SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

HIROSHI KANEKO
SHINTARO NAKAO

A note on approximation for stochastic differential equations

Séminaire de probabilités (Strasbourg), tome 22 (1988), p. 155-162
<http://www.numdam.org/item?id=SPS_1988__22__155_0>

© Springer-Verlag, Berlin Heidelberg New York, 1988, tous droits réservés.

L'accès aux archives du séminaire de probabilités (Strasbourg) (http://portail.mathdoc.fr/SemProba/) implique l'accord avec les conditions générales d'utilisation (http://www.numdam.org/conditions). Toute utilisation commerciale ou impression systématique est constitutive d'une infraction pénale. Toute copie ou impression de ce fichier doit contenir la présente mention de copyright.

NUMDAM
Article numérisé dans le cadre du programme
Numérisation de documents anciens mathématiques
http://www.numdam.org/

A NOTE ON APPROXIMATION FOR STOCHASTIC DIFFERENTIAL EQUATIONS

By H. Kaneko and S. Nakao

1. Introduction

Let $\sigma(t,x)$ be an $M^{d\times r}$-valued measurable function and $b(t,x)$ be an R^d-valued measurable function which are defined on $[0,\infty)\times R^d$, here $M^{d\times r}$ is the set of all $d\times r$ matrices. Consider the following stochastic differential equation

(1) $dX(t) = \sigma(t,X(t))dB(t) + b(t,X(t))dt,$

where $B(t)$ is a given r-dimensional $(\mathcal{F}_t)$-Brownian motion on a usual filtered probability space $(\Omega,\mathcal{F},P,(\mathcal{F}_t))$. Throughout this paper we assume that the equation (1) has a pathwise unique solution $X(x,t)$ for each initial value $x \in R^d$ with respect to $B(t)$.

We shall consider two problems on the approximation for the solution of the stochastic differential equation (1). The first one is an approximation by the coefficients. This problem was treated by Kawabata-Yamada [2] and Le Gall [4].

Consider a sequence of stochastic differential equations with measurable coefficients

(2) $dX(t) = \sigma_n(t,X(t))dB(t) + b_n(t,X(t))dt,\quad n = 1,2,\cdots$

Suppose that for an arbitrary initial value $x \in R^d$ each equation possesses a solution $X_n(x,t)$ with respect to the Brownian motion $B(t)$, then we can show the folowing theorem:

Theorem A. Suppose that $\sigma_n(t,x)$ and $b_n(t,x)$ $(n = 1,2,\cdots)$ are continuous. Further suppose that for each $T \geqq 0$, there exists $L_T > 0$ such that

$$\sup_n \sup_{t\leqq T} (\|\sigma_n(t,x)\| + |b_n(t,x)|) \leqq L_T(1 + |x|),$$

and that

$$\lim_{n\to\infty} \sup_{t\leqq T} \sup_{x\in K} (\|\sigma_n(t,x) - \sigma(t,x)\| + |b_n(t,x) - b(t,x)|) = 0\ ,$$

for each $T \geqq 0$ and compact subset K in R^d. If the pathwise uniqueness of solutions for (1) holds, then we have

$$\lim_{n\to\infty} \sup_{x\in K} E[\max_{0\leqq t\leqq T} |X_n(t,x) - X(t,x)|^2] = 0$$

for every $T \geqq 0$ and compact set K in R^d.

In §2 we will give the proof of Theorem A and state two variants of the theorem (Theorem B and C), including in the case in which the reflecting boundary condition is involved. In §3 the polygonal approximation will be treated and we will get a similar theorem for this approximation. This problem was considered by Yamada [9] in the case of one-dimensional stochastic differential equations whose coefficients satisfy a certain Hölder continuity. Finally we construct a unique strong solution possessing a simple measurability for the equation (1) by using the polygonal approximation.

2. Proof and variants of Theorem A

The following lemma can be proved using the same technique in the proof of Theorem 4.6 in [6].

Lemma 1. Under the assumption of Theorem A, there exists, for each $T \geqq 0$ and compact set K in R^d, a positive constant $C = C_{T,K}$ such that

$$\sup_{x\in K} E[\max_{s\leqq u,v\leqq t} |X(x,u) - X(x,v)|^4] \leqq C|t - s|^2 \qquad 0 \leqq s < t \leqq T,$$

and

$$\sup_n \sup_{x\in K} E[\max_{s\leqq u,v\leqq t} |X_n(x,u) - X_n(x,v)|^4] \leqq C|t - s|^2, \qquad 0 \leqq s < t < T .$$

Proof of Theorem A. Assume that the conclusion of our theorem is not true. Then there exist positive constants δ, T, a subsequence $\{n'\}$ and a sequence $\{x_{n'}\}$ in R^d contained in some compact set such that

$$\inf_{n'} E[\max_{0\leqq t\leqq T} |X_{n'}(x_{n'},t) - X(x_{n'},t)^{\,2}] \geqq \delta .$$

Without loss of generality, we may assume that $\{n'\} = \{n\}$ and $\{x_n\}$ converges in R^d.

From Lemma 1, we know that the family of the processes $\{X(x_n,t), X_n(x_n,t), B(t)\}_{n=1}^{\infty}$ is tight (Theorem 4.2 and Theorem 4.3 in [1, Chapter

I]). Therefore, there exist some probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ and a sequence of stochastic processes $\{\hat{X}_n(t), \hat{Y}_n(t), \hat{B}_n(t)\}_{n=0}^{\infty}$ on the probability space $(\hat{\Omega}, \hat{\mathcal{F}}, \hat{P})$ which enjoy the properties:

(i) The probability law of $\{\hat{X}_n, \hat{Y}_n, \hat{B}_n\}$ coincides with the law of $\{X(x_n), X_n(x_n), B\}$ for each $n = 1,2,\cdots$.

(ii) There exists a subsequence $\{n'\}$ such that $\{\hat{X}_{n'}, \hat{Y}_{n'}, \hat{B}_{n'}\}$ converges to $\{\hat{X}_0, \hat{Y}_0, \hat{B}_0\}$ uniformly on every finite time interval a.s.

Here we may assume that (ii) holds without subtracting subsequence of $\{\hat{X}_n, \hat{Y}_n, \hat{B}_n\}_{n=1}^{\infty}$. By virtue of uniformly integrability, we obtain

$$
\begin{aligned}
\delta &\leqq \liminf_{n\to\infty} E[\max_{0\leqq t\leqq T} |X(x_n,t) - X_n(x_n,t)|^2] \\
&= \liminf_{n\to\infty} \hat{E}[\max_{0\leqq t\leqq T} |\hat{X}_n(t) - \hat{Y}_n(t)|^2] \\
&= \hat{E}[\max_{0\leqq t\leqq T} |\hat{X}_0(t) - \hat{Y}_0(t)|^2],
\end{aligned}
\tag{3}
$$

where $\hat{E}$ stands for the expectation with respect to the probability measure $\hat{P}$.

On the other hand, because of the coincidence (i) of probability laws, we have for $n = 1,2,\cdots$

$$\hat{X}_n(t) - x_n = \int_0^t \sigma(s,\hat{X}_n(s))d\hat{B}_n(s) + \int_0^t b(s,\hat{X}_n(s))ds$$

and

$$\hat{Y}_n(t) - x_n = \int_0^t \sigma_n(s,\hat{Y}_n(s))d\hat{B}_n(s) + \int_0^t b_n(s,\hat{Y}_n(s))ds.$$

By letting $n \to \infty$ (Skorohod [7]), it turn out that not only $\hat{X}_0(t)$ but also $\hat{Y}_0(t)$ are solutions of (1) with respect to the Brownian motion $\hat{B}_0(t)$.

Because the pathwise uniqueness of solutions for (1) holds and clearly $\hat{X}_0(0) = \hat{Y}_0(0)$, we arrive at $\hat{X}_0(t) = \hat{Y}_0(t)$ which contradicts (3). q.e.d.

We state a variant of Theorem A in the case that the coefficients σ and b are not always continuous and $d = r$.

<u>Theorem B.</u> Suppose that the coefficients $\sigma_n(t,x)$ and $b_n(t,x)$ are uniformly bounded in t,x and $n = 1,2,\cdots$ and there exists $\delta > 0$ such that

$$(\sigma_n(t,x)\xi,\xi) \geqq \delta|\xi|^2, \qquad \xi \in R^d$$

holds for $t \geqq 0$, $x \in R^d$, $n = 1,2,3,\cdots$. Further suppose $\lim_{n\to\infty} \sigma_n(t,x) = \sigma(t,x)$ and $\lim_{n\to\infty} b_n(t,x) = b(t,x)$ in $L_{2d+1,loc}([0,\infty) \times R^d)$. If the pathwise uniqueness of solutions for (1) holds, then we have

$$\lim_{n\to\infty} \sup_{x\in K} E[\max_{0\leq t\leq T} |X_n(x,t) - X(x,t)|^2] = 0,$$

for each $T \geq 0$ and compact set K in R^d.

The proof can be perfomed in the similar way as in the proof of Theorem A, but we need the limiting procedure introduced in the proof of Theorem 1 of Krylov [3; Chapter 2].

Another variant concerns the case of the reflecting boundary condition.

Let D be a convex domain in R^d. We then consider the following stochastic differential equation with reflecting boundary condition for D (for the detailed definition see Tanaka [8]):

$$(4) \qquad dX(t) = \sigma(t,X(t))dB(t) + b(t,X(t))dt + d\Phi(t),$$

where σ and b are continuous functions on $[0,\infty) \times R^d$. We can define the notion of the pathwise uniqueness of solutions for (4) similarly. Let $\{D_n\}_{n=1}^{\infty}$ be a sequence of convex domains of R^d. Let $\{\sigma_n\}_{n=1}^{\infty}$ and $\{b_n\}_{n=1}^{\infty}$ be sequences of continuous coefficients and consider a family of stochastic differential equations with reflecting boundary conditions for D_n.

$$(5) \qquad dX(t) = \sigma_n(t,X(t))dB(t) + b_n(t,X(t))dt + d\Phi(t),$$
$$n = 1,2,3,\cdots.$$

We denote by $(X_n(x,t),\Phi_n(x,t))$ the solution for the n-th equation of (5) with initial value x. Then we have the following theorem:

<u>Theorem C.</u> Suppose that for each $T \geq 0$ there exists $L_T > 0$ such that

$$\sup_{0\leq t\leq T} \sup_n (\|\sigma_n(t,x)\| + |b_n(t,x)|) \leq L_T(1 + |x|),$$

and that

$$\lim_{n\to\infty} \sup_{0\leq t\leq T} \sup_{x\in K} (\|\sigma_n(t,x) - \sigma(t,x)\| + |b_n(t,x) - b(t,x)|) = 0$$

holds for each $T \geq 0$ and compact set K in R^d. Further suppose that D_n converges to D in the following sense:

(i) $\quad x_n \in D_n$ and $\lim_{n\to\infty} x_n = x \Longrightarrow x \in \bar{D}$.

(ii) For any compact set $K \subset D$, $K \subset D_n$ holds for sufficiently large n.

If the equation (4) has a pathwise unique solution $(X(x,t),\Phi(x,t))$ for every starting point $x \in D$, then we have

$$\lim_{n\to\infty} \sup_{x\in K} E[\max_{0\leqq t\leqq T} |X_n(x,t) - X(x,t)|^2] = 0$$

for each $T \geqq 0$ and compact set K in D.

<u>Proof.</u> Applying a method in Tanaka [8], we can obtain that for each $T \geqq 0$ and compact set K in R^d, there exists a positive constant $C = C_{T,K}$ such that

$$\sup_n \sup_{x\in K} E[\max_{s\leqq u,v\leqq t} |X_n(x,u) - X_n(x,v)|^4] \leqq C|t-s|^2,$$

$$0 \leqq s < t \leqq T.$$

Therefore $\{X_n(x,t),X(x,t),B(t),\Phi_n(x,t),\Phi(x,t)\}_{n=1}^{\infty}$ is tight. Assuming the contrary to the conclusion of Theorem C, we can derive a contradiction in the same manner as in the proof of Theorem A. q.e.d.

<u>Remark.</u> P. L. Lions and A. S. Sznitman [5] introduced the notion of the admissible domain D in R^d and they considered the stochastic differential equation with reflecting boundary condition for D. In this case we can get the same result as Theorem C.

3. Polygonal approximation and unique strong solution

Again, we will consider the stochastic differential equation (1) in §1. For an arbitrary partition $\Delta : 0 = t_0 < t_1 < t_2 < \cdots < t_n < \cdots \longrightarrow \infty$, define the polygonal approximation process $X_\Delta(x,t)$ for (1) by

$$X_\Delta(x,t) - x = \int_0^t \sigma(\phi_\Delta(s),X_\Delta(x,\phi_\Delta(s)))dB(s)$$

$$+ \int_0^t b(\phi_\Delta(s),X_\Delta(x,\phi_\Delta(s)))ds,$$

where $\phi_\Delta(s) = t_i$, if $t_i \leqq s < t_{i+1}$. Put $\|\Delta\| = \sup_n(t_{n+1} - t_n)$. We then have the following theorem:

<u>Theorem D.</u> Suppose that $\sigma(t,x)$ and $b(t,x)$ are continuous. Further suppose that for each $T \geqq 0$ there exists $L_T > 0$ such that

$$\sup_{0\leqq t\leqq T} (\|\sigma(x,t)\| + |b(x,t)|) \leqq L_T(1 + |x|).$$

If the pathwise uniqueness of solutions for (1) holds, then we have

$$\lim_{\|\Delta\|\to 0} \sup_{x\in K} E[\max_{0\leqq t\leqq T} |X_\Delta(x,t) - X(x,t)|^2] = 0$$

for each $T \geqq 0$ and compact set K in R^d.

The proof can be performed using the same method as the preceding proof.

If the pathwise uniqueness of solutions for (1) holds, then (1) has a unique strong solution (Ikeda-Watanabe [1]). But the measurability of the strong solution in [1] is very complicated. On the contrary Theorem D enables us to construct a unique strong solution with a simple measurability.

In what follows, let $W^d = C([0,\infty) \longrightarrow R^d)$ and $W_0^r = \{w \in C([0,\infty) \longrightarrow R^r);\ w(0) = 0\}$. By the uniform convergence on every finite interval, we can regard W^d and W_0^r as Polish spaces. Let $\mathcal{B}(W^d)$ (resp. $\mathcal{B}(W_0^r)$) be the topological Borel field on W^d (resp. W_0^r) and $\mathcal{B}_t(W^d)$ (resp. $\mathcal{B}_t(W_0^r)$) be the sub-σ-field of $\mathcal{B}(W^d)$ (resp. $\mathcal{B}(W_0^r)$) generated by $w(s)$, $0 \leqq s \leqq t$. Consider the standard Wiener measure P^W on $(W_0^r, \mathcal{B}(W_0^r))$ and put $\mathcal{N} = \{N \in \mathcal{B}(R^d) \times \mathcal{B}(W_0^r);\ P^W(N_x) = 0$ for every $x \in R^d\}$, where $N_x = \{w \in W_0^r;\ (x,w) \in N\}$. We denote by $\mathcal{H}_t$ the σ-field generated by $\mathcal{B}(R^d) \times \mathcal{B}_t(W_0^r) \cup \mathcal{N}$.

<u>Corollary.</u> Under the assumption of Theorem D, there exists a $\mathcal{B}(R^d) \times \mathcal{B}(W_0^r)/\mathcal{B}(W^d)$-measurable function $F(x,w): R^d \times W_0^r \longrightarrow W^d$ which satisfies the following properties:

(i) $F(x,w)$ is $\mathcal{H}_t/\mathcal{B}_t(W^d)$-measurable for each $t \geqq 0$.

(ii) If $B(t)$ is an r-dimensional $(\mathcal{F}_t)$-Brownian motion defined on a usual filtered probability space $(\Omega, \mathcal{F}, P, (\mathcal{F}_t))$ and ξ is an $\mathcal{F}_0$-measurable R^d-valued random variable on $(\Omega, \mathcal{F}, P)$, then $F(\xi, B(\cdot))$ is the unique solution of (1) with initial value ξ.

(iii) If $X(t)$ is a solution of (1) with respect to a Brownian motion $B(t)$, then $X(\cdot) = F(X(0), B(\cdot))$ a.s.

<u>Proof.</u> The polygonal approximation $X_\Delta(x,t,w)$ with respect to the standard Brownian motion $(W_0^r, \mathcal{B}(W_0^r), P^W)$ in Theorem D defines a measurable function $F_\Delta(x,w): R^d \times W_0^r \longrightarrow W^d$ by setting $F_\Delta(x,w) = X_\Delta(x,\cdot,w)$. Theorem D implies that there exists a sequence $\{\Delta_n\}_{n=1}^{\infty}$ of partitions such that

$$\lim_{n\to\infty} \|\Delta_n\| = 0$$

and for every $x \in R^d$ $\{F_{\Delta_n}(x,w)\}$ converges in W^d a.s., because Borel-Cantelli's lemma works uniformly on each compact x-set in R^d.

Put

$$\Lambda = \{(x,w) \in R^d \times W_0^r;\ F_{\Delta_n}(x,w) \text{ converges in } W^d\}$$

and define

$$F(x,w) = \begin{cases} \lim_{n\to\infty} F_{\Delta_n}(x,w) & \text{if } (x,w) \in \Lambda \\ (0,\cdots,0) & \text{if } (x,w) \notin \Lambda . \end{cases}$$

Then it is easy to see that $F(x,w)$ satisfies the desired conditions. q.e.d.

REFERENCES

[1] N. Ikeda and S. Watanabe, Stochastic Differential Equations and Diffusion Processes, North-Holland and Kodansha, Tokyo, 1981.

[2] S. Kawabata and T. Yamada, On some limit theorems for solutions of stochastic differential equations, Séminaire de Probabilités XVI, Lecture Notes in Math., 920, Springer Verlag, Berlin, 1980/81.

[3] N. V. Krylov, Controlled Diffusion Processes, Springer Verlag, Berlin, 1980.

[4] J. F. Le Gall, Application du temps local aux équations differentielles stochastiques unidimensionnelles, Séminaire de Probabilités XVII, Lecture Notes in Math., 986, Springer Verlag, Berlin, 1981/82.

[5] P. L. Lions and A. S. Sznitman, Stochastic differential equation with reflecting boundary conditions, Comm. Pure Appl. Math., 37(1984), 511-537.

[6] R. S. Liptser and A. N. Shiryayev, Statistics of Random Processes 1, Springer Verlag, Berlin, 1977.

[7] A. V. Skorohod, Studies in the Theory of Random Processes, Addison-Wesley, Washington, 1965.

[8] H. Tanaka, Stochastic differential equations with reflecting boundary conditions in convex regions, Hiroshima Math. J., 9(1979), 163-177.

[9] T. Yamada, Sur une construction des solutions d'équations différentielles stochastiques des les cas non-Lipschitzien, Séminaire de Probabilités XII, Lecture Notes in Math., 649, 1976/77.

Hiroshi Kaneko
Department of Mathematical Sciences
College of Engineering
University of Osaka Prefecture
Mozu-Umemachi, Sakai, Osaka 591,
Japan

Shintaro Nakao
Department of Mathematics
Osaka University
Toyonaka, Osaka 560,
Japan