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Remarks on Absolute Continuity, Contiguity and Convergence in Variation of Probability Measures*

by

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Let (Ω^n, F^n), $n\geqslant 1$, be measurable spaces with right-continuous filtrations $\underline{F}^n=(\underline{\underline{F}}^n_t)_{t>0}$, and $\underline{\underline{F}}^n = \bigvee_{t\geqslant 0}\underline{\underline{F}}^n_t$. Let P^n and $\tilde{P}^n$ be probability measures defined on $\underline{\underline{F}}^n$. From [2]-[4], it is known that Hellinger processes are the main tools for the study of absolute continuity, contiguity and convergence in variation of probability measures. In §1, by using the results about the convergence of submartingales at infinity, we give the Lebesgue's decomposition between measures. Then the conditions for absolute continuity and singularity can be deduced immediately. These facts are easy, but they supplement the known results completely. In §2 and §3, we give new proofs of the conditions for contiguity and convergence in variation respectively. These proofs start directly from derivative processes, don't need the deeper properties of Hellinger processes. Hence, they are straightforward and can be easily followed. All results are applied to semimartingale cases.

1. Absolute Continuity

1.1Preliminaries. We'll adopt all denotations of [1] without specification. For the sake of convenience, we always omit the index n. It appears only in the case, where it is indispensable.

Set $Q = \frac{1}{2}(P+\tilde{P})$. Suppose that $(\Omega, \underline{\underline{F}}, Q)$ is a complete probability space, and under Q $\underline{F} =(\underline{\underline{F}}_t)$ satisfies the usual conditions. Let Z and $\tilde{Z}$ be the derivative processes of P and $\tilde{P}$ with respect to Q respectively,

$$Z = \left(E^Q[\tfrac{dP}{dQ} \mid \underline{\underline{F}}_t] \right), \qquad \tilde{Z} = \left(E^Q[\tfrac{d\tilde{P}}{dQ} \mid \underline{\underline{F}}_t] \right),$$

$$T_k = \inf\{t: Z_t \leqslant 1/k\}, \qquad \tilde{T}_k = \inf\{t: \tilde{Z}_t \leqslant 1/k\},$$

$$T = \sup_k T_k = \inf\{t: Z_t = 0\}, \qquad \tilde{T} = \sup_k \tilde{T}_k = \inf\{t: \tilde{Z}_t = 0\},$$

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$$\Gamma = \bigcup_k \llbracket 0, T_k \rrbracket, \qquad \widetilde{\Gamma} = \bigcup_k \llbracket 0, \widetilde{T}_k \rrbracket,$$

$$S_k = T_k \wedge \widetilde{T}_k, \qquad S = T \wedge \widetilde{T}, \qquad \Gamma \cap \widetilde{\Gamma} = \bigcup_k \llbracket 0, S_k \rrbracket .$$

Denote by μ the jump measure of Z, by ν the compensator of μ under Q: $\nu = \mu^{p,Q}$. Set

$$\lambda = 1 + x/Z_-, \qquad \widetilde{\lambda} = 1 - x/\widetilde{Z}_-,$$

then $\lambda, \widetilde{\lambda} \in \widetilde{\underline{\underline{P}}}$, and (see[3])

$$\mu^{p,P} = \lambda . \nu, \qquad \mu^{p,\widetilde{P}} = \widetilde{\lambda} . \nu,$$

(Obviously, $1_{\lambda<0} . \nu = 1_{\widetilde{\lambda}<0} . \nu = 0$.)

From now on all discussions proceed under the probability measure Q, unless otherwise specified. We have

$$Z = Z^c + x*(\mu - \nu), \qquad \widetilde{Z} = \widetilde{Z}^c - x*(\mu - \nu),$$

$$Z + \widetilde{Z} = 2, \qquad Z^c + \widetilde{Z}^c = 2, \qquad \Delta Z + \Delta \widetilde{Z} = 0.$$

The Hellinger process (with index 1/2) of P and $\widetilde{P}$ is

$$H = \frac{1}{8} (1/Z_- + 1/\widetilde{Z}_-)^2 . \langle Z^c \rangle + \frac{1}{2} (\sqrt{\lambda} - \sqrt{\widetilde{\lambda}})^2 * \nu \tag{1.1}$$

and $1_{(\Gamma \cap \widetilde{\Gamma})^c} . H = 0$.

1.2. **Theorem.** Set $N = \{S < \infty \text{ or } H_\infty = \infty\}$. Then

(1) $\widetilde{P} \perp P$ on N,

(2) $\widetilde{P} \sim P$ on N^c.

Proof. We have (see [1])

$$\{Z_\infty > 0\} = \{T = \infty, \ \frac{1}{Z_-^2} . \langle Z^c \rangle + (1 - \sqrt{\lambda})^2 * \nu_\infty < \infty\} \tag{1.2}$$

$$\{\widetilde{Z}_\infty > 0\} = \{\widetilde{T} = \infty, \ \frac{1}{\widetilde{Z}_-^2} . \langle Z^c \rangle + (1 - \sqrt{\widetilde{\lambda}})^2 * \nu_\infty < \infty\} \tag{1.3}$$

Since 1 is between $\sqrt{\lambda}$ and $\sqrt{\widetilde{\lambda}}$,

$$(1 - \sqrt{\lambda})^2 \leqslant (\sqrt{\lambda} - \sqrt{\widetilde{\lambda}})^2, \qquad (1 - \sqrt{\widetilde{\lambda}})^2 \leqslant (\sqrt{\lambda} - \sqrt{\widetilde{\lambda}})^2$$

Comparing (1.1) with (1.2) and (1.3), we get

$$\{S = \infty, \ H_\infty < \infty\} = \{Z_\infty > 0, \ \widetilde{Z}_\infty > 0\} \tag{1.4}$$

$$\{S < \infty \text{ or } H_\infty = \infty\} = \{Z_\infty = 0\} \cup \{\widetilde{Z}_\infty = 0\} \tag{1.5}$$

But $P(Z_\infty = 0) = \widetilde{P}(\widetilde{Z}_\infty = 0) = 0$. The conclusions follow from (1.4) and (1.5).

1.3. **Corollary.** ([2],[3]) $\widetilde{P} \ll P$ iff

(i) $\widetilde{P}_0 \ll P_0$,

(ii) $\widetilde{P}(H_\infty < \infty) = 1$,

(iii) $\widetilde{P}(1_{\lambda=0} * \nu_\infty = 0) = 1$.

Proof. Since $\widetilde{P} \ll P$ iff $\widetilde{P}(N) = 0$, but

$$\widetilde{P}(N) = \widetilde{P}(T < \infty \text{ or } H_\infty = \infty)$$

and

$$\{T < \infty\} \cup \{ H_\infty = \infty \} = \{ T=0 \} \cup \{ 0<T<\infty,\ H_{T-} < \infty \} \cup \{ H_\infty = \infty \}.$$

Also notice that

$$\{ T = 0 \} = \{ Z_0 = 0 \},$$

$$\{ 0<T<\infty,\ H_{T-}<\infty \} = \{ 0<T<\infty,\ Z_{T-}>0 \} = \{ 1_{\lambda=0} * \mu_\infty > 0 \}.$$

Hence

$$\{ T < \infty \} \cup \{ H_\infty = \infty \} = \{ Z_0 = 0 \} \cup \{ 1_{\lambda=0} * \mu_\infty > 0 \} \cup \{ H_\infty = \infty \},$$

but

$$\widetilde{P}(1_{\lambda=0} * \mu_\infty > 0) = 0 \iff \widetilde{E}(1_{\lambda=0} * \mu_\infty) = 0 \iff \widetilde{E}(1_{\lambda=0} \widetilde{\lambda} * \nu)_\infty = 0 \iff$$

$$\iff E^Q(1_{\lambda=0} * \nu_\infty) = 0 \iff \widetilde{P}(1_{\lambda=0} * \nu_\infty > 0) = 0,$$

therefore the Corollary holds.

1.4. Remark. The condition (iii) in Corollary 1.3 is equivalent to the following

(iii') $\forall A \in \widetilde{\underline{\underline{P}}},\ 1_A * \mu_\infty^{p,P} = 0$ a.s. $\widetilde{P} \Rightarrow 1_A * \mu_\infty^{p,\widetilde{P}} = 0$ a.s. $\widetilde{P}$.

Proof. (iii) => (iii'). If $A \in \widetilde{\underline{\underline{P}}}$ and $1_A \lambda * \nu_\infty = 1_A * \mu_\infty^{p,P} = 0$ a.s. $\widetilde{P}$, then $1_{A\{\lambda>0\}} * \nu_\infty = 0$ a.s. $\widetilde{P}$. By (iii)

$$1_A * \mu_\infty^{p,\widetilde{P}} = 1_A \widetilde{\lambda} * \nu_\infty = 1_{A\{\lambda>0\}} \widetilde{\lambda} * \nu_\infty = [\widetilde{\lambda} * (1_{A\{\lambda>0\}} * \nu)]_\infty = 0 \text{ a.s.} \widetilde{P}.$$

(iii') => (iii). Obviously, we have $\lambda 1_{\lambda=0} * \nu_\infty = 0$. By (iii')

$$1_{\lambda=0} \widetilde{\lambda} * \nu_\infty = 0 \text{ a.s. } \widetilde{P}.$$

$$1_{\lambda=0}(Z_- \lambda + \widetilde{Z}_- \widetilde{\lambda}) * \nu_\infty = 0 \text{ a.s. } \widetilde{P} \tag{1.6}$$

Since $Z_- \lambda + \widetilde{Z}_- \widetilde{\lambda} = 2$, from (1.6) we get

$$1_{\lambda=0} * \nu_\infty = 0 \text{ a.s. } \widetilde{P}.$$

1.5. Corollary. $\widetilde{P} \perp P$ iff

$$\widetilde{P}(Z_0 = 0 \text{ or } H_\infty = \infty \text{ or } 1_{\lambda=0} * \mu_\infty > 0) = 1.$$

Proof. It is sufficient to notice that $\widetilde{P} \perp P$ iff $\widetilde{P}(N) = 1$ and similarly to the proof of Corollary 1.3. we have $\widetilde{P}(N) = \widetilde{P}(Z_0 = 0 \text{ or } H_\infty = 0 \text{ or } 1_{\lambda=0} * \mu > 0)$, hence the Corollary holds.

1.6. Application to semimartingales. Suppose that $\bar{Q}$ is a probability measures on $\underline{\underline{F}}$

such that $P \ll \bar{Q}$ and $\tilde{P} \ll \bar{Q}$. ($\bar{Q}$ is not necessarily Q, but $Q \ll \bar{Q}$. This is the difference from the assumption in [5].) Suppose $X = (X_t)_{t \geq 0}$ is a semimartingale under $\bar{Q}$ (and so under P and $\tilde{P}$). The predictable characteristics of X under P, $\tilde{P}$ and $\bar{Q}$ are (B,C,ν) , $(\tilde{B},\tilde{C},\tilde{\nu})$ and $(\bar{B},\bar{C},\bar{\nu})$ respectively, and

$$\nu = \rho . \bar{\nu}, \qquad \tilde{\nu} = \tilde{\rho} . \bar{\nu} \qquad \rho, \tilde{\rho} \in \tilde{\underline{\underline{P}}}^{+}$$

Set $a = (a_t)$, $\tilde{a} = (\tilde{a}_t)$ and $\bar{a} = (\bar{a}_t)$:

$$a_t = \nu([t] \times \mathbb{R}), \qquad \tilde{a}_t = \tilde{\nu}([t] \times \mathbb{R}), \qquad \bar{a}_t = \bar{\nu}([t] \times \mathbb{R}).$$

Define

$$\tau = \inf \{ t: \mathrm{Var}_t(B-\tilde{B}) + |x| 1_{|x| \leq 1} * \mathrm{Var}(\nu - \tilde{\nu})_t = \infty \}$$

$$A_t = \begin{cases} 0, & \tau = 0, \\ \tilde{B}_t - B_t - x 1_{|x| \leq 1} * (\tilde{\nu} - \nu)_t, & t < \tau, \tau > 0, \\ +\infty, & t \geq \tau, \tau > 0. \end{cases}$$

$$K = \frac{dA}{d\bar{C}} \in \underline{\underline{P}}.$$

(If on $[0, t]$ A is not absolutely continuous with respect to $\bar{C}$, $K_t = +\infty$.)

$$N = \{ Z_o \tilde{Z}_o = 0 \} \cup \{ C \neq \tilde{C} \} \cup \{ \tau = 0 \} \cup$$
$$\{ K^2 . \bar{C}_\infty + (\sqrt{\rho} - \sqrt{\tilde{\rho}})^2 * \bar{\nu}_\infty + S_\infty(\sqrt{1 - a} - \sqrt{1 - \tilde{a}})^2 = \infty \} \cup$$
$$\{ (1_{\rho\tilde{\rho}=0} * \mu_X)_\infty > 0 \} \cup \{ S_\infty(1_{\Delta X=0, a=1 \text{ or } \tilde{a}=1}) > 0 \},$$

where μ_X is the jump measure of X.

Suppose that under $\bar{Q}$ the derivative processes of P and $\tilde{P}$ with respect to $\bar{Q}$ (still denoted by Z and $\tilde{Z}$) have predictablerepresentation:

$$Z = L.X^c + W*(\mu_X - \bar{\nu}), \qquad \tilde{Z} = \tilde{L}.X^c + \tilde{W}*(\mu_X - \bar{\nu}) \qquad (1.7)$$

where $L, \tilde{L} \in \underline{\underline{P}}$, $W, \tilde{W} \in \tilde{\underline{\underline{P}}}$. Applying Theorem 1.2, we have

(1) $\tilde{P} \perp P$ on N,

(2) $\tilde{P} \sim P$ on N^c.

The conclusion (1) about singularity needn't the assumption of predictable representation (1.7). But the conclusion (2) need it in order to represent the Hellinger process as

$$H = 1_{\Gamma \cap \tilde{\Gamma}} \cdot \{ \tfrac{1}{8} K^2 . \bar{C} + \tfrac{1}{2}(\sqrt{\rho} - \sqrt{\tilde{\rho}})^2 * \bar{\nu} + \tfrac{1}{2} S(\sqrt{1 - a} - \sqrt{1 - \tilde{a}})^2 .$$

2. Contiguity

2.1. $(\tilde{P}^n)$ is contiguous to (P^n), if $\forall A^n \in \underline{\underline{F}}^n$

$$P^n(A^n) \to 0 \;\Rightarrow\; \tilde{P}^n(A^n) \to 0$$

and denoted by $(\tilde{P}^n) \lhd (P^n)$. The main result on contiguity is the following ([2],[3])

2.2. <u>**Theorem.**</u> $(\tilde{P}^n) \lhd (P^n)$ iff

(i) $(\tilde{P}^n_o) \lhd (P^n_o)$

(ii) $\lim\limits_{N\to\infty} \overline{\lim\limits_{n\to\infty}}\, \tilde{P}^n(H^n_\infty \geqslant N) = 0,$

(iii) $\forall \eta>0,\ \lim\limits_{N\to\infty} \overline{\lim\limits_{n\to\infty}}\, \tilde{P}^n(i^n_\infty(N) \geqslant \eta) = 0,$

where P^n_o $(\tilde{P}^n_0)$ is the restriction of P^n $(\tilde{P}^n)$ on $\underline{\underline{F}}_0$, and

$$i^n(N) =(\tilde{\lambda}^n 1_{\{N\lambda^n < \tilde{\lambda}^n\}} * \nu^n, \quad N \geqslant 2.$$

The proof of necessity, given in [2], is already very simple, needn't improving further. We'll give another proof for sufficiency. Our proof is based on the following lemma, as in [3]. But the procedure after that is greatly simpler than that in [3].

3.3. <u>**Lemma.**</u> ([3]) $(\tilde{P}^n) \lhd (P^n)$ iff

$$\lim_{N\to\infty} \overline{\lim_{n\to\infty}}\, \tilde{P}^n\Big(\lim_{k\to\infty} (\tilde{Z}/Z)^{*}_{S^n_k} \geqslant N \Big) = 0 \tag{2.1}$$

(Denote by Z^* the supremum process of Z: $Z^*_t = \sup\limits_{s\leqslant t}|Z_s|$.)

2.4. <u>The proof of sufficiency.</u> From exponential formula, on $[\![0, S_k]\!]$,

$$\begin{aligned}
\tilde{Z}/Z &= \tilde{Z}_o/Z_o \exp\Big\{(1/\tilde{Z}_-).\tilde{Z} - \tfrac{1}{2}(1/\tilde{Z}^2_-).\langle\tilde{Z}^c\rangle + S(\log(1+\Delta\tilde{Z}/\tilde{Z}_-) - \Delta\tilde{Z}/\tilde{Z}_-) \\
&\qquad -(1/Z_-).Z + \tfrac{1}{2}(1/Z^2_-).\langle Z^c\rangle - S(\log(1 + \Delta Z/Z_-) - \Delta Z/Z_-)\Big\} \\
&= \tilde{Z}_o/Z_o \exp\Big\{ -(1/Z_- + 1/\tilde{Z}_-).Z^c - \tfrac{1}{2}(1/\tilde{Z}^2_- - 1/Z^2_-).\langle Z^c\rangle + \\
&\qquad + (\tilde{\lambda}-\lambda)*(\mu-\nu) + (\log\tilde{\lambda} - (\tilde{\lambda} - 1))*\mu - (\log\lambda - (\lambda - 1))*\mu\Big\} \\
&= \tilde{Z}_o/Z_o \exp\Big\{ -(1/Z_- + 1/\tilde{Z}_-).Z^{c,\tilde{P}} + \tfrac{1}{2}(1/Z_- + 1/\tilde{Z}_-)^2.\langle Z^c\rangle + \\
&\qquad + (\tilde{\lambda}-\lambda)*(\mu-\nu) + (\log\tfrac{\tilde{\lambda}}{\lambda} -(\tilde{\lambda}-\lambda))*\mu \Big\} \\
&= Z_o/Z_o \exp\{ A + B \}
\end{aligned} \tag{2.2}$$

where $Z^{c,\tilde{P}} = Z^c - 1/\tilde{Z}_-.\langle Z^c,\tilde{Z}^c\rangle = Z^c + 1/\tilde{Z}_-.\langle Z^c\rangle$ is the continuous local martingale part of Z under $\tilde{P}$. Set $\rho = \lambda/\tilde{\lambda}$, $\tilde{\nu} = \mu^{p,\tilde{P}}$, and $0 < b < 1$ is a constant. Then

$$A = -(1/Z_- + 1/\tilde{Z}_-).Z^{c,\tilde{P}} + \tfrac{1}{2}(1/Z_- + 1/\tilde{Z}_-)^2.\langle Z^c\rangle$$

$$\begin{aligned}
B &= (\tilde{\lambda}-\lambda)*(\mu-\nu) + (\log\tfrac{\tilde{\lambda}}{\lambda} - (\tilde{\lambda}-\lambda))*\mu \\
&= 1_{|\rho-1|>b} \log\tfrac{1}{\rho} *\mu + 1_{|\rho-1|>b}(\rho -1)*\tilde{\nu} \\
&\quad + 1_{|\rho-1|\leqslant b} \log\tfrac{1}{\rho} *(\mu-\tilde{\nu}) + 1_{|\rho-1|\leqslant b}(\log\tfrac{1}{\rho} - (1-\rho))*\tilde{\nu} \\
&= B^1 + B^2 + B^3 + B^4
\end{aligned} \tag{2.3}$$

In the sequel, we'll discuss under $\widetilde{P}$, and estimate (2.2) term by term in order to get (2.1).

1^o since $(\widetilde{P}^n_0) \lhd (P^n_o)$, we have

$$\lim_{N\to\infty} \overline{\lim_{n\to\infty}} \widetilde{P}^n(\widetilde{Z}^n_o/Z^n_o \geqslant N) = 0 \tag{2.4}$$

2^o $\langle Z^{c,\widetilde{P}}\rangle = \langle Z^c\rangle$, and by Lenglart's inequality

$$\widetilde{P}(\ ((1/Z_- + 1/\widetilde{Z}_-).Z^{c,\widetilde{P}})^*_{S_k} \geqslant N\ ) \leqslant L/N^2 + P(8H_\infty \geqslant L\ )$$

$$\widetilde{P}(A^*_{S_k} \geqslant 2N) \leqslant L/N^2 + \widetilde{P}(8H_\infty \geqslant L) + \widetilde{P}(4H_\infty \geqslant N)$$

Let $k\to\infty$, $n\to\infty$, $N\to\infty$, $L\to\infty$ successively, we get

$$\lim_{N\to\infty} \overline{\lim_{n\to\infty}} \widetilde{P}^n(\ \lim_{k\to\infty} \ (A^n)^*_{S^n_k} \geqslant 2N) = 0 \tag{2.5}$$

3^o Using $|\log(1+x)| < |x|/(1-|x|)$ for $|x| < 1$, we have

$$\langle 1_{|\rho-1|\leqslant b}\ \log\frac{1}{\rho} * (\mu - \widetilde{\nu})\rangle \leqslant 1_{|\rho-1|\leqslant b}\ \log^2\rho * \widetilde{\nu}$$
$$\leqslant 1_{|\rho-1|\leqslant b} \frac{(\rho-1)^2}{(b-1)^2} * \widetilde{\nu} \leqslant (\frac{1+\sqrt{1+b}}{1-b})^2(\sqrt{\rho} - 1\ )^2 * \widetilde{\nu} \leqslant C_b H$$

where C_b is a constant, dependent on b only. By Lenglart's inequality

$$\widetilde{P}(\ (B^3)^*_{S_k} \geqslant N) \leqslant L/N^2 + P(C_b H_\infty \geqslant L)$$

Set $k\to\infty$, $n\to\infty$, $N\to\infty$, $L\to\infty$ successively, we get

$$\lim_{N\to\infty} \overline{\lim_{n\to\infty}} \ \widetilde{P}^n(\ \lim_{k\to\infty} \ (B^{n,3})^*_{S^n_k} \geqslant N) = 0 \tag{2.6}$$

4^oUsing $|\log(1 + x) - x| < x^2/2(1 - |x|)$ for $|x|<1$, we have

$$|B^4| \leqslant 1_{|\rho-1|<b}|-\log\rho + \rho - 1| * \widetilde{\nu}$$
$$\leqslant 1_{|\rho-1|\leqslant b}\frac{(\rho- 1)^2}{2(1-b)} * \widetilde{\nu} \leqslant \frac{(1 + \sqrt{1 + b})^2}{2(1- b)}(\sqrt{\rho} - 1)^2 * \widetilde{\nu} \leqslant C_b H$$

$$\widetilde{P}(\ (B^4)^*_{S_k} \geqslant N) \leqslant \widetilde{P}(\ C_b H_\infty \geqslant N)$$

Hence

$$\lim_{N\to\infty} \overline{\lim_{n\to\infty}} \ \widetilde{P}^n(\ \lim_{k\to\infty} (B^{n,4})^*_{S^n_k} \geqslant N) = 0 \tag{2.7}$$

5^o

$$|B^2| \leqslant \ 1_{|\rho-1|>b}\left|\frac{\sqrt{\rho} + 1}{\sqrt{\rho} - 1}\right|(\sqrt{\rho} - 1)^2 * \widetilde{\nu}$$
$$\leqslant (\ 1 + 2/(-1 + \sqrt{1 + b}))(\sqrt{\rho} - 1)^2 * \widetilde{\nu} \leqslant C_b H \tag{2.8}$$
$$\widetilde{P}(\ (B^2)^*_{S_k} \geqslant N\ ) \leqslant \widetilde{P}(\ C_b H_\infty \geqslant N\ )$$

Hence

$$\lim_{N\to\infty} \overline{\lim_{n\to\infty}} \tilde{P}^n\left(\lim_{k\to\infty} (B^{n,2})^{*}_{S^n_k} \geqslant N \right) = 0 \qquad (2.9)$$

6^o Take $0 < \delta < 1 - b$

$$B^1 \leqslant 1_{\delta<\rho<1-b}(\log^+ \tfrac{1}{\rho})*\mu + 1_{\rho\leqslant\delta}\log^+\tfrac{1}{\rho} * \mu$$

$$\leqslant (\log \tfrac{1}{\delta}\, 1_{\rho<1-b})*\mu + (1_{\rho\leqslant\delta}\, \log^+ \tfrac{1}{\rho})*\mu$$

$$\tilde{P}\left((e^{B^1})^{*}_{S_k} \geqslant N \right) \leqslant \tilde{P}\left((1_{(\rho<1-b)}*\mu)_{S_k} \geqslant \frac{\log N}{\log 1/\delta}\right) + \tilde{P}\left((1_{\rho\leqslant\delta}*\mu)_{S_k}>0\right) \qquad (2.10)$$

$$1_{\{\rho<1-b\}}*\tilde{\nu} \leqslant 1_{\{|\rho-1|>b\}}*\tilde{\nu} \leqslant \frac{1}{b}\, 1_{|\rho-1|>b}|\rho-1| *\tilde{\nu} \leqslant C_bH$$

By Lenglart's inequality

$$\tilde{P}\left((1_{\rho<1-b}*\mu)_{S_k} \geqslant \frac{\log N}{\log 1/\delta} \right) \leqslant L\, \frac{\log 1/\delta}{\log N} + \tilde{P}(C_bH_\infty \geqslant L) \qquad (2.11)$$

On the other hand,

$$\begin{aligned}1_{\rho\leqslant\delta} * \tilde{\nu} &\leqslant 1_{K\lambda<\tilde{\lambda}} * \tilde{\nu} + 1_{0<\tilde{\lambda}\leqslant K\lambda,\, \rho\leqslant\delta} * \tilde{\nu} \\ &\leqslant i(K) + K\, 1_{\tilde{\lambda}>0,\, \rho\leqslant\delta}\,\rho * \tilde{\nu} \\ &\leqslant i(K) + K\delta/(1-\sqrt{\delta})^2(\sqrt{\rho} - 1)^2 * \tilde{\nu} \\ &\leqslant i(K) + 2K\delta/(1 -\sqrt{\delta})^2H \qquad (2.12)\end{aligned}$$

Notice that μ is integer-valued, again by Lenglart's inequality

$$\begin{aligned}\tilde{P}\left((1_{\rho\leqslant\delta} * \mu)_{S_k} > 0 \right) &= \tilde{P}\left((1_{\rho\leqslant\delta} * \mu)_{S_k} \geqslant 1 \right) \\ &\leqslant \eta + \tilde{P}\left((1_{\rho\leqslant\delta} * \tilde{\nu})_{S_k} \geqslant \eta\right) \qquad (2.13)\end{aligned}$$

From (2.10) - (2.13) we get

$$\begin{aligned}\tilde{P}\left((e^{B^1})^{*}_{S_k} \geqslant N \right) \leqslant{} & L \log\tfrac{1}{\delta}/\log N + \tilde{P}(C_bH_\infty \geqslant L) + \eta + \tilde{P}(i_\infty(K) > \tfrac{1}{2}\eta) \\ & + \tilde{P}(H_\infty \geqslant \eta(1 - \sqrt{\delta})^2/4K\delta)\end{aligned}$$

Set $k\to\infty$, $n\to\infty$, $N\to\infty$, $L\to\infty$, $\delta\to 0$, $K\to\infty$, $\eta\to 0$ successively, we get

$$\lim_{N\to\infty} \overline{\lim_{n\to\infty}} \tilde{P}^n\left(\lim_{k\to\infty} (e^{B^{n,1}})^{*}_{S^n_k} \geqslant N \right) = 0 \qquad (2.14)$$

then (2.1) follows from (2.2) - (2.7),(2.9) and (2.14).

2.5. Remark. In Theorem 2.2 the condition (iii) can be substituted by

(iii') $\forall\, A^n \in \underline{\underline{F}}^n$

$$1_{A^n}*(\mu^n)^{p,P}_\infty \to 0 \text{ in } (\tilde{P}^n) \Rightarrow 1_{A^n}*(\mu^n)^{p,\tilde{P}}_\infty \to 0 \text{ in } (\tilde{P}^n).$$

Proof. (iii) => (iii')

$$\begin{aligned}\tilde{\lambda}^n 1_{A^n} * \nu^n_\infty &\leqslant 1_{N\lambda^n<\tilde{\lambda}^n}\, \tilde{\lambda}^n * \nu^n_\infty + 1_{A^n\{N\lambda^n\geqslant\tilde{\lambda}^n\}}\, \tilde{\lambda}^n * \nu^n \\ &\leqslant i^n_\infty(N) + N\, 1_{A^n} \lambda^n * \nu^n_\infty .\end{aligned}$$

(iii') + (ii) => (iii).

$$1_{N\lambda^n<\tilde{\lambda}^n}\lambda^n * \nu^n_\infty \leqslant (\sqrt{N}-1)^{-2} H^n_\infty ,$$

Hence for each sequence $N_n \to \infty$, take $A^n = \{N_n\lambda^n < \tilde{\lambda}^n\}$, we have $i^n_\infty(N_n) \to 0$ in $(\tilde{P}^n)$, therefore (iii) holds.

2.6. Application to semimartingales. We make the assumptions, as in 1.6. Applying Theorem 2.2, we have the conclusion (see [5]):

$(\tilde{P}^n) \lhd (P^n)$ iff

(i) $(\tilde{P}^n_o) \lhd (P^n_o)$,

(ii) $\lim\limits_{N\to\infty} \overline{\lim\limits_{n\to\infty}}\ \tilde{P}^n((K^n)^2.\bar{C}^n_\infty + (\sqrt{\rho^n}-\sqrt{\tilde{\rho}^n})^2 * \bar{\nu}^n_\infty + S_\infty(\sqrt{1-a^n}-\sqrt{1-\tilde{a}^n})^2 \geqslant N) = 0$,

(iii) $\forall \eta>0$,

$$\lim_{N\to\infty} \overline{\lim_{n\to\infty}}\ \tilde{P}^n(1_{N\rho^n<\tilde{\rho}^n} * \tilde{\nu}^n_\infty + S_\infty((1-\tilde{a}^n)1_{N(1-a^n)<1-\tilde{a}^n}) \geqslant \eta) = 0.$$

In fact, we have

$$i(N) = 1_{\Gamma\cap\tilde{\Gamma}}\cdot\{1_{N\rho<\tilde{\rho}} * \tilde{\nu} + S((1-\tilde{a})1_{N(1-a)<(1-\tilde{a})})\}.$$

Similar to Remark 2.5, (iii) can be substituted by following

(iii') (a) $\forall A^n \in \underline{\underline{\tilde{P}}}^n$

$$1_{A^n} * \nu^n_\infty \to 0 \text{ in } (\tilde{P}^n) \Rightarrow 1_{A^n} * \tilde{\nu}^n_\infty \to 0 \text{ in } (\tilde{P}^n),$$

(b) $\forall A^n \in \underline{\underline{P}}^n$

$$(1_{A^n}.S((1-a^n)1_{[\bar{a}^n>0]}))_\infty \to 0 \text{ in } (\tilde{P}^n)$$

$$\Rightarrow (1_{A^n}.S((1-\tilde{a}^n)1_{[\bar{a}^n>0]}))_\infty \to 0 \text{ in} (\tilde{P}^n).$$

3. Convergence in Variation

3.1. Lemma. The following statements are equivalent:

(1) $\|P^n - \tilde{P}^n\| \to 0$.

(2) $(Z^n - 1)^*_\infty \to 0$ in (P^n).

(3) $(Y^n - 1)^*_\infty \to 0$ in(P^n), where $Y^n = \sqrt{Z^n\tilde{Z}^n}$.

Proof. Since $\|P^n - \tilde{P}^n\| = E^Q|Z^n_\infty - \tilde{Z}^n_\infty| = 2E^Q|Z^n_\infty - 1|$, $|Z^n_\infty - 1| \leqslant 1$, so

$$\|P^n - \tilde{P}^n\| \to 0 \iff |Z^n_\infty - 1| \to 0 \text{ in } (Q^n).$$

(1) => (2) By maximal inequality of martingales, for $\varepsilon>0$

$$Q^n((Z^n-1)^*_\infty \geqslant \varepsilon) \leqslant \frac{1}{\varepsilon} E^{Q^n}|Z^n_\infty - 1|$$

Hence, $(Z^n - 1)^*_\infty \to 0$ in (Q^n) and (P^n).

(2) => (1). Obviously, $Z^n_\infty - 1 \to 0$ in (P^n). For given $\varepsilon > 0$, and $0 < \delta < \varepsilon < 1$,

$$Q^n(|Z^n_\infty - 1| \leqslant \varepsilon) \geqslant \int_{|Z^n_\infty - 1| \leqslant \delta} 1/Z^n_\infty \, dP^n \geqslant 1/(1+\delta)\, P^n(|Z^n_\infty - 1| \leqslant \delta)$$

Set $n \to \infty$, $\delta \to 0$ successively, we get $Z^n_\infty - 1 \to 0$ in (Q^n).

Note that $1 - (Y^n)^2 = (1 - Z^n)^2$ and $0 \leqslant Y^n \leqslant 1$, we have

$$(1 - Y^n)^* \leqslant (1 - Z^n)^{*2} \leqslant 2(1 - Y^n)^* ,$$

(2) <=> (3) follows.

3.2 Theorem([4]). The following statements are equivalent:

(1) $\|P^n - \tilde{P}^n\| \to 0$.

(2) (a) $\|P^n_0 - \tilde{P}^n_0\| \to 0$,

(b) $H^n_\infty - 1 \to 0$ in (Q^n).

(3) (a) $\|P^n_0 - P^n_0\| \to 0$,

(b) $H^n_\infty - 1 \to 0$ in (P^n).

Proof. (1) => (2). (a) is trivial. Suppose that the Doob-Meyer decomposition of $Y^n = \sqrt{Z^n \tilde{Z}^n}$ is

$$Y^n = Y^n_0 + M^n - A^n \tag{3.1}$$

where M^n is a martingale with $M^n_0 = 0$, $A^n = Y^n_- . H^n$. By Lemma 3.1, $(Y^n - Y^n_0)^*_\infty \to 0$ in (Q^n). A^n is dominated by $(Y^n - Y^n_0)^*$. $\Delta(Y^n - Y^n_0)^* \leqslant |\Delta Y^n| \leqslant 1$. By Lenglart's inequality, we have $A^n_\infty \to 0$ in (Q^n). On $\{\inf_{t \geqslant 0} Y^n_t \geqslant \frac{1}{2}\}$,

$$H^n_\infty = (1/Y^n_- - 1).A^n_\infty + A^n_\infty \leqslant 2(Y^n - 1)^*_\infty A^n_\infty + A^n_\infty$$

On $\{\inf_{t > 0} Y^n_t < \frac{1}{2}\}$, $(Y^n - 1)^*_\infty \geqslant \frac{1}{2}$. Therefore, $\forall \varepsilon > 0$

$$Q^n(H^n_\infty \geqslant \varepsilon) \leqslant Q^n((Y^n - 1)^*_\infty \geqslant \tfrac{1}{2}) + Q^n((2(Y^n - 1)^*_\infty + 1)A^*_\infty \geqslant \varepsilon)$$

Hence, $H^n_\infty \to 0$ in (Q^n). (2) => (3) is trivial.

(3) => (1). At first, observe that

$$\begin{aligned} 2H_\infty &\geqslant (\sqrt{\lambda} - \sqrt{\tilde{\lambda}})^2 * \nu_\infty \geqslant \lambda(\sqrt{\tilde{\lambda}/\lambda} - 1)^2 \, 1_{N\tilde{\lambda}<\lambda} * \nu_\infty \\ &\geqslant \lambda(\sqrt{1/N} - 1)^2 \, 1_{N\tilde{\lambda}<\lambda} * \nu_\infty \end{aligned}$$

Hence, $1_{(N\tilde{\lambda}^n < \lambda^n)} \lambda^n * \nu^n_\infty \to 0$ in (P^n). Applying Theorem 2.2, we have $(P^n) \lhd (\tilde{P}^n)$. Since $P^n(T^n_k < \infty) \leqslant 1/k$. $\tilde{P}^n(\tilde{T}^n_k \leqslant \infty) \leqslant 1/k$,

$$\lim_{k \to \infty} \overline{\lim_{n \to \infty}} \, P^n(S^n_k < \infty) = 0 \tag{3.2}$$

Now define $L = 1/Y_- . Y = 1/Y_- . M - H$. Using Ito's formula. on $\tilde{\Gamma} \cap \Gamma$ we get

$$(1/Y\_).M = \tfrac{1}{2}((1/Z\_).Z + (1/\tilde{Z}\_).\tilde{Z}) - \tfrac{1}{2}(\sqrt{\lambda} - \sqrt{\tilde{\lambda}})^2 * (\mu - \nu)$$

$$= \tfrac{1}{2}(1/Z\_ - 1/\tilde{Z}\_).Z^{c,P} + (\sqrt{\lambda\tilde{\lambda}} - 1) * (\mu - \mu^{p,P}) +$$

$$+ \tfrac{1}{2}(1/Z\_ - 1/\tilde{Z}\_)(1/Z\_).\langle Z^c \rangle + (\lambda - 1)(\sqrt{\lambda\tilde{\lambda}} - 1) * \nu \qquad (3.3)$$

where $Z^{c,P}$ is the continuous local martingale part of Z under P. It is easy to see, on $\llbracket 0 , S_k \rrbracket$,

$$|(1/Z\_ - 1/\tilde{Z}\_)\ 1/Z\_|.\langle Z^c \rangle \leqslant (1/Z\_ + 1/\tilde{Z}\_)^2.\langle Z^c \rangle \leqslant 8H \qquad (3.4)$$

$$|\sqrt{\lambda\tilde{\lambda}} - 1| \leqslant |\sqrt{\lambda} - \sqrt{\tilde{\lambda}}|$$

$$|(\lambda - 1)(\sqrt{\lambda\tilde{\lambda}} - 1)| \leqslant (\sqrt{\lambda} + 1)|\sqrt{\lambda} - 1||\sqrt{\lambda} - \sqrt{\tilde{\lambda}}| \leqslant (\sqrt{\lambda} + 1)(\sqrt{\tilde{\lambda}} - \sqrt{\lambda})^2$$

$$|(\lambda - 1)(\sqrt{\lambda\tilde{\lambda}} - 1) * \nu| \leqslant (\sqrt{\lambda} + 1)(\sqrt{\lambda} - \sqrt{\tilde{\lambda}})^2 * \nu \leqslant 2(\sqrt{1+2k} + 1)H \qquad (3.5)$$

Under P we have

$$\langle \tfrac{1}{2}(1/Z\_ - 1/\tilde{Z}\_).Z^{c,P} + (\sqrt{\lambda\tilde{\lambda}} - 1) * (\mu - \mu^{p,P}) \rangle \leqslant$$

$$\leqslant \tfrac{1}{4}(1/Z\_ + 1/\tilde{Z}\_)^2.\langle Z^c \rangle + (2k+ 1)(\sqrt{\lambda} - \sqrt{\tilde{\lambda}})^2 * \nu < (2 + 4k)H \qquad (3.6)$$

From (3.3) -(3.6) and Lenglart's inequality, we obtain

$$(L^n)^*_{S^n_k} \to 0 \quad \text{in } (P^n) \qquad (3.7)$$

By exponential formula, on $\llbracket 0,S_k \rrbracket$

$$Y = Y_o\, \mathcal{E}(L) = Y_o \exp\{L + S(\log(1 + \Delta L) - \Delta L)\}$$

Note that $0 \leqslant x - \log(1+x) \leqslant x^2$ for $|x| < \tfrac{1}{2}$, and $\Delta L = \sqrt{(1+\Delta Z/Z\_)(1+\Delta\tilde{Z}/\tilde{Z}\_)} - 1$ we have

$$0 \leqslant \sum_{|\Delta L| \leqslant \frac{1}{2}} (\Delta L - \log(1 + \Delta L)) \leqslant \sum_{|\Delta L| \leqslant \frac{1}{2}} (\Delta L)^2$$

$$\leqslant (\sqrt{\lambda\tilde{\lambda}} - 1)^2 * \mu_\infty \leqslant (\sqrt{\lambda} - \sqrt{\tilde{\lambda}})^2 * \mu_\infty$$

Since $(\sqrt{\lambda} - \sqrt{\tilde{\lambda}})^2 * \mu^{p,P} \leqslant 2(1+2k)H$, using Lenglart's inequality again, we obtain

$$(S((\Delta L^n - \log(1 + \Delta L^n))\ 1_{|\Delta L^n| \leqslant \frac{1}{2}})_{S^n_k} \to 0 \quad \text{in}(P^n) \qquad (3.8)$$

$\forall \varepsilon > 0$,

$$\{(S(|\log(1 + \Delta L) - \Delta L|))_{S_k} \geqslant \varepsilon\} \subset$$

$$\subset \{(\Delta L)^*_{S_k} > \tfrac{1}{2}\} \cup \{(S((\Delta L - \log(1 + \Delta L))1_{|\Delta L| \leqslant \frac{1}{2}}))_{S_k} \geqslant \varepsilon\} \qquad (3.9)$$

According to (3.7), $(\Delta L^n)^*_{S^n_k} \to 0$ in (P^n), and from (3.8), (3.9) we get

$$(L^n)^*_{S^n_k} + (S(|\log(1 + L^n) - L^n|))_{S^n_k} \to 0 \qquad \text{in } (P^n) \qquad (3.10)$$

By (i), $Y^n_o - 1 \to 0$ in (P^n), now

$$Y^n - 1 = Y^n_o - 1 + Y^n_o\{\exp(L^n + S(\log(1 + \Delta L^n) - \Delta L^n)) - 1\}$$

and from (3.10) we have

$$(Y^n - 1)^*_{S^n_k} \to 0 \quad \text{in } (P^n) \tag{3.11}$$

For given $\varepsilon > 0$,

$$P^n((Y^n - 1)^*_\infty \geqslant \varepsilon) \leqslant P^n(S^n_K < \infty) + P^n((Y^n - 1)^*_{S^n_k} \geqslant \varepsilon)$$

Set $n \to \infty$ and $k \to \infty$ successively, from (3.11) and (3.2) we know

$$(Y^n - 1)^*_\infty \to 0 \quad \text{in } (P^n)$$

At last, $\|P^n - \tilde{P}^n\| \to 0$ follows from Lemma 3.1.

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