the corresponding σ-algebra is almost trivial, so we shall consider $t = 1$. First (Step 0), we choose an increasing sequence $(m(1,k))_{k\geq 0}$ of non-negative integers $m(1,k)$, say $m(1,0) = 8$, $m(1,1) = 31$,.... Next we apply the approximation scheme to X_1 (Step 1) which yields the sequence $(\mathcal{A}_k(1))_{k\geq 0}$ of sub-σ-algebras of $\sigma(X_1)$.

Step 2 gives

$$\mathcal{F}_0^{(0)} = \{\varnothing, \Omega\}$$

$$\mathcal{F}_1^{(0)} = \mathcal{A}_8(1) ,$$

and the random vectors

$$\xi_0^{(0)} = X_0 ,$$

$$\xi_1^{(0)} = E_p[X_1 \mid \mathcal{F}_1^{(0)}] .$$

Finally, Step 3 yields the continuous-time process

$$\xi_t^{(0)} = \begin{cases} \xi_0^{(0)} & \text{if } 0 \le t < 1 \\ \xi_1^{(0)} & \text{if } t = 1 \end{cases}$$

n = 1. $D_1 = \{0, 1/2, 1\}$. Choose $(m(1/2,k))_{k\geq 1}$, say $m(1/2,1) = 5$, $m(1/2,2) = 12,\ldots$. Apply the approximation scheme to $X_{1/2}$ and let $(\mathcal{A}_k(1/2))_{k\geq 0}$ denote the resulting sequence of sub-σ-algebras of $\sigma(X_{1/2})$.
Step 2 gives

$$\mathcal{F}_0^{(1)} = \{\varnothing, \Omega\} ,$$

$$\mathcal{F}_{1/2}^{(1)} = \mathcal{A}_5(1/2) ,$$

$$\mathcal{F}_1^{(1)} = \sigma(\mathcal{A}_5(1/2) \cup \mathcal{A}_{31}(1)) ,$$

and the random vectors

$$\xi_0^{(1)} = X_0 ,$$

$$\xi_{1/2}^{(1)} = E_p[X_{1/2} \mid \mathcal{F}_{1/2}^{(1)}] ,$$

$$\xi_1^{(1)} = E_p[X_1 \mid \mathcal{F}_1^{(1)}] .$$

Note that $\mathcal{F}_1^{(0)}$ and $\mathcal{F}_1^{(1)}$ differ not only by the presence of $\mathcal{A}_5(1/2)$ but also by the choice of a *further* element in the sequence $(\mathcal{A}_k(1))_{k\geq 0}$, namely $\mathcal{A}_{31}(1) \supset \mathcal{A}_8(1)$. Finally, Step 3 yields the continuous-time process

$$\xi_t^{(1)} = \begin{cases} \xi_0^{(1)} \text{ if } 0 \leq t < 1/2 \\ \xi_{1/2}^{(1)} \text{ if } 1/2 \leq t < t \\ \xi_1^{(1)} \text{ if } t = 1 \end{cases}$$

A practical application of the construction requires choosing a set of "free parameters," namely, the sequences $m(t,\cdot)_{t\in D}$. Intuitively, by choosing these sequences, we are able to control the amount of information to be included at each step of the construction. We achieve this way a first, rather limited, capability for dealing with the fine structure of the underlying filtration. Theorem 4.3.1 below states that any choice of free parameters leads to continuous-time skeletons of the pair $(\mathbf{F}^X, X)$. Moreover, it shows that one can choose sequences $(m(t,\cdot))_{t\in D}$ such that the construction yields continuous-time skeleton-approximations in the sense of Definition 4.2.2.

Theorem 4.3.1. In the above construction one can choose sequences $(m(t,\cdot))_{t\in D}$ such that the resulting sequence $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})_{n\geq 0}$ defines a continuous-time skeleton-approximation of $(\mathbf{F}^X, X)$.

Proof. 1) For any choice of sequences $m(t,\cdot))_{t\in D}$ (in accordance with Step 0 of the construction), each triplet $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$ in the resulting sequence $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})_{n\geq 0}$ satisfies the defining properties (4.2.1a) - (4.2.1c) of a continuous-time skeleton of $(\mathbf{F}^X, X)$ by the very construction. Moreover, $I \equiv \cup_{n\geq 0} I^{(n)} = D$, and property (4.2.2a) is obviously satisfied. Also, the proof of property (4.2.2b) holds for arbitrary sequences $(m(t,\cdot))_{t\in D}$ of increasing, non-negative integers and uses Theorem 2.2.1(3) and the continuity of the sample paths of $X = (X_t : 0 \leq t \leq T)$. In fact, for $t\in[0,T]$, Theorem 2.2.1(3) implies $\mathcal{F}_t^{(n+1)} \supseteq \mathcal{F}_t^{(n)}$ and moreover, by definition of $\mathcal{F}_t^{(n)}$ and $\mathcal{A}_{m(s,\cdot)}(s)$ $(s,t\in D)$, we can write

$$\bigvee_{n\geq 0} \mathcal{F}_t^{(n)} = \sigma\Big(\bigcup_{n\geq 0} \mathcal{F}_t^{(n)}\Big)$$

$$= \sigma\Big(\bigcup_{n\geq 0} \sigma\Big(\bigcup_{\substack{s\leq t \\ s\in D_n}} \mathcal{A}_{m(s,n)}(s)\Big)\Big)$$

$$= \sigma(\bigcup_{n\geq 0} \sigma(\bigcup_{\ell=0}^{n} \sigma(\bigcup_{\substack{s\leq t \\ s\in D_\ell}} \mathcal{A}_{m(s,n)}(s))))$$

$$= \sigma(\bigcup_{\ell\geq 0} \sigma(\bigcup_{n\geq \ell} \sigma(\bigcup_{\substack{s\leq t \\ s\in D_\ell}} \mathcal{A}_{m(s,n)}(s))))$$

$$= \sigma(\bigcup_{\ell\geq 0} \sigma(\bigcup_{\substack{s\leq t \\ s\in D_\ell}} \sigma(\bigcup_{n\geq \ell} \mathcal{A}_{m(s,n)}(s))))$$

$$= \sigma(\bigcup_{\ell\geq 0} \sigma(\bigcup_{\substack{s\leq t \\ s\in D_\ell}} \sigma(X_s)))$$

$$= \sigma(\bigcup_{\ell\geq 0} \sigma(X_s : s\leq t, s\in D_\ell))$$

$$= \sigma(X_s : s\leq t, s\in D) \text{ (up to } P\text{-null sets)} .$$

Now use continuity of X in order to conclude that for $0 \leq t \leq T$,

$$\mathcal{G}_t \equiv \sigma(X_s : s\leq t) = \bigvee_{n\geq 0} \mathcal{F}_t^{(n)} \text{ (up to } P \text{ null sets).}$$

Since $\mathcal{F}_t$ is the completion of $\mathcal{G}_t$, we have $\mathcal{G}_t = \mathcal{F}_t$ (up to P -null sets), $0 \leq t \leq T$.

2) We now show that one can choose sequences $m(t,\cdot))_{t\in D}$ such that the resulting sequence $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})_{n\geq 0}$ also satisfies property (4.2.2c), and hence defines a continuous-time skeleton-approximation of the pair $(\mathbf{F}^X, X)$. To this end, let $(b_n)_{n\geq 0}$ denote a sequence of positive real numbers converging to 0. Then for each $n\geq 0$, Theorem 3.2.1 guarantees that one can choose $m(t,n)_{t\in D_n}$ in accordance with Step 0 of the construction such that

$$\text{(4.3.1)} \quad P[\{\omega\in\Omega: \sup_{t\in D_n} |\, X_t(\omega) - \xi_t^{(n)}(\omega) \,| > b_n \}] \leq 2^{-n} .$$

However,

$$\{\omega\in\Omega: \limsup_{n\to\infty} \sup_{0\leq t\leq T} |\, X_t(\omega) - \xi_t^{(n)}(\omega) \,| > 0\}$$

$$\subseteq \{\omega \in \Omega: \sup_{t \in D_n} \mid X_t(\omega) - \xi_t^{(n)}(\omega) \mid > b_n \ i.o.\}$$

and since

$$\sum_{n \geq 0} P[\{\omega \in \Omega: \sup_{t \in D_n} \mid X_t(\omega) - \xi_t^{(n)} \mid > b_n \}] \leq \sum_{n \geq 0} 2^{-n} < \infty,$$

an application of the Borel-Cantelli Lemma yields

$$P[\{\omega \in \Omega: \limsup_{n \to \infty} \sup_{0 \leq t \leq T} \mid X_t(\omega) - \xi_t^{(n)}(\omega) \mid > 0\}] = 0,$$

that is,

$$P[\{\omega \in \Omega: \lim_{n \to \infty} \sup_{0 \leq t \leq T} \mid X_t(\omega) - \xi_t^{(n)}(\omega) \mid = 0\}] = 1. \quad \square$$

4.4 The Special Construction

We now provide a refinement of the general construction which uses the continuity of the time parameter more effectively. We must, however, impose an additional continuity condition on $\mathbf{F}^X$, namely, we require $\mathbf{F}^X$ to satisfy

(4.4.1) $\mathcal{F}_{t-}^X = \mathcal{F}_t^X$, where $\mathcal{F}_{t-} \equiv \sigma(\bigcup_{s<t} \mathcal{F}_s^X)$ $(\mathcal{F}_{o-}^X = \mathcal{F}_o^X, 0 \leq t \leq T)$

Note that right-continuity (i.e. $\mathcal{F}_t^X = \mathcal{F}_{t+}^X$) holds by (4.1.1). The additional restriction to so-called *semi-continuous* filtration is necessary to rule out the existence of events "that can take us by surprise" (see Section 5.4). For an alternate definition of "continuous" information structures, see Huang (1985).

Special Construction of continuous-time skeletons of $(\mathbf{F}^X, X)$ when $d = 1$

For each $n = 0,1,...$ do the following four steps $(D_{-1} = \emptyset)$.

Step 0. For each $t \in D_n \backslash D_{n-1}$ choose an increasing sequence $m(t,k)_{k \geq n}$ of non-negative integers with $m(t,k) \geq k$ for all $k \geq n$.

Step 1. For each $t \in D_n \backslash D_{n-1}$ apply the approximation scheme of Section to $(\Omega, \sigma(X_t), P)$ and X_t. Consider the resulting sequences $\mathcal{A}_k(t)_{k \geq 0}$.

Step 2. For $k = 0,1,...,2^n$ set $t(k) = kT/2^n$ (i.e. $D_n = \{t(k): 0 \le k \le 2^n\}$) and for $\ell = 0,1,...,m(t(k),n)-1$ set $s(\ell) = \ell/2^{2m(t(k),n)}$. Then for each $k = 0,1,...,2^n$ and for each $\ell = 0,1,...,m(t(k),n)-1$ set

$$\mathcal{F}^{(n)}_{t(k)+s(\ell)} = \mathcal{F}^{(n)}_{t(k-1)+(m(t(k-1),n)-1)/2^{2m(t(k-1),n)}} \vee \mathcal{A}_{\ell+1}(t(k)), \left[\mathcal{F}^{(n)}_{t(-1)} \equiv \{\emptyset,\Omega\} \right]$$

and

$$\xi^{(n)}_{t(k)+s(\ell)} = E_P \left[X_{t(k)} \mid \mathcal{F}^{(n)}_{t(k)+s(\ell)} \right] \quad P\text{-}a.s.$$

Step 3. Define the triplet $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$ where

$$I^{(n)} = \{t(k) + s(\ell): k = 0,1,...,2^n;\ \ell = 0,1,...,m(t(k),n)-1\},$$

$$\mathbf{F}^{(n)} = (\mathcal{F}^{(n)}_t: 0 \le t \le T^{(n)} \equiv T + (m(T,n)-1)/2^{2m(T,n)})$$

with $\mathcal{F}^{(n)}_t = \mathcal{F}^{(n)}_{t(\xi^{(n)})}$ if $t \in [t(\xi^{(n)},i),\ t(\xi^{(n)},i+1)),\ 0 \le i < N^{\xi^{(n)}}$, and

$$\xi^{(n)} = (\xi^{(n)}_t: 0 \le t \le T^{(n)})$$

with $\xi^{(n)}_t = \xi^{(n)}_{t(\xi^{(n)},i)}$ if $t \in [t(\xi^{(n)},i), t(\xi^{(n)},i+1)),\ 0 \le i < N^{\xi^{(n)}}$.

Note that in order for Step 2 of the Special Construction to be well-defined, we set $X_t = X_T$ whenever $t > T$. Thus, the triplet $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$ and the process $(\mathbf{F}, X)$ are both assumed to be defined on $[0, T^{(n)}]$ $(n \ge 1)$, where $T^{(n)} \to T$ as $n \to \infty$. For ease of presentation, the special construction is formulated only for the case $d = 1$. For some examples and results for $d > 1$, see Section 5.

Comparing the Special Construction and the general construction (Section 4.3) shows that when "producing" the n^{th} skeleton at time $t \in D_n$, the latter provides all the necessary information at once; i.e.,

$$\mathcal{F}^{(n)}_n = \text{"past"} + \mathcal{A}_{m(t,n)}(t),$$

where $\mathcal{A}_{m(t,n)}(t)$ is some element of the approximation sequence $(\mathcal{A}_k(t))_{k \ge n}$ of the σ-algebra $\sigma(X_t)$. The Special Construction, on the other hand, distributes that same information in a more *piecemeal* fashion. The construction shows explicitly how this can be done. At any dyadic point $t \in D_n$, we provide the past information and the coarse information $\mathcal{A}_1(t)$ about X_t which we update bit-wise (in an either/or-fashion), at our chosen rate $2^{-2m(t,n)}$, so as to be certain that all the information about X_t needed for the construction of the n^{th} skeleton, namely $\mathcal{A}_{m(t,n)}(t)$, is fully available by time $t+(m(t,n)-1)2^{-2m(t,n)}$, and hence by the next dyadic time $t+2^{-n}$. Thus we obtain skeletons of $(\mathbf{F}^X, X)$ with a *binary* tree-structure; that is, each non-terminal node of the tree corresponding to the n^{th} skeleton ($n \geq 0$) branches in two (or does not branch at all if no relevant information is received).

The following result states that there exist sequences $(m(t,\cdot))_{t \in D}$ yielding skeleton-approximations of $(\mathbf{F}, X)$ by using the Special Construction. In other words, there exist sequences of "binary" skeletons which approximate X pathwise and for which the corresponding binary skeleton-filtrations converge in the sense of (4.2.2b). The Special Construction reveals the fine structure of the filtration $\mathbf{F}^X$ in terms of binary tree-structures. Such an explicit description of the underlying filtration is possible for vector-valued processes ($d > 1$) as well (see Section 5).

Theorem 4.4.1. Suppose $\mathbf{F}^X$ satisfies condition (4.4.1). Then in Step 0 of the Special Construction, one can choose sequences $m(t,\cdot)_{t \in D}$ such that the resulting skeleton sequence $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})_{n \geq 0}$ defines a continuous-time skeleton-approximation of $(\mathbf{F}^X, X)$, and such that, for each $n \geq 0$, $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$ satisfies the following condition

$(C_{\xi^{(n)}}')$ For each $t' \equiv t(\xi^{(n)}, k-1)$, $t \equiv t(\xi^{(n)}, k) \in I^{(n)}$ $(0 < k \leq N^{\xi^{(n)}})$,

and for each $A \in \mathcal{P}_{t'}^{\xi^{(n)}}$ with $P[A] > 0$,

$$\dim(\mathrm{span}(\{\xi_t^{(n)}(\omega) - \xi_{t'}^{(n)}(\omega) \colon \omega \in A \setminus A^0(t,A)\}))$$

$$= \mathrm{cardinality}\,(A' \in \mathcal{P}_t^{\xi^{(n)}} \colon A' \subseteq A \setminus A^0(t,A)) - 1$$

$$= \begin{cases} 0 & \text{if } A^0(t,A) \neq \varnothing \\ 1 & \text{if } A^0(t,A) = \varnothing \end{cases}$$

where $A^0(t,A) = \cup\{A'' \in \mathcal{P}_t^{\xi(n)} : A'' \subseteq A, P[A''] = 0\}$.

Proof. 1) For any allowable sequences $(m(t,\cdot))_{t \in D}$, each triplet $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$, $n \geq 0$, satisfies by construction properties (4.2.1a)-(4.2.1c) of a continuous-time skeleton.

2) Next we show that for any choice of sequences $m(t,\cdot)_{t \in D}$ (in accordance with Step 0 of the construction), the resulting sequence $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})_{n \geq 0}$ has property (4.2.2b). (Property (4.2.2a) is obvious, and the existence of sequences $(m(t,\cdot))_{t \in D}$ such that the resulting sequence of skeletons also satisfies property (4.2.2c) follows than exactly as in the proof of Theorem 4.3.1.) Fix $t \in D$ and consider the construction of $(\mathcal{F}_t^{(n)})_{n \geq 0}$.

First we prove that for each $n \geq 0$, $\mathcal{F}_t^{(n+1)} \supseteq \mathcal{F}_t^{(n)}$. In fact, let k denote the first index n with $t \in D_n$. Then, for $n \geq k$, we have

$$\mathcal{F}_t^{(n+1)} = \Big(\bigvee_{\substack{s<t \\ s \in D_{n+1}}} \mathcal{A}_{m(s,n+1)}(s) \Big) \vee \mathcal{A}_1(t) \quad \text{(by construction)}$$

$$\supseteq \Big(\bigvee_{\substack{s<t \\ s \in D_n}} \mathcal{A}_{m(s,n)}(s) \Big) \vee \mathcal{A}_1(t) \quad (m(s,n+1) > m(s,n),\ D_{n+1} \supseteq D_n)$$

$$= \mathcal{F}_t^{(n)} \quad \text{(by construction)}.$$

For $n < k$, set $t(n) = \max\{s \in D_n : s \leq t\}$. Then there are two cases:

Case (i). $t(n) \neq t(n+1)$, i.e. $t(n+1) = t(n) + 2^{-(n+1)}$:

$$\mathcal{F}_t^{(n+1)} = \Big(\bigvee_{\substack{s<t(n+1) \\ s \in D_{n+1}}} \mathcal{A}_{m(s,n+1)}(s) \Big) \vee \mathcal{A}_\ell(t(n+1)) \text{ for some } 0 < \ell < m(t(n+1),n+1)$$

$$\supseteq \bigvee_{\substack{s \leq t(n) \\ s \in D_n}} \mathcal{A}_{m(s,n)}(s) \quad (m(s,n+1) > m(s,n), D_{n+1} \supseteq D_n)$$

$$= \mathcal{F}_t^{(n)} .$$

Case (ii). $t(n) = t(n+1)$: since $\ell 2^{-2m(t(n),n)} > \ell 2^{-2m(t(n),n+1)}$ $(0 < \ell < m(t(n),n))$,

$$\mathcal{F}_t^{(n+1)} = \Big(\bigvee_{\substack{s<t(n) \\ s\in D_{n+1}}} \mathcal{A}_{m(s,n+1)}(s)\Big) \vee \mathcal{A}_k(t(n)) \quad (\text{for some } 0 < k < m(t(n),n+1))$$

$$\supseteq \Big(\bigvee_{\substack{s<t(n) \\ s\in D_n}} \mathcal{A}_{m(s,n)}(s)\Big) \vee \mathcal{A}_\ell(t(n)) \quad (\text{for } 0 < \ell < k < m(t(n),n+1))$$

$$= \mathcal{F}_t^{(n)} \qquad (\text{for some } 0 < \ell < m(t(n),n)) .$$

Thus, for each $n \ge 0$, we have $\mathcal{F}_t^{(n+1)} \supseteq \mathcal{F}_t^{(n)}$.

Next for $t \in D$ and k such that $t \in D_k$, we have, on the one hand,

$$\bigvee_{n\ge 0} \mathcal{F}_t^{(n)} \equiv \sigma\Big(\bigcup_{n\ge 0} \mathcal{F}_t^{(n)}\Big)$$

$$= \sigma\Big(\bigcup_{n\ge k} \mathcal{F}_t^{(n)}\Big) \qquad (\mathcal{F}_t^{(n)} \subseteq \mathcal{F}_t^{(n+1)})$$

$$\subseteq \sigma\Big(\bigcup_{n\ge k} \mathcal{F}^{(n)}_{t+(m(t,n)-1)/2^{2m(t,n)}}\Big) \quad (\mathcal{F}^{(n)}_{t+(m(t,n)-1)/2^{2m(t,n)}} \supseteq \mathcal{F}_t^{(n)})$$

$$= \sigma\Big(\bigcup_{n\ge k} \sigma\Big(\bigcup_{\substack{s\le t \\ s\in D_n}} \mathcal{A}_{m(s,n)}(s)\Big)\Big) \quad (\text{by definition of } \mathcal{F}^{(n)}_{t+(m(t,n)-1)/2^{2m(t,n)}})$$

$$= \sigma\Big(\bigcup_{\substack{s\le t \\ s\in D}} \sigma(X_s)\Big)$$

$$= \sigma(X_s : s \le t, s \in D)$$

$$= \sigma(X_s : s \leq t) \quad \text{(up to } P\text{-null sets)} \quad \text{(by continuity of } X)$$

so $\bigvee_{n \geq 0} \mathcal{F}_t^{(n)} \subseteq \mathcal{F}_t^X$ for any $0 \leq t \leq T$ by (4.1.1).

On the other hand,

$$\bigvee_{n \geq 0} \mathcal{F}_t^{(n)} = \sigma(\bigcup_{n \geq k} \mathcal{F}_t^{(n)})$$

$$\supseteq \sigma(\bigcup_{n \geq k} \mathcal{F}_{t-1/2^{2m(t',n)}}^{(n)}) \quad \text{(where } t' = t - 2^{-n})$$

$$= \sigma(\bigcup_{n \geq k} \sigma(\bigcup_{\substack{s<t \\ s \in D_n}} \mathcal{A}_{m(s,n)}(s)))$$

$$= \sigma(\bigcup_{\substack{s<t \\ s \in D}} \sigma(X_s))$$

$$= \sigma(X_s : s < t, s \in D)$$

$$= \sigma(X_s : s < t) \quad \text{(up to } P\text{-null sets)}$$

so $\bigvee_{n \geq 0} \mathcal{F}_t^{(n)} \supseteq \mathcal{F}_{t-}^X$ for any $0 \leq t \leq T$ by (4.1.1). Using (4.4.1), we get

$$\mathcal{F}_t^X = \bigvee_{n \geq 0} \mathcal{F}_t^{(n)} \supseteq \cdots \supseteq \mathcal{F}_t^{(n)} \supseteq \cdots \supseteq \mathcal{F}_t^{(0)} \quad \text{(up to } P\text{-null sets)},$$

and (4.2.2b) holds.

3) Finally, we show that $(C_{\xi^{(n)}}')$ holds for each skeleton $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$. (We write $t(n,\cdot)$ instead of $t(\xi^{(n)},\cdot)$, for simplicity.) Since $d = 1$, $\xi_{t(n,k+1)}^{(n)} - \xi_{t(n,k)}^{(n)}$ is real-valued and hence the dimension of its span is at most one. It is equal to one on a partition set $A \in \mathcal{P}_{t(n,k)}^{(n)}$ with $P[A] > 0$, if $\xi_{t(n,k+1)}^{(n)} - \xi_{t(n,k)}^{(n)}$ takes two different values on A and zero

if $\xi^{(n)}_{t(n,k+1)} - \xi^{(n)}_{t(n,k)}$ takes only one value on A. By construction, any set $A \in \mathcal{P}^{(n)}_{t(n,k)}$ with $P[A] > 0$ is split into exactly two subsets (A_1 and $A_2 \in \mathcal{P}^{(n)}_{t(n,k+1)}$, say) when going from $t(n,k)$ to the next point in time $t(n,k+1)$. Thus if $P[A \mid A_1] \neq 0$ and $P[A \mid A_2] \neq 0$ then $\xi^{(n)}_{t(n,k+1)} - \xi^{(n)}_{t(n,k)}$ takes two distinct values on A. On the other hand, if $P[A \mid A_1] = 0$ or $P[A \mid A_2] = 0$ (note that both probabilities cannot be zero since $P[A] > 0$), then only one value of $\xi^{(n)}_{t(n,k+1)} - \xi^{(n)}_{t(n,k)}$ on A is defined and of interest. Since n and $t(n,k) \in I^{(n)}$ were arbitrary, condition ($C_{\xi^{(n)}}{}'$) holds for each continuous-time skeleton $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$. □

Note that regardless of the underlying filtration $\mathbf{F}$, the two methods of constructing continuous-time skeleton-approximations of X *always* provide a sequence of skeletons which approximate X pathwise. However, in the case $\mathbf{F} \supset \mathbf{F}^X$, convergence of information (i.e. $\mathbf{F}^{(n)} \uparrow \mathbf{F}$ as $n \to \infty$) no longer generally holds since skeleton-approximations are typically not capable of dealing with information other than that provided by X.

Pathwise approximations of continuous-time stochastic processes are not very common in the probability literature, even in the special case of a one-dimensional Brownian motion. An exception is Knight (1981) (see also Ito and McKean (1965)), who uses stopping-time techniques and intrinsic properties of Brownian motion to arrive at a pathwise approximation via simple random-walk-like processes. However, his use of stopping times destroys any monotonicity of the corresponding information-approximation of Brownian filtration.

Finally we observe that the results of the skeleton-approach in general, and the Special Construction in particular (i.e. Theorem (4.4.1), are invariant under an equivalent change of probability measure. Thus the actual value $P[A]$ of the probability of event $A \in \mathcal{F}$ is irrelevant so long as $P[A] \neq 0$ or 1 (see, for example condition ($C_{\xi^{(n)}}{}'$).

5. Examples and variations of the Special Construction

5.1 Standard Brownian motion in one dimension

Let $W = (W_t : 0 \leq t \leq 1)$ denote standard Brownian motion defined on some stochastic base $(\Omega, \mathcal{F}, P, \mathbf{F}^W)$ where $\mathbf{F}^W$ denotes the minimal filtration satisfying the "usual conditions". Clearly, $(\mathbf{F}^W, W)$ has continuous sample paths and satisfies assumption (4.4.1). Thus Theorem 4.4.1 applies and the Special Construction yields continuous-time skeleton-approximations of W with corresponding binary skeleton-filtrations. Since the sample paths of a Brownian motion on any subinterval are never constant, we get:

Corollary 5.1.1 There exists a continuous-time skeleton-approximation $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})_{n \geq 0}$ of $(\mathbf{F}^W, W)$ such that for each $n \geq 0$, $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$ satisfies: for each $t(n,k-1), t(n,k) \in I^{(n)}$ $(0 < k \leq N^{(n)})$, and for each $A \in \mathcal{P}^{(n)}_{t(n,k-1)}$ with $P[A] > 0$,

$$\dim(\text{span}(\{\xi^{(n)}_{t(n,k)}(\omega) - \xi^{(n)}_{t(n,k-1)}(\omega) : \omega \in A\}))$$

$$= \text{cardinality}(A' \in \mathcal{P}^{(n)}_{t(n,k)} : A' \subseteq A) - 1 = 1 \ .$$

It is illuminating to present the first few elements in the sequence of skeletons used to obtain a skeleton-approximation of Brownian motion. For ease of presentation, we take $m(t,n) = n$ for all t and n, although that choice may not be adequate.

<u>$n = 1$</u>. $I^{(1)} = \{0, 1/2, 1\}$.

Step 1 of the Special Construction yields:

for $t = 1/2$: $\mathcal{A}_0(1/2) = \{\varnothing, \Omega\}$

$$\mathcal{A}_1(1/2) = \sigma(\{\omega \in \Omega : W_{1/2}(\omega) > 0\}, \{\omega \in \Omega : W_{1/2}(\omega) \leq 0\})$$

$$\mathcal{A}_2(1/2) = \sigma(\{W_{1/2} > x_1\}, \{0 < W_{1/2} \leq x_1\},$$

$$\{x_2 < W_{1/2} \leq 0\}, \{W_{1/2} \leq x_2\})$$

where $x_1 = E[W_{1/2} \mid W_{1/2} > 0]$,

$x_2 = E[W_{1/2} \mid W_{1/2} \leq 0]$,

etc.

for $t = 1$: $\mathcal{A}_0(1) = \{\varnothing, \Omega\}$

$$\mathcal{A}_1(1) = \sigma(\{W_1 > 0\}, \{W_1 \leq 0\})$$

$$\mathcal{A}_2(1) = \sigma(\{W_1 > y_1\},\ \{0 < W_1 \le y_1\},\ \{y_2 < W_1 \le 0\},\ \{W_1 \le y_2\})$$

$$\text{where } y_1 = E[W_1 \mid W_1 > 0] \ ,$$

$$y_2 = E[W_1 \mid W_1 \le 0] \ ,$$

etc.

Step 2 defines the first skeleton ($I^{(1)}$, $\mathbf{F}^{(1)}$, $\xi^{(1)}$) by setting

$$\mathcal{F}_0^{(1)} = \{\varnothing, \Omega\} \ ,$$

$$\mathcal{F}_{1/2}^{(1)} = \mathcal{F}_0^{(0)} \vee \mathcal{A}_1(1/2) = \sigma(\{W_{1/2} > 0\},\ \{W_{1/2} \le 0\}) \ ,$$

$$\mathcal{F}_1^{(1)} = \mathcal{F}_{1/2}^{(1)} \vee \mathcal{A}_1(1) = \sigma(\{W_{1/2} > 0\},\ \{W_{1/2} \le 0\},\ \{W_1 > 0\},\ \{W_1 \le 0\})$$

and

$$\xi_0^{(1)} = 0 \ ,$$

$$\xi_{1/2}^{(1)}(\omega) = \begin{cases} E[W_1 \mid W_{1/2} > 0] & \text{if } \omega \in \{W_{1/2} > 0\} \ , \\ E[W_1 \mid W_{1/2} \le 0] & \text{if } \omega \in \{W_{1/2} \le 0\} \ , \end{cases}$$

$$\xi_1^{(1)}(\omega) = \begin{cases} E[W_1 \mid W_{1/2} > 0, W_1 > 0] & \text{if } \omega \in \{W_{1/2} > 0, W_1 > 0\} \ , \\ E[W_1 \mid W_{1/2} > 0, W_1 \le 0] & \text{if } \omega \in \{W_{1/2} > 0, W_1 \le 0\} \ , \\ E[W_1 \mid W_{1/2} \le 0, W_1 > 0] & \text{if } \omega \in \{W_{1/2} \le 0, W_1 > 0\} \ , \\ E[W_1 \mid W_{1/2} \le 0, W_1 \le 0] & \text{if } \omega \in \{W_{1/2} \le 0, W_1 \le 0\} \ . \end{cases}$$

<u>$n = 2$</u>. $I^{(2)} = \{0,\ 1/16,\ 1/4,\ 1/4 + 1/16,\ 1/2,\ 1/2 + 1/16,\ 3/4,\ 3/4 + 1/16,\ 1,\ 1 + 1/16\}$.

Step 1 of the Special Construction yields:

for $t = 1/4$: $\mathcal{A}_0(1/4) = \{\varnothing, \Omega\}$,

$$\mathcal{A}_1(1/4) = \sigma(\{W_{1/4} > 0\},\ \{W_{1/4} \leq 0\})\ ,$$

$$\mathcal{A}_2(1/4) = \sigma(\{W_{1/4} > u_1\},\ \{0 < W_{1/4} \leq u_1\}\ ,$$

$$\{u_2 < W_{1/4} \leq 0\},\ \{W_{1/4} \leq u_2\})$$

where $u_1 = E[W_{1/4} \mid W_{1/4} > 0]$,

$$u_2 = E[W_{1/4} \mid W_{1/4} \leq 0]\ .$$

etc.

for $t = 3/4$: $\mathcal{A}_0(3/4) = \{\varnothing, \Omega\}$,

$$\mathcal{A}_1(3/4) = \sigma(\{W_{3/4} > 0\},\ \{W_{3/4} \leq 0\})\ ,$$

$$\mathcal{A}_2(3/4) = \sigma(\{W_{3/4} > v_1\},\ \{0 < W_{3/4} \leq v_1\}\ ,$$

$$\{v_2 < W_{3/4} \leq 0\},\ \{W_{3/4} \leq v_2]\ ,$$

where $v_1 = E[W_{1/4} \mid W_{1/4} > 0]$,

$$v_2 = E[W_{1/4} \mid W_{1/4} \leq 0]\ .$$

etc.

Step 2 defines the second skeleton $(I^{(2)}, \mathbf{F}^{(2)}, \xi^{(2)})$ by setting

$$\mathcal{F}_0^{(2)} = \{\varnothing, \Omega\} \quad ,$$

$$\mathcal{F}_{1/16}^{(2)} = \{\varnothing, \Omega\} \quad ,$$

$$\mathcal{F}_{1/4}^{(2)} = \mathcal{F}_{1/16}^{(2)} \vee \mathcal{A}_1(1/4) = \sigma(\{W_{1/4} > 0\},\ \{W_{1/4} \le 0\}) \quad ,$$

$$\mathcal{F}_{1/4+1/16}^{(2)} = \mathcal{F}_{1/4}^{(2)} \vee \mathcal{A}_2(1/4) = \sigma(\{W_{1/4} > u_1\},\ \{0 < W_{1/4} \le u_1\} \quad ,$$

$$\{u_2 < W_{1/4} \le 0\},\ \{W_{1/4} \le u_2\})$$

$$\mathcal{F}_{1/2}^{(2)} = \mathcal{F}_{1/4+1/16}^{(2)} \vee \mathcal{A}_1(1/2) = \sigma(\{W_{1/4} > u_1, W_{1/2} > x_1\} \quad ,$$

$$\{W_{1/4} > u_1, 0 < W_{1/2} \le x_1\},\ \{W_{1/4} > u_1, x_2 < W_{1/2} \le 0\} \quad ,$$

$$\{W_{1/4} > u_1, W_{1/2} \le x_2\},\ \ldots,\ \{W_{1/4} \le u_2, W_{1/2} \le x_2\})$$

etc.

$$\mathcal{F}_{1+1/16}^{(2)} = \mathcal{A}_2(1/4) \vee \mathcal{A}_2(1/2) \vee \mathcal{A}_2(3/4) \vee \mathcal{A}_2(1)$$

(generated by $2^8 = 256$ partition sets of Ω)

and

$$\xi_0^{(2)} = 0 \quad ,$$

$$\xi_{1/16}^{(2)} = 0 \quad ,$$

$$\xi^{(2)}_{1/4}(\omega) = \begin{cases} E[W_1 \mid W_{1/4} > 0] & \text{if } \omega \in \{W_{1/4} > 0\} \\ E[W_1 \mid W_{1/4} \le 0] & \text{if } \omega \in \{W_{1/4} \le 0\} \ , \end{cases}$$

$$\xi^{(2)}_{1/4+1/16}(\omega) = \begin{cases} E[W_1 \mid W_{1/4} > u_1] & \text{if } \omega \in \{W_{1/4} > u_1\} \\ E[W_1 \mid 0 < W_{1/4} \le u_1] & \text{if } \omega \in \{0 < W_{1/4} \le u_1\} \\ E[W_1 \mid u_2 < W_{1/4} \le 0] & \text{if } \omega \in \{u_2 < W_{1/4} \le 0\} \\ E[W_1 \mid W_{1/4} \le u_2] & \text{if } \omega \in \{W_{1/4} \le u_2\} \cdot , \end{cases}$$

etc.

$\xi^{(2)}_{1/2} = E[W_1 \mid \mathcal{F}^{(2)}_{1/2}]$ takes 8 different values,

$\xi^{(2)}_{1+1/16} = E[W_1 \mid \mathcal{F}^{(2)}_{1+1/16}]$ takes 256 different values.

Although the notation gets out of hand rapidly, Figure 5.1.1 indicates how pathwise approximation is achieved as $n \to \infty$: at each step in the construction, the skeleton-filtration "closes in" on the correct $\omega \in \Omega$ by measuring with greater precision the position of the path $\{W_t(\omega):\ 0 \le t \le 1\}$ in addition to measuring it at more and more points in time.

From a practical point of view, observe the following. In order to determine the partition-sets of the skeleton-filtration, we use the approximation scheme of Section 2, which requires *one-dimensional* integration (for calculating objects of the form $E[W_t \mid x < W_{t'} \le y])$. An explicit calculation of the values of each skeleton, however, requires *multi-dimensional* integration, despite the Markovian nature of W and the Markov-property of the approximation scheme (see Remark 2.2.1). A typical object for multi-dimensional integration looks like $E[W_1 \mid x_1 < W_{t_1} \le y_1, \ldots, x_n < W_{t_n} \le y_n]$ and cannot be easily simplified. We also note that in many cases it is not necessary to have complete knowledge about the rapidly expanding tree-structure corresponding to the skeleton $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$; for example, when interest focuses on a subset A of Ω, the skeletons can easily be restricted to this set by "cutting off" those branches of $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$ not involving A. The latter comment is important when using skeletons to compute approximations to stochastic integrals (see Willinger and Taqqu (1987)).

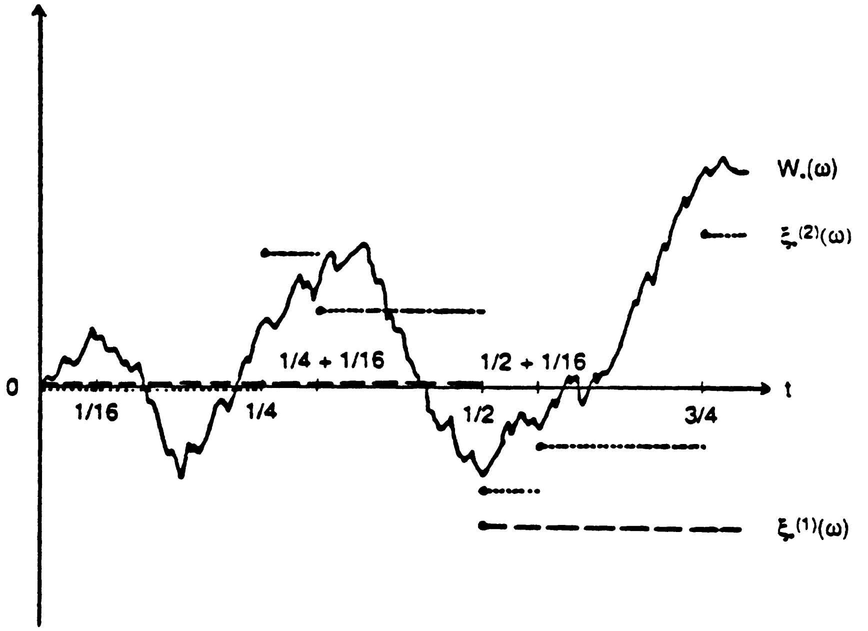


Figure 5.1.1 Skeleton-approximation of W via the Special Construction
$\omega \in \{0 < W_{1/4} \le u_1, x_2 < W_{1/2} \le 0, W_{3/4} > v_1,\ \ 0 < W_1 \le y_1\}$

5.2 Standard Brownian motion in $d > 1$ dimensions

This example illustrates modifications of the Special Construction for higher dimensional Brownian motion. For ease of exposition, we consider $d = 2$. Let $W = (W^1, W^2)^T$ denote 2-dimensional standard Brownian motion; that is, W^1 and W^2 are two independent (real-valued) standard Brownian motions defined on $(\Omega, \mathcal{F}, P)$, and take $\mathbf{F} = \mathbf{F}^W$.

First we show that the Special Construction can be modified so that the resulting skeleton-filtrations are binary. This is achieved by being more stingy with available information then in the case $d = 1$ but by releasing this information at a faster rate.

Assume that we are building - via the original Special Construction - the n^{th} skeleton-filtration between times $t' = kT/2^n$ and $t = (k+1)T/2^n$ where $\mathcal{F}_{t'}^{(n)}$ and $\mathcal{A}_{m(t,n)}(t)$ represent the presently available information. (The sequence $(\mathcal{A}_k(t))_{k\geq 0}$ results from applying the approximation scheme of Section 2 to the set-up $(\Omega, \sigma(X(t)), P)$ and $X(t) = (W_t^1, W_t^2)^T$ where W_t^1 and W_t^2 are two independent normal random variables each with mean 0 and variance t.) The approximating sequence $(X_k(t))_{k\geq 0}$ can be roughly pictured as in Figure 5.2.1. In particular, we observe that each partition set in $\mathcal{P}_k(t)$ is split into four subcells of $\mathcal{P}_{k+1}(t)$ when going from step k to step $k+1$ in our approximation procedure of (W_t^1, W_t^2). Therefore, setting

$$\mathcal{F}_{t+s(\ell)}^{(n)} = \mathcal{F}_{t'}^{(n)} \vee \mathcal{A}_{\ell+1}(t) \qquad (\ell = 0, 1, \ldots, n-1) \quad ,$$

as we did in the Special Construction for $d = 1$, would produce skeleton-filtrations with the following property

$$\dim(\text{span}\,(\xi_{t+s(\ell)}^{(n)}(\omega) - \xi_{t+s(\ell-1)}^{(n)}(\omega) : \omega \in A\,)) = 2$$

Further, since each cell is split in four,

$$\text{cardinality}\,(A' \in \mathcal{P}_{t+s(\ell)}(n) : A \supseteq A') = 4 \quad .$$

where $A \in \mathcal{P}_{t+s(\ell-1)}^{(n)}$ with $P[A] > 0; \ell = 1, \ldots, n-1$. Thus, $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$ does not satisfy condition $(C_{\xi^{(n)}}')$ when using the (unmodified) Special Construction.

To produce skeletons satisfying $(C_{\xi^{(n)}}')$, we introduce an additional point in time between $t + s(\ell)$, and $t + s(\ell+1)$, e.g. $t + s(\ell) + 2^{-3m(t,n)}$ $(\ell = 0, 1, \ldots, n-1)$. Then, instead of providing, for example, all the information contained in $\mathcal{A}_1(t)$ *at once* at time t, we first "reduce" this information by introducing some *dependence* between W_t^1 and

W_t^2 (see below) and construct $\mathcal{F}_t^{(n)}$. Subsequently, the coupling-effect between W_t^1 and W_t^2 is released and $\mathcal{F}^{(n)}_{t+2^{-3m(t,n)}}$ is defined to contain $\mathcal{A}_1(t)$. This same procedure is repeated at each point $t + s(\ell)$ $(\ell = 1, \ldots, n-1)$.

We have considerable freedom in choosing the degree of coupling between W_t^1 and W_t^2. One choice discussed below is to "open" one dimension at a time in order to obtain binary skeleton-filtrations. Let t' denote the last time in $I^{(n)}$ preceding $t \in D_n$, and consider $A \in \mathcal{P}_{t'}^{(n)}$ with $P[A] > 0$. Before determining the subcells of A that will belong to $\mathcal{P}_t^{(n)}$ and $\mathcal{P}^{(n)}_{t+2^{-3m(t,n)}}$, respectively, observe that the relevant information at time t is provided by $\mathcal{F}_{t'}^{(n)}$ and $\mathcal{A}_1(t)$; that is, by the set A and the partition sets $A_{11} = \{W_t^1 > 0,\ W_t^2 > 0\}$, $A_{12} = \{W_t^1 > 0,\ W_t^2 \le 0\}$, $A_{21} = \{W_t^1 \le 0,\ W_t^2 > 0\}$, $A_{22} = \{W_t^1 \le 0,\ W_t^2 \le 0\}$ of $\mathcal{A}_1(t)$. Now subdivide A in two stages (see Figure 5.2.2).

(1) Subcells of A that belong to $\mathcal{P}_t^{(n)}$:

$$A_1 = \{\omega \in A : W_t^1(\omega) > 0,\ W_t^2(\omega) \in \mathbf{R}\},$$

$$A_2 = \{\omega \in A : W_t^1(\omega) \le 0,\ W_t^2(\omega) \in \mathbf{R}\}.$$

(2) Subcells of A that belong to $\mathcal{P}^{(n)}_{t+2^{-3m(t,n)}}$:

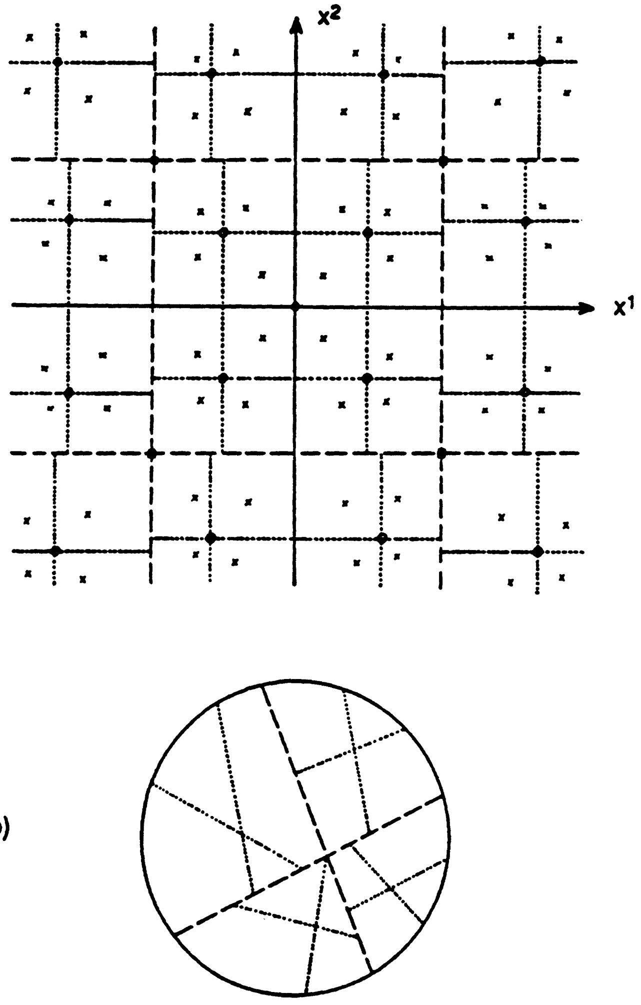
$$A_{11} = \{\omega \in A_1 : W_t^2(\omega) > 0\} = \{\omega \in A : W_t^1(\omega) > 0, W_t^2(\omega) > 0\},$$

$$A_{12} = \{\omega \in A_1 : W_t^2(\omega) \le 0\} = \{\omega \in A : W_t^1(\omega) > 0,\ W_t^2(\omega) \le 0\},$$

$$A_{21} = \{\omega \in A_2 : W_t^2(\omega) > 0\} = \{\omega \in A : W_t^1(\omega) \le 0, W_t^2(\omega) > 0\},$$

$$A_{22} = \{\omega \in A_2 : W_t^2(\omega) \le 0\} = \{\omega \in A : W_t^1(\omega) \le 0, W_t^2(\omega) \le 0\}.$$

(a)



(b)

Figure 5.2.1 Approximation scheme for the random vector $(W_t^1, W_t^2)^T$

(a) Partition of $\mathbf{R}^2$ after three steps.

(b) Partition of Ω after three steps.

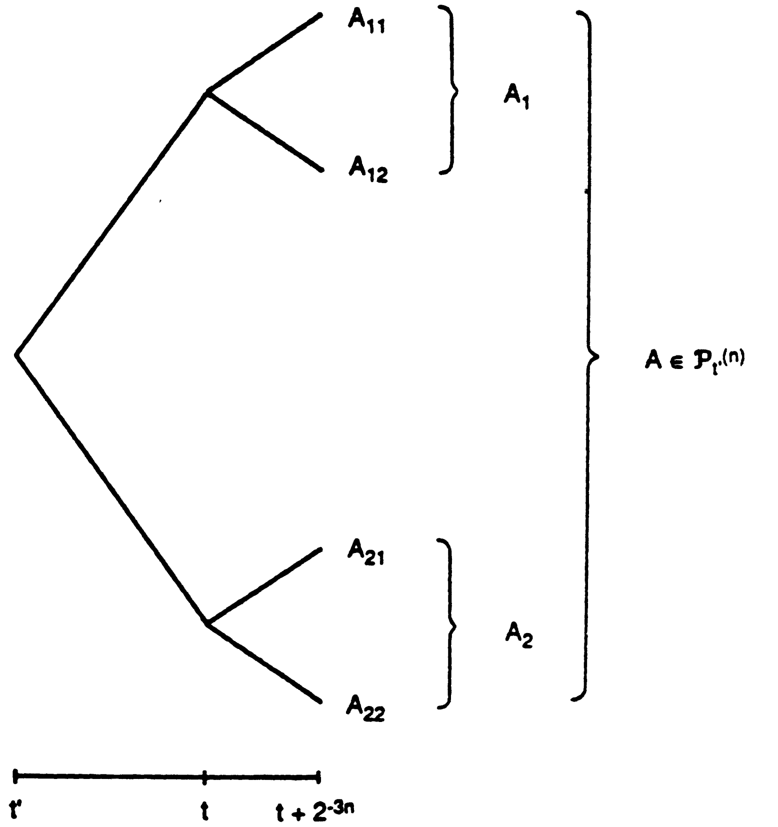


Figure 5.2.2 Binary skeleton-filtration.

Repeating this procedure at every point in the original index set $I^{(n)}$ implies that condition $(C_{\zeta^{(n)}}')$ holds at any point in the extended index set consisting of $I^{(n)}$ plus the additional points of the form $t + s(\ell) + 2^{-3m(t,n)}$. We abuse notation and denote the extended index set by $I^{(n)}$ as well.

Clearly this method applies to any dimension $1 \leq d < \infty$ and yields the following.

Corollary 5.2.1 Let $W = (W_t : 0 \le t \le 1)$ denote d-dimensional standard Brownian motion $(1 \le d < \infty)$. Then there exists a continuous-time skeleton-approximation $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})_{n \ge 0}$ of $(\mathbf{F}^W, W)$ such that for each $n \ge 0$, $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$ satisfies: for each $t(n,k-1), t(n,k) \in I^{(n)}$ $(0 < k \le N^{(n)})$, and for each $A \in \mathcal{P}^{(n)}_{t(n,k-1)}$ with $P[A] > 0$,

$$\dim(\text{span}(\{\xi^{(n)}_{t(n,k-1)}(\omega) - \zeta^{(n)}_{t(n,k-1)}(\omega) : \omega \in A\}))$$

$$= \text{cardinality}(A' \in \mathcal{P}^{(n)}_{t(n,k)} : A' \subseteq A) - 1 = 1 .$$

Next we show that the Special Construction can be modified to yield continuous-time skeleton-approximations of $(\mathbf{F}^W, W)$ where each skeleton has a *ternary* tree-structure (in the case $d = 2$). A ternary tree represents a filtration with the property that each event splits in three from one time point to the next.

To prove our claim, we modify the approximation scheme of Section 2 so that it looks like Figure 5.2.3 when applied to $(\Omega, \sigma(X(t)), P)$ and $X(t) = (W_t^1, W_t^2)^T$, and use this modified version in Step 1 of the Special Construction. For example, in order to obtain the first approximation step one can partition $\mathbf{R}^2$ in the three regions defined by 120° wedges. The first approximation of $X(t)$ would then yield a random vector that takes three different values each one equal to the conditional expectation of $X(t)$ over one of the three regions. Without going into details, it should be clear how to obtain an algorithm for such an approximation scheme with the same probabilistic and geometric properties as desired in Section 2 (see Theorems 2.2.1 and 2.2.2). Each partition set $A \in \mathcal{P}_k(t)$ is now split into *three* subcells of $\mathcal{P}_{k+1}(t)$ and execution of Step 2 of the Special Construction now yields skeleton-filtrations with the properties:

$$\dim(\text{span}(\{\zeta^{(n)}_{t(n,k)}(\omega) - \xi^{(n)}_{t(n,k-1)}(\omega) : \omega \in A\})) = 2$$

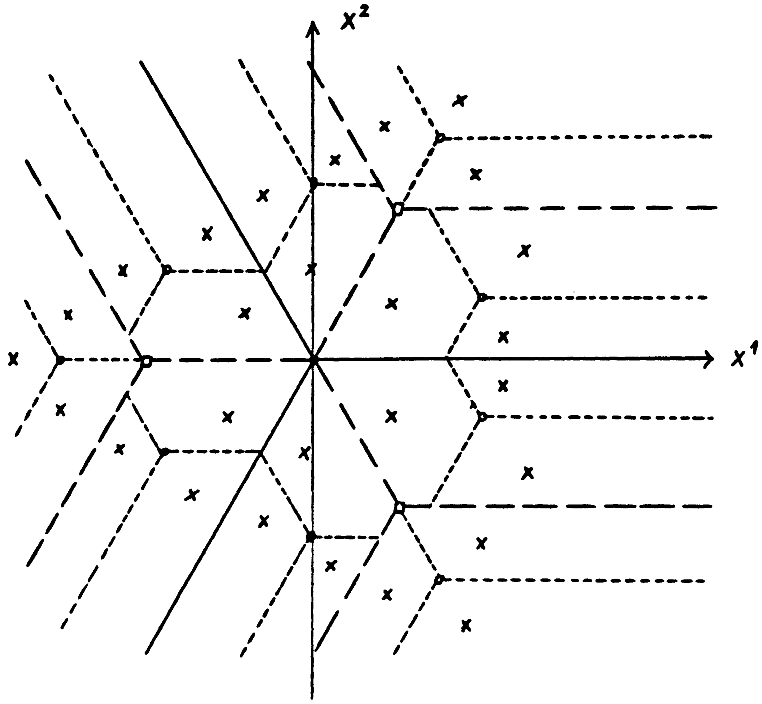
and

$$\text{cardinality}\ (A' \in \mathcal{P}^{(n)}_{t(n,k)} : A' \subseteq A) = 3$$

for all $A \in \mathcal{P}^{(n)}_{t(n,k-1)}$ with $P[A] > 0$ and $t(n,k), t(n,k-1) \in I^{(n)}$ $(0 < k \le N^{(n)})$.

Clearly, this method of modification works for any given dimension d and establishes the following result which explicitly relates the dimensionality d to the fine structure of the filtration $\mathbf{F}^W$. This relationship is only implicit in Corollary 5.2.1. For an economic interpretation of this remarkable result, see Willinger and Taqqu (1987).

(a)



(b)

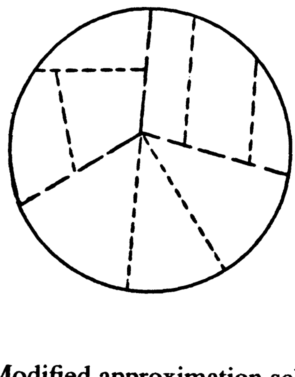


Figure 5.2.3 Modified approximation scheme for $(W_t^1, W_t^2)^T$.

(a) Partition of $\mathbf{R}^2$ after three steps.

(b) Partition of Ω after three steps.

Corollary 5.2.2 Let $W = (W_t : 0 \le t \le 1)$ denote d-dimensional standard Brownian motion $(1 \le d < \infty)$ Then there exists a continuous-time skeleton-approximation $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})_{n \ge 0}$ of $(\mathbf{F}^W, W)$ such that for each $n \ge 0$, $(I^{(n)}, \mathbf{F}^{(n)}, \xi^{(n)})$ satisfies: for each $t(n,k-1), t(n,k) \in I^{(n)}$ $(0 < k \le N^{(n)})$, and for each $A \in \mathcal{P}^{(n)}_{t(n,k-1)}$ with $P[A] > 0$,

$$\dim (\mathrm{span} (\{ \zeta^{(n)}_{t(n,k)}(\omega) - \zeta^{(n)}_{t(n,k-1)}(\omega) : \omega \in A \}))$$

$$= \mathrm{cardinality}\,(A' \in \mathcal{P}^{(n)}_{t(n,k)} : A' \subseteq A) - 1 = d \quad .$$

5.3 Continuous martingales with values in $\mathbf{R}^d$ $(1 \le d < \infty)$

Let $M = (M_t : 0 \le t \le T)$ be an $\mathbf{R}^d$-valued martingale defined in a stochastic base $(\Omega, \mathcal{F}, P, \mathbf{F})$. Assume that M has continuous sample paths and take $\mathbf{F} = \mathbf{F}^M$ where $\mathbf{F}^M$ denotes the (completed) minimal filtration satisfying the continuity-assumption (4.4.1). Note that each continuous-time skeleton $(I^{(n)}, \mathbf{F}^{(n)}, \mu^{(n)})$ of $(\mathbf{F}^M, M)$ produced by either the general constructions or the Special Construction (or its variations) defines a "skeleton-martingale", i.e. $\mu^{(n)}$ is a $(\mathbf{F}^{(n)}, P)$-martingale. This result is proved exactly as in the discrete-time case (see Example 3.2.1) and demonstrates the structure-preserving character of these constructions. Moreover, using the variations of the Special Construction presented in the previous subsection, we can easily extend the results of Corollaries 5.2.1 an 5.2.2 to continuous martingales satisfying assumption (4.4.1).

Corollary 5.3.1 There exists a continuous-time skeleton-approximation $(I^{(n)}, \mathbf{F}^{(n)}, \mu^{(n)})_{n \ge 0}$ of the $(\mathbf{F}^M, P)$-martingale M such that or each $n \ge 0$, the following two properties hold:

(i) $(I^{(n)}, \mathbf{F}^{(n)}, \mu^{(n)})$ is a $(\mathbf{F}^{(n)}, P)$-martingale,

(ii) for each $t(n,k-1), t(n,k) \in I^{(n)}$ $(0 < k \le N^{(n)})$ and for each $A \in \mathcal{P}^{(n)}_{t(n,k-1)}$ with
$P[A] > 0$,

$$\dim(\mathrm{span}(\{\mu^{(n)}_{t(n,k)}(\omega)-\mu^{(n)}_{t(n,k-1)}(\omega):\omega\in A\backslash A^{\circ}(t(n,k),A)\}))$$

$$=\mathrm{cardinality}\,(A'\in \mathcal{P}^{(n)}_{t(n,k)}:A'\subseteq A\backslash A^{\circ}(t(n,k),A))-1$$

$$=\begin{cases}0 & \text{if } A^{\circ}(t(n,k),A)\neq\varnothing\\ 1 & \text{if } A^{\circ}(t(n,k),A)=\varnothing\end{cases},$$

where $A^{\circ}(t(n,k),A)=\bigcup\{A''\in \mathcal{P}^{(n)}_{t(n,k)}:A''\subseteq A, P[A'']=0\}$.

Corollary 5.3.2 There exists a continuous-time skeleton-approximation $(I^{(n)}, \mathbf{F}^{(n)}, \mu^{(n)})_{n\geq 0}$ of the $(\mathbf{F}^{M}, P)$-martingale M such that for each $n\geq 0$, the following two properties hold:

(i) $(I^{(n)}, \mathbf{F}^{(n)}, \mu^{(n)})_{n\geq 0}$ is a $(\mathbf{F}^{(n)}, P)$-martingale,

(ii) for each $t(n,k-1), t(n,k)\in I^{(n)}$ $(0<k\leq N^{(n)})$, and for each $A\in \mathcal{P}^{(n)}_{t(n,k-1)}$, with $P[A]>0$,

$$\dim(\mathrm{span}(\{\mu^{(n)}_{t(n,k)}(\omega)-\mu^{(n)}_{t(n,k-1)}(\omega):\omega\in A\backslash A^{\circ}(t(n,k),A)\}))$$

$$=\mathrm{cardinality}\,(A'\in \mathcal{P}^{(n)}_{t(n,k)}:A'\subseteq A\backslash A^{\circ}(t(n,k),A))-1\leq d$$

where $A^{\circ}(t(n,k),A)=\bigcup\{A''\in \mathcal{P}^{(n)}_{t(n,k)}:A''\subseteq A, P[A'']=0\}$.

Compared to the results for Brownian motion (i.e. Corollaries 5.2.1 and 5.2.2), property (ii) of Corollary 5.3.1 and Corollary 5.3.2 expresses the fact (first discovered by Fisk (1965)) that the sample paths of continuous martingales on many subinterval of $[0,T]$ are either of unbounded variation or constants. Of course, (P-almost) all sample paths of Brownian motion are known to be of unbounded variation over each subinterval of $[0,T]$ (Billingsley (1979)) and, therefore, property (ii) of Corollaries 5.3.1 and 5.3.2 is satisfied with equality for Brownian motion.

For an application of these martingale approximation results, see Willinger and Taqqu (1987). They show that the completeness property can be maintained along skeleton-approximations and use this to provide a pathwise construction of stochastic integrals relative to continuous martingales.

5.4 Brownian motion with a random variance ($d = 1$)

This example appears in a related context in Harrison and Pliska (1981) and illustrates the importance of the continuity-of-information assumption (4.4.1). We show that events "that take us by surprise" cannot be incorporated in convergent, binary skeleton-filtrations and, therefore, destroy the convergence-of-information property (4.2.2b). However, if such events can be made "observable" by adding suitable component processes, then convergence of information is possible and skeleton-approximations will exist.

Specifically, let $W = (W_t : 0 \le t \le 1)$ denote 1-dimensional, standard Brownian motion and consider an independent stochastic process $\sigma = (\sigma_t : 0 \le t \le 1)$ such that

$$\sigma_t(\omega) = \begin{cases} 2 & \text{if } 0 \le t < 1/2 \quad \text{for all } \omega \in \Omega \ , \\ 1 & \text{if } 1/2 \le t \le 1 \quad \text{and if } \omega \in A \ , \\ 3 & \text{if } 1/2 \le t \le 1 \quad \text{and if } \omega \notin A \ , \end{cases}$$

where $P[A] = 1/2$. Next define

$$X_t(\omega) = 1_{[0,1/2)}(t)\, 2W_t(\omega) + 1_{[1/2,1]}(t)\, (1_A(\omega)\, W_t(\omega) + 1_A(\omega)\, 3W_t(\omega)),\ 0 \le t \le 1 \ .$$

The stochastic process $X = (X_t : 0 \le t \le 1)$ evolves as a driftless Brownian motion with variance parameter $\sigma^2 = 4$ over the time interval $[0, 1/2)$. Then depending on the outcome of an independent Bernoulli-trial at $t = 1/2$, the variance parameter increases to $\sigma^2 = 9$ if A^c occurs, or decreases to $\sigma^2 = 1$ if A occurs. Set $\mathbf{F} = \mathbf{F}^X$ and note that $\mathbf{F}$ is identical to a (completed) Brownian filtration augmented by the outcome of an independent Burnoulli-trial for times $1/2 \le t \le 1$.

Clearly, $\mathbf{F}$ satisfies the "usual conditions" (4.1.1) and P-almost all sample paths of X are continuous. Continuity of $\mathbf{F}$ (i.e. assumption (4.4.1)), however, does not hold at time $t = 1/2$; in fact, $A \notin \mathcal{F}_{1/2-}$, $A \in \mathcal{F}_t$ $(1/2 \le t \le 1)$ and, therefore, $\mathcal{F}_{1/2-} \ne \mathcal{F}_{1/2} = \mathcal{F}_{1/2+}$. In this sense, A represents an event "that takes us by surprise" and prevents the construction of convergent binary skeleton-filtration (see part 2) of the proof of Theorem 4.4.1). Intuitively, in order to determine the variance parameter of X at time $t = 1/2$, we have to observe a sample path of X over an infinitesimal interval $[1/2, 1/2+\varepsilon)$ $(\varepsilon > 0)$; that is, we need the information contained in $\mathcal{F}_{1/2+} = \mathcal{F}_{1/2}$. But this "limiting" information cannot be handled by binary, approximating skeleton-filtrations.

Convergence of information can be achieved, however, by adding a suitable component-process $Z^2 = (Z_t^2 : 0 \le t \le 1)$. Specifically, let

$$Z_t^1(\omega) = X_t(\omega), \quad \omega \in \Omega, 0 \le t \le 1 \quad ,$$

$$Z_t^2(\omega) = \begin{cases} 1 & \text{if } 0 \le t < 1/2 \quad , \omega \in \Omega \; , \\ 0 & \text{if } 1/2 \le t \le 1 \quad \text{and } \sigma_t(\omega) = 1 \; , \\ 2 & \text{if } 1/2 \le t \le 1 \quad \text{and } \sigma_t(\omega) = 3 \; . \end{cases}$$

and set $Z = (Z^1, Z^2)$, $\mathbf{F} = \mathbf{F}^Z = \mathbf{F}^X$. Although neither Z is continuous nor does $\mathbf{F}$ satisfy (4.4.1), a variation of the Special Construction ($d = 2$), produces convergence of information and continuous-time skeleton-approximations of $(\mathbf{F}^Z, Z)$. This is because jumps of Z^2 occur at fixed points in time (namely, at $t = 1/2$) and do not create problems for the Special Construction and its various modifications. More importantly, the discontinuity of $\mathbf{F}^Z$ at time $t = 1/2$ is fully explained by the "observable" component-process Z^2.

Acknowledgement. This work is based in part on the first author's dissertation under the direction of the second author. We acknowledge Harry Kesten for pointing out some technical problems in an early version of Willinger (1987) and for helpful discussions.

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