SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

PEI HSU
PETER MARCH

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Séminaire de probabilités (Strasbourg), tome 22 (1988), p. 502-507
<http://www.numdam.org/item?id=SPS_1988__22__502_0>

Brownian Excursions From Extremes

*Pei Hsu**

Courant Institute of Mathematical Sciences, New York.

*P. March***

McGill University, Montréal.

Let $B = (B_t, \mathcal{F}_t, P; t \geq 0)$ be standard Brownian motion starting at zero and define its extreme processes as

$$M_t = \max_{0 \leq s \leq t} B_s \qquad \text{and} \qquad m_t = \min_{0 \leq s \leq t} B_s.$$

The point of this note is to observe a mapping property of Brownian motion and use it to derive some results about excursions of B from its extremes which are related to the work of Groeneboom [4], Bass[1] and Pitman[9] and of Imhof[7]. It must be pointed out that these results are consequences of general excursion theory as expounded by Getoor [2],[3] and Jacobs[8], for example. However this mapping property is new and its application to excursions is direct.

Let $r_t = M_t - m_t$ be the range process and for each $\epsilon > 0$ define the increasing processes

$$a(t, \epsilon) = \int_\epsilon^t 4 r_s^{-2} ds$$

and

$$\tau(t, \epsilon) = \inf\{s : a(s, \epsilon) > t\}.$$

Let

$$X_t = \frac{2B_t - M_t - m_t}{M_t - m_t} \tag{1}$$

and define $X_t^\epsilon = X_{\tau(t,\epsilon)}$.

Proposition 1. *The process $X^\epsilon = (X_t^\epsilon, \mathcal{F}_{\tau(t,\epsilon)}, P; t \geq 0)$ is a reflecting Brownian motion on $[-1, 1]$. Its local times at ± 1 are*

$$\phi_t^{\epsilon,+} = \int_\epsilon^{\tau(t,\epsilon)} 4 r_s^{-1} dM_s$$

and

$$\phi_t^{\epsilon,-} = \int_\epsilon^{\tau(t,\epsilon)} 4 r_s^{-1} d(-m_s) \qquad \textit{respectively}$$

Proof. We may write equation (1) as $X_t = F(B_t, M_t, m_t)$ where

$$F(x, y, z) = (2x - y - z)/(y - z).$$

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* Research supported in part by the grant NSF-MCS-82-01599.

** Research supported in part by the Institute for Mathematics and Its Applications with funds provided by the National Science Foundation and the Army Research Office and by the grant NSERC U0523.

Since F is smooth on $\{y \neq z\}$, we may apply Itô's formula there to obtain

$$(2) \qquad dX_s = \frac{2dB_s}{M_s - m_s} + \frac{2(M_s - B_s)d(-m_s)}{(M_s - m_s)^2} - \frac{2(B_s - m_s)dM_s}{(M_s - m_s)^2}.$$

Because each $\tau(t,\epsilon)$ is an $\mathcal{F}_t$-stopping time we may write (2) in the integrated form

$$(3) \qquad X_{\tau(t,\epsilon)} = X_\epsilon + W_t^\epsilon + \frac{1}{2}\phi_t^{\epsilon,-} - \frac{1}{2}\phi_t^{\epsilon,+}$$

where

$$(4) \qquad \begin{cases} W_t^\epsilon = \int_\epsilon^{\tau(t,\epsilon)} 2r_s^{-1} dB_s \\ \phi_t^{\epsilon,-} = \int_\epsilon^{\tau(t,\epsilon)} 4r_s^{-1} d(-m_s) \\ \phi_t^{\epsilon,+} = \int_\epsilon^{\tau(t,\epsilon)} 4r_s^{-1} dM_s \end{cases}$$

To finish the proof we check that W^ϵ is a standard Brownian motion and that (3) is its Skorohod equation (Tanaka[11]). Clearly W^ϵ is an $\mathcal{F}_{\tau(t,\epsilon)}$-martingale and

$$[W^\epsilon]_t = \int_\epsilon^{\tau(t,\epsilon)} 4r_s^{-2} ds = a(\tau(t,\epsilon),\epsilon) = t.$$

By Lévy's criterion, W^ϵ is a Brownain motion, independent of X_ϵ. Now $\phi_t^{\epsilon,\pm}$ are continuous, incereasing $\mathcal{F}_{\tau(t,\epsilon)}$-adapted processes which increase only when B attains a new extremum, that is only when $X^\epsilon = \pm 1$. ◇

As the next propostion shows, we may write the extreme processes in terms of the local times $\phi^{\epsilon,\pm}$. Set $\phi_t^\epsilon = \phi_t^{\epsilon,+} + \phi_t^{\epsilon,-}$.

Proposition 2. *For $t \geq \epsilon$,*

$$(i) \qquad M_t = M_\epsilon + r(\epsilon) \int_0^{a(t,\epsilon)} \exp\{\phi_s^\epsilon/4\}\, d\phi_s^{\epsilon,+}$$

$$(ii) \qquad m_t = m_\epsilon + r(\epsilon) \int_0^{a(t,\epsilon)} \exp\{\phi_s^\epsilon/4\}\, d\phi_s^{\epsilon,-}$$

where $a(t,\epsilon) = \inf\{s : \tau(s,\epsilon) > t\}$ and

$$\tau(t,\epsilon) = \epsilon + \frac{1}{4} r(\epsilon)^2 \int_0^t \exp\{\phi_s^\epsilon/2\}\, ds.$$

Proof. By (4) we have

$$\phi_t^\epsilon = \int_\epsilon^{\tau(t,\epsilon)} 4r_s^{-1} dr_s = 4 \log \frac{r(\tau(t,\epsilon))}{r(\epsilon)},$$

hence

(5) $$r(\tau(t,\epsilon)) = r(\epsilon)\exp\{\phi_t^\epsilon/4\}$$

Since $a(\tau(t,\epsilon),\epsilon) = t$ it follows that $d\tau(t,\epsilon) = r(\tau(t,\epsilon))^2 dt/4$. Thus by (5),

$$\begin{aligned}\tau(t,\epsilon) =&\tau(0,\epsilon) + \frac{1}{4}\int_0^t r(\tau(t,\epsilon))^2 ds\\ =&\epsilon + \frac{1}{4}r(\epsilon)^2\int_0^t \exp\{\phi_s^\epsilon/2\}\, ds.\end{aligned}$$

It follows that τ and hence a are defined solely in terms of M_ϵ, m_ϵ and $\phi^{\epsilon,\pm}$.

Next, by (4)

(6) $$\phi_t^{\epsilon,+} = \int_\epsilon^{\tau(t,\epsilon)} 4r_s^{-1} dM_s = \int_0^t 4r(\tau(s,\epsilon))^{-1} dM_{\tau(s,\epsilon)},$$

and so

(7a) $$M_{\tau(t,\epsilon)} = M_\epsilon + \frac{1}{4}r_\epsilon\int_0^t \exp\{\phi_s^\epsilon/4\}\, d\phi_s^{\epsilon,+},$$

Similarly, we have

(7b) $$m_{\tau(t,\epsilon)} = m_\epsilon + \frac{1}{4}r_\epsilon\int_0^t \exp\{\phi_s^\epsilon/4\}\, d\phi_s^{\epsilon,-},$$

and the proposition follows from a time change in (7a) and (7b).◇

These propositions allow us to compare excursions of B from its extremes with excursions of reflecting Brownian motion in $[-1,1]$. To be precise, let

(8) $$f(t,\epsilon) = \inf\{s : \phi_s^\epsilon > t\}$$

be the inverse of boundary local time and let q^ϵ be the point process of excursions of X^ϵ. That is, let

$$D_{q^\epsilon} = \{s : f(s,\epsilon) > f(s-,\epsilon)\}$$

and for each $s \in D_{q^\epsilon}$ let

(9) $$q_s^\epsilon(u) = X^\epsilon(f(s-,\epsilon) + u\wedge l_s^\epsilon), \qquad u \geq 0$$

where $l_s^\epsilon = f(s,\epsilon) - f(s-,\epsilon)$ is the duration of the excursion. Similarly, consider the point process p of excursions of B from its extremes. Let

(10) $$\mu(t) = \inf\{s : r(s) > t\},$$

let the domain of p be $D = \{t : \mu(t) > \mu(t-)\}$ and for each $t \in D$ let

(11) $$p_t(u) = B(\mu(t-) + u\wedge\lambda(t)), \qquad u \geq 0$$

where $\lambda(t) = \mu(t) - \mu(t-)$. Proposition 4 provides a formula for p in terms of q^ϵ. To ensure the formula is well defined we need the

Lemma 3. *Let* $D^\epsilon = \{t : \mu(t) > \mu(t-) \text{ and } \mu(t) > \epsilon\}$. *Then*
(i) $f(t,\epsilon) = a(\mu(r(\epsilon)e^{t/4}), \epsilon)$
(ii) $D_{q^\cdot} = \{s(u,\epsilon) : u \in D^\epsilon\}$ *where* $s(t,\epsilon) = 4\log(tr(\epsilon)^{-1})$.

Proof. The lemma follows easily from the equality $\tau(f(t,\epsilon),\epsilon) = \mu(r(\epsilon)e^{t/4})$, which we now show. Let $\alpha(t)$ and $\beta(t)$ denote the left and right side of this equality, respectively. On the one hand, by (7a) and (7b), we have

$$r(\alpha(t)) = r(\epsilon)\exp\{\phi^\epsilon(f(t,\epsilon))\} = r(\epsilon)e^{t/4} \overset{\text{def.}}{=} g(t).$$

On the other hand, by definition $r(\beta(t)) = g(t)$. Thus $r(\alpha(t)) = r(\beta(t))$. Since $g(t)$ is strictly increasing, for any $\delta > 0$

$$\begin{aligned}\alpha(t+\delta) \geq & \mu(r(\alpha(t+\delta))-) = \mu(g(t+\delta)-)\\ \geq & \mu(g(t)) = \mu(r(\beta(t))) = \beta(t).\end{aligned}$$

Letting $\delta \downarrow 0$ we get $\alpha(t) \geq \beta(t)$. Since the reverse inequality is similar, the lemma is proved.◇

Proposition 4. *Let* $\{p_t; t \in D\}$ *be the point process of excursions of B from its extremes. For each* $t \in D^\epsilon$

$$p_t(u) = \frac{t}{2} q^\epsilon_{s(t,\epsilon)}\left(\frac{4u}{t^2}\right) + \frac{1}{2}\left(M_{\mu(t)} + m_{\mu(t)}\right)$$

Proof. First note that the statement makes sense, by Lemma 3. Let $s \in D_{q^\cdot}$ where $s = s(t,\epsilon)$ and $t \in D^\epsilon$. The durations $l^\epsilon(s)$ of q^ϵ_s and $\lambda(t)$ of p_t are related, according to Lemma 3, by

$$\begin{aligned}l^\epsilon(s) =& f(s(t,\epsilon),\epsilon) - f(s(t,\epsilon)-,\epsilon)\\ =& a(\mu(t),\epsilon) - a(\mu(t)-,\epsilon)\\ =& 4\int_{\mu(t-)}^{\mu(t)} r_u^{-2}\,du\\ =& \frac{\mu(t)-\mu(t-)}{r(\mu(t))^2} = 4\frac{\lambda(t)}{t^2}.\end{aligned} \tag{12}$$

Thus by the formulas

$$X_u = \frac{2B_u - M_u - m_u}{M_u - m_u}, \qquad X^\epsilon_v = X_{\tau(v,\epsilon)}$$

and the definition of q^ϵ and p we get

$$q^\epsilon_{s(t,\epsilon)}(u) = \frac{1}{t}\left(2p_t\left(\frac{t^2u}{4}\right) - M_{\mu(t)} - m_{\mu(t)}\right),$$

from which the proposition follows. ◇

An immediate corollary is the identification of the conditional law of excursions of B from its extremes. Indeed, let $-\infty < c < d < \infty$ and introduce the transition density of Brownian motion in $[c,d]$ with absorption at the endpoints (Port-Stone[10]):

$$p_0^{c,d}(t,x,y) = \frac{2}{d-c}\sum_{n=0}^{\infty} \sin\left(n\pi\frac{x-c}{d-c}\right)\sin\left(n\pi\frac{y-c}{d-c}\right)\exp\left\{-\frac{n^2\pi^2}{(d-c)^2}\frac{t}{2}\right\} \tag{14}$$

as well as the functions

$$(15)\qquad \begin{cases} g^{c,d}(t,y;a) = \dfrac{1}{2}\dfrac{\partial}{\partial n_a} p_0^{c,d}(t,a,y), & a = c,d \\ \theta^{c,d}(t,a,b) = \dfrac{1}{4}\dfrac{\partial^2}{\partial n_a \partial n_b} p_0^{c,d}(t,a,b), & a,b = c,d. \end{cases}$$

There exist unique probability laws $P_{c,d}^{a,b;l}$ on $C([0,\infty),[c,d])$ with absolute distribution:

$$(16)\qquad P_{c,d}^{a,b;l}\left(e(u) \in dy\right) = \frac{g^{c,d}(u,y;a)g^{c,d}(l-u,y;b)}{\theta^{c,d}(l,a,b)}dy, \qquad 0 \le u \le l$$

and transition density

$$(17)\quad P_{c,d}^{a,b;l}\left(e(v) \in dy \middle| e(u) = x\right) = p_0^{c,d}(v-u,x,dy)\frac{g^{c,d}(l-v,y;b)}{g^{c,d}(l-u,x;b)} \qquad 0 \le u < v \le l.$$

Indeed, if $X^{c,d}$ is reflecting Brownian motion in $[c,d]$ then $P_{c,d}^{a,b;l}$ is just the law of the excursion process of $X^{c,d}$ conditioned to begin at a, end at b and have duration l. This is a simple extension of the well-known case of one reflecting barrier (e.g. Ikeda-Watanabe[6]) and also can be proved by imitating the calculations of Hsu[5]. Finally let us note a scaling property of the laws $P_{c,d}^{a,b;l}$ which follows from the invariance of the family $\{p_0^{c,d}, -\infty < c < d < \infty\}$ under affine changes of variable:

(18) If $Z = \{Z(t); 0 \le t \le l\}$ has the law $P_{c,d}^{a,b;l}$ then $\{\alpha Z(\alpha^{-2}t) + \beta; 0 \le t \le \alpha^2 l\}$ has the law $P_{\alpha c+\beta,\alpha d+\beta}^{\alpha a+\beta,\alpha b+\beta;\alpha^2 l}$.

Theorem 5. *Let $t \in D$. Let $-\infty < c < d < \infty$ and let $l > 0$. Then conditional on the event $\xi = [m_{\mu(t)} = c, M_{\mu(t)} = d, p_t(0) = a, p_t(\lambda(t)) = b, \lambda(t) = l]$, the law of the excursion process $p_t(\cdot)$ is $P_{c,d}^{a,b;l}$.*

Proof. Fix some ϵ with $t \in D^\epsilon$ and let $s = s(t,\epsilon)$. By (9) and (12), we have

$$\xi = [q_s^\epsilon(0) = \mathrm{sgn}(a), q_s^\epsilon(l^\epsilon(s)) = \mathrm{sgn}(b), l^\epsilon(s) = |d-c|^2 l/4]$$

But then conditional on ξ, the process $q_s^\epsilon(\cdot)$ has law $P_{-1,1}^{e,f;m}$ with $e = \mathrm{sgn}(a), f = \mathrm{sgn}(b)$ and $m = |d-c|^2 l/4$. So by Proposition 4 and the invariance property (18) we find that conditional on ξ, $p_t(\cdot)$ has the law $P_{c,d}^{a,b;l}$. ◇

It is known that if X is reflecting Brownian motion in an interval then conditional on the σ-field generated by the boundary local time of X, the various excursions of X from the boundary are mutually independent. This is evident from the construction of the excursions law characterizing the excursion point process in the one reflecting barrier case (Ikeda-Watanabe[6]). Or again, one can either imitate the argument of Hsu[5] or simply quote the results in Jacobs[8]. Let us show that this conditional independence property is shared by excursions of Brownian motion B from its extremes, conditional on $\sigma\{M_s, m_s; s \ge 0\}$.

Lemma 6. *Let $\mathcal{B}_\epsilon = \sigma\{\phi_s^{\epsilon,+}, \phi_s^{\epsilon,-}; s \ge 0\}$ and $\mathcal{B} = \sigma\{M_s, m_s; s \ge 0\}$. Then $\mathcal{B}_\epsilon \subset \mathcal{B}$ and $\lim_{\epsilon \to 0} \mathcal{B}_\epsilon = \mathcal{B}$.*

Proof. Since Proposition 2 exhibits M and m as explicit functions of $\phi^{\epsilon,\pm}$, we have the inclusions

$$\sigma\{M_s - M_\epsilon, m_s - m_\epsilon; s \geq \epsilon\} \subset \sigma\{M_\epsilon, m_\epsilon, \phi_s^{\epsilon,\pm}; s \geq 0\} \subset \sigma\{M_s, m_s; s \geq \epsilon\}$$

and the lemma follows from this.◇

Theorem 7. *Conditional on* $\mathcal{B} = \sigma\{M_s, m_s; s \geq 0\}$, *the excursions* $[p_t(\cdot); t \in D]$ *are mutually independent.*

Proof. For $n \geq 1$ consider functionals $F : C([0,\infty), R)^n \to R$ of the form

$$F(\omega_1, \omega_2, \ldots, \omega_n) = \prod_{j=1}^{n} f_j(\omega_j(s_{j,1}), \ldots, \omega_j(s_{j,m(j)}))$$

for bounded continuous functions f_j. Let $t_1, \ldots, t_n \in D$. Using Proposition 1, for all sufficiently small ϵ,

$$E\left[F(p_{t_1}, \ldots, p_{t_n})\middle|\mathcal{B}_\epsilon\right] = \prod_{j=1}^{n} E\left[f_j(p_{t_j}(s_{j,1}), \ldots, p_{t_j}(s_{j,m(j)}))\middle|\mathcal{B}_\epsilon\right]$$

by the conditional independence property of q^ϵ. Thus by the martingale convergence theorems and Lemma 6; taking the limit as $\epsilon \downarrow 0$ yields

$$E\left[F(p_{t_1}, \ldots, p_{t_n})\middle|\mathcal{B}\right] = \prod_{j=1}^{n} E\left[f_j(p_{t_j}(s_{j,1}), \ldots, p_{t_j}(s_{j,m(j)}))\middle|\mathcal{B}\right]. \qquad \diamond$$

We close by remarking that Theorem 5 and 7 show that Brownian motion consists of conditionally independent Brownian excursions properly interpolated between endpoints of flat stretches of the extreme process M and m.

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