MICHAEL I. TAKSAR

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STATIONARY MARKOV SETS

M. I. Taksar*

Department of Statistics
Florida State University
Tallahassee, Florida
United States

1. Introduction

If one looks at the set of times when a strong Markov process visits a point in the state space, then this set is a regenerative set. It forms a replica of itself after each stopping time whose graph lies in this set. Closed regenerative sets have been studied for a long time (see Hoffman-Jørgensen [4], Maisonneuve [6], Meyer [10] and others).

Since the studies of regenerative sets were motivated by the theory of Markov processes, such sets were originally called (strong) Markov. In addition it was always supposed that any regenerative set M is a subset of the positive half-line and $P\{0 \in M\} = 1$.

However, if one considers visiting times of a stationary strong Markov process, then the corresponding set M is stationary, that is the probability law of the set $M+t$ is the same as the one of M. The "natural" state space for stationary sets would be the set of closed subsets of a real line and the condition $0 \in M$ a.s. should be dropped. The first study of such sets was done in Taksar [12]. It was shown that all such sets are in one-to-one correspondence with the weak limits of the ranges (closures of the images) of the processes with independent increments having finite expectation.

The paper of Maisonneuve [8] gives a simple and comprehensive approach to the regenerative sets on a real line. It also give an easy proof of the main results of [12]. Further development of the theory of regenerative sets on a real line is done in the recent work of Fitzsimmons, Frisdedt and Maisonneuve [3].

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All regenerative set have a (weak) Markov property. The "future" after time t of such set and its "past" are conditionally independent given "resent". In this context "future" after time t means the intersection of the random set with $]t,\infty[$. The "present" stands for the infimum of the "future". The "past" is the compliment of the "future". A Markov set is a set for which conditional independence of the "future" and the "past" holds, but stronger regenerative property might not be true.

Apparently, Markov sets form a larger class than regenerative sets. In a stationary case, however, the difference is not as big as one could expect. It was shown in [12] that staionary Markov sets are "almost" regenerative. There are two types of regeneration after each point t; one occurs if the point t belongs to the set and the other type of regeneration takes place if t does not belong to the set. In particular, every stationary Markov set which almost surely has Lebesgue measure zero, is regenerative, (see [12] Theorem 2).

In this paper we will describe all closed stationary Markov sets. We will show that each stationary Markov set which is not regenerative can be constructed from two special regenerative sets, by either taking a mixture of these regenerative sets or taking a "superposition" of two regenerative sets. One of the two regenerative sets is thin (that is having a.s. Lebesgue measure zero) and the other is "rather thick". In the case of mixture the second set is the entire real line. In the case of the superposition the "thick" regenerative set consists of a union of closed intervals with the exponential iid lengths with the spacings between these intervals having any iid distributions.

Superposition can be described loosely as follows. We take the real line $\mathbb{R}_1$ with a thin set M_1 and a real line $\mathbb{R}_2$ with a thick set M_2, which consists of a countable number of closed intervals $\ldots, I_{-1}, I_0, I_1, I_2, \ldots$. The real line $\mathbb{R}_1$ is cut in a segments of iid lengths, exponentially distributed in the <u>local time</u> of the set M_1. The line $\mathbb{R}_2$ is cut at the left end of each interval I_k. Then we combine $\mathbb{R}_1$ and $\mathbb{R}_2$ into one line by alternating pieces from $\mathbb{R}_1$ and $\mathbb{R}_2$ (i.e., inserting intervals I_k with their right spacings into the cuts of the set M_1). The union of the cut offs from M_1 and M_2 will be the superposition of the sets M_1 and M_2.

In the case in which M_1 is a discrete set one can describe the resulting Markov set in operations research/reliability vernacular. Consider a serviceman who is regularilly called on site for inspection of a working device. At each inspection there is a probability p of discovering a defect. While the defect is not detected, the intervals between successive calls are iid random variables with distribution F_1. If the inspection reveals the defect then the serviceman stay for a repair which has an exponential distribution. The time of the next inspection after the repair is decided by the serviceman and it has distribution F_2 which might be different form F_1. The set of times when the serviceman is on site, that is, the inspection times and the repair time, is a Markov set. However this set is neither regenerative nor is a mixture of two regenerative sets.

Although the description of the superposition in terms of cutting and recombining the lines is more intuitively understandable, we would rather use an equivalent definition in terms of processes with independent increments, which is more useful from the technical point of view.

The paper is structured as follows. In section 2 we give definitions and formulate the main results. In section 3 we establish the main properties of stationary Markov sets. Section 4 studies the operations which transforms a stationary Markov set into a stationary regenerative set. Section 5 analyses those stationary Markov sets which are neither regenerative nor are mixtures of regenerative sets. In section 6 we study the "residual life" process associated with the stationary Markov set, and find its stationary distribution. The last section is devoted to reversibility properties. We outline a necessary and sufficient condition for the set -M to have the same distribution as M.

2. Basic definition. Formulation of the main result.

In our definition and notations we follow Maisonneuve [8] and Fitzimmons, Fristedt, and Maisonneuve [3] (following slight corrections suggested by Maisonneuve in [9]). Let Ω^o be the set of all closed sets in $\mathbb{R}$. For each $\omega^o \in \Omega^o$ and $t \in \mathbb{R}$ put (assuming $\inf \emptyset = \infty$, $\sup \emptyset = -\infty$)

$d_t(\omega^o) \triangleq \inf\{s>t:\ s\in\omega^o\}$, $\ell_t(\omega^o) \triangleq \sup\{u<t:\ u\in\omega^o\}$,

$r_t(\omega^o) \triangleq d_t(\omega^o) - t$, $n_t(\omega^o) \triangleq t - \ell_t(\omega^o)$,

$\tau_t(\omega^o) \triangleq \overline{\{s-t:\ s>t,\ s\in\omega^o\}}$,

$\rho_t(\omega^o) \triangleq \overline{\{s-t:\ s<t\ s\in\omega^o\}}$, where the bar above the set stands for closure.

Let G^o (G^o_t respectively) be the σ-field generated by all functions d_s, $s\in\mathbb{R}$ ($s\le t$ respectively). Let J^o (J^o_t respectively) be the σ-field generated by all functions ℓ_u, $u\in\mathbb{R}$ ($u\ge t$ respectively). It is easy to see that G^o_t is an increasing and J^o_t is a decreasing filtration and $J^o = G^o$.

A closed random set M on a space (Ω,F) is a measurable mapping of (Ω,F) into (Ω^o,G^o).

In this paper we will deal only with closed random sets, so in the sequel we will not write "closed" each time. Put

$$D_t \triangleq d_t\circ M,\quad R_t \triangleq r_t\circ M,$$

$$L_t \triangleq \ell_t\circ M,\quad N_t \triangleq n_t\circ M,$$

$$M^t \triangleq \tau_{D_t}\circ M,\quad M_t \triangleq \rho_{L_t}\circ M.$$

It is obvious that all the mappings D_t, R_t, L_t and N_t are measurable and so are M^t and M_t.

Let (Ω,F,P) be a complete probability space and M be a random set on this space. Let G, G_t and J_t be the preimages in F of the σ-fields G^o, G^o_t and J^o_t under the mapping M.

(2.1) A set M is called right Markov (r.M.) if for any two bounded measurable functions f and g on (Ω^o,G^o)

$$P\{f(M\cap[D_t,\infty[)g(M\cap]-\infty,D_t])|D_t\}=P\{f(M\cap[D_t,\infty[)|D_t\}P\{g(M\cap]-\infty,D_t])|D_t\}.$$

(2.2) A set M is called left Markov (l.M.) if for any two bounded measurable functions f and g on (Ω^o,G^o)

$$P\{f(M\cap[L_t,\infty[)g(M\cap]-\infty,L_t])|L_t\}=P\{f(M\cap[L_t,\infty[)|L_t\}P\{g(M\cap]-\infty,L_t])|L_t\}.$$

For brevity here and in sequel we write equations with conditional expectations without adding a.s. after equalities. Given a random set M, we denote by $M+s$ the set $\{t+s: t \in M\}$.

(2.3) A set M is called stationary if for any bounded measurable function f on (Ω^o, G^o) and any $s \in \mathbb{R}$

$$P\{f(M+s)\} = P\{f(M)\}.$$

Our aim is to describe all stationary r.M. sets. We will need results from the theory of regenerative sets. The precise notion of regenerative set used in this paper is due to Maisonneuve [8] (with slight corrections according to [9]).

(2.4) A random set M is right regenerative (r.r.) if there exists a measure P_0 on (Ω^o, G^o) such that for each $f \in bG^o$ (set of bounded G^o-measureable functions)

$$P\{f \circ M^t | G_t\} = P_0\{f\} \text{ on } \{D_t < \infty\}.$$

Following [8], the measure P_0 is called the law of (right) regeneration of M.

(2.5) A set M is left regenerative (l.r.) if there exists a measure P^0 on (Ω^o, G^o) such that for each $f \in bG^o$

$$P\{f \circ M_t | J_t\} = P^0\{f\} \quad \text{on} \quad \{L_t > -\infty\}.$$

In the sequel we will sometimes use the term regenerative (r.) and Markov (M.) instead of right regenerative and right Markov respectively.

Increasing processes with independent increments (subordinators) play an important role in the description of regenerative sets and, as we will see in the sequel, stationary Markov sets as well. Each subordinator z is characterized by a constant $\alpha < 0$ and a measure Π on $]0, \infty[$. We call such a subordinator an (α, Π)-process. Let $z_t(\omega)$, $t \geq 0$, be a stochastic process on a probability space (Ω, F, P). The image M of this process is defined as

$$M(\omega) = \overline{z_{\mathbb{R}_+}(\omega)}$$

If z is a subordinator, then the image of z is a right regenerative set. If z is a decreasing process with independent increments then the image of z is a left regenerative set.

Let us recall the main results of [8] and [12] regarding stationary regenerative sets. There is one-to-one correspondence between all stationary r.r. sets

M and all pairs (α, Π) defined up to proportionality, where α and Π are characteristics of a subordinator subject to

$$\int_0^\infty x \, \Pi(dx) < \infty .$$

The stationary set M which corresponds to the pair (α, Π) is called (α, Π)-generated. Any stationary r.r. set M is also l.r. Moreover the set $-M$ has the same distribution in (Ω^o, G^o) as M.

Since the definition of r.M. set is weaker than that of r.r. set, any r.r. set is r.M., however the opposite is not true.

An example of a stationary r. M. set which is not r.r. was constructed in [12]. Any mixture of a $(0, \Pi)$-generated set and a real line $\mathbb{R}$ with "weights" $0 < p < 1$ and $q = 1 - p$ is a r. M. set but not a r.r. set.

DEFINITION. Right Markov sets of the first type are right regenerative sets. Right Markov sets which can be represented as a mixture of a $(0, \Pi)$-generated and a real line are called r.M. sets of the second type. Right Markov sets which are neither of the first or the second type are called right Markov sets of the third type.

Markov processes provide good examples of different types of stationary Markov sets. If x_t is a strong Markov process and b is a point in the state space then the "visiting set"

$$M = \overline{\{t: x_t = b\}}$$

is regenerative and if in addition x_t is stationary, then M is stationary.

To obtain a Markov set of the second type, consider a strong Markov process x_t^1, for which $P\{x_t^1 = b\} = 0$ for each t, but point b is not a polar set and a process x_t^2 which stays deterministically at the point b. The mixture x_t of the processes x_t^1 and x_t^2 will be a Markov (but not a strong Markov) process. The visiting times of b by x_t is a Markov set of the second type, and if x_t^1 is stationary then so is the visiting times set.

To give an example of a Markov set of the third type, consider a particle moving on the positive half line according to a diffusion law. An infinitely thin elastic screen is placed at the origin. The particle is reflected from this screen until time

$$\tau = \{\inf t: \Lambda_t \geq S\},$$

where Λ_t is the local time at zero of the reflected diffusion and S is a random variable with exponential distribution independent of the process x_t. At the moment τ the particle moves to the other side of the screen where it stays for time X, where X is another exponential random variable independent of x. and S. At the time $X+\tau$ the particle is placed back to a random point on the positive half line and the whole process starts anew. The closure of the set of times when this particle visits the origin is a Markov set of the third type. If this Markov process is stationary (which can be easily achieved, provided that there exists a constant downward drift, or there exists a reflecting upper barrier)then this Markov set is stationary.

In the remainder of this section we define rigorously the superposition of two regenerative sets and formulate the main result. The definition in introduction might be convinient but we find it more useful to define the superposition by means of the processes with independent increments.

In the sequel we will use $\mathbb{R}_+$ and $[0,\infty[$ interchangably. If a measure Π is defined on $]0,\infty[$ then it is assumed to be extended to $\mathbb{R}_+$ by setting $\Pi\{0\}=0$.

Let Π be a measure on $]0,\infty]$ and μ be a probability measure on $[0,\infty[$ and λ and α be two positive constants. Let y_t be a $(0,\Pi)$-process and $\{S_k\}$, $k=1, 2, \ldots,$ $\{X_k\}$ and $\{Y_k\}$, $k=0, 1, 2, \ldots$ be three sequences of iid random variables, independent of y_t and independent of each other. The distributions of S_i and X_k are exponential with parameters α and λ respectively. The distribution of Y_j is given by μ. Consider a subordinator x_t of a pure jump type constructed in the following manner (we assume below $\sigma_0=0$)

$$\sigma_k = \sum_{i\le k} S_i, \quad k = 1,\ 2,\ \dots,$$

$$x_t = \sum_{k\ge 1} (Y_k + X_k) 1_{\sigma_k \le t}, \qquad (2.6)$$

Put

$$Z_t = y_t + x_t,$$

$$L = \bigcup_{k=1}^{\infty} \{x : Z_{\sigma_k^-} \le x \le Z_{\sigma_k^-} + X_k\}, \qquad (2.7)$$

$$M = \overline{Z_{\mathbb{R}_+} \cup L}. \qquad (2.8)$$

The set M defined by (2.8) is called $(\Pi, \alpha, \lambda, \mu)$-set. (Note that there are many $(\Pi, \alpha, \lambda, \mu)$-sets corresponding to different initial distributions of the process y_t).

Let μ' be the restriction of μ on $]0,\infty[$. We say that quadruple $(\Pi, \alpha, \lambda, \mu)$ is equivalent to $(\Pi_1, \alpha_1, \lambda_1, \mu_1)$ if there exists a constant c such that

$$(\Pi, \alpha) = c(\Pi_1, \alpha_1), \qquad (2.9)$$

$$\mu' - \mu_1' = \frac{\mu\{0\} - \mu_1\{0\}}{\Pi(\mathbb{R}_+)} \Pi, \qquad (2.10)$$

$$\lambda(1 - \alpha\mu\{0\}/(\alpha + \Pi(\mathbb{R}_+)) = \lambda(1 - \alpha_1\mu_1\{0\}/(\alpha_1 + \Pi_1(\mathbb{R}_+)). \qquad (2.11)$$

In particular, when Π is an infinite measure, equivalency of $(\Pi, \alpha, \lambda, \mu)$ and $(\Pi_1, \alpha_1, \lambda_1, \mu_1)$ means proportionality of (Π, α) and (Π_1, α_1) and equality of (λ, μ) and (λ_1, μ_1).

It is easy to see that if $\Pi(\mathbb{R}_+) = \infty$ and quadruples $(\Pi, \alpha, \lambda, \mu)$ and $(\Pi_1, \alpha_1, \lambda_1, \mu_1)$ are equivalent then every $(\Pi, \alpha, \lambda, \mu)$-set is a $(\Pi_1, \alpha_1, \lambda_1, \mu_1)$-set as well. In fact, if we construct processes x, y and Z by (2,6) and (2,7), then processes $x_t' = x_{ct}$, $y_t' = y_{ct}$ and $Z_t' = Z_{ct}$ generate the same set M given by (2.8). However, the Levy's measure of the process y_{ct} is $c\Pi$ and the rate of jumps of the process x_{ct} is $c\,\alpha$, which shows that $(\Pi, \alpha, \lambda, \mu)$-set is $(c\Pi, c\,\alpha, \lambda, \mu)$-set as well.

If Π is a finite measure then x_t and y_t are Poisson processes with jump rates α and $\Pi(\mathbb{R}_+)$ respectively. In particular

$$p \triangleq P\{y_{\sigma_1} = y_0\} = \alpha/(\alpha + \Pi(\mathbb{R}_+))$$

(see (2.6) for definition of σ_1). The set M given by (2.7) consists of the intervals of L and discrete points of the image of Z. The length of the first interval I_1 of L is equal to $X_1 + X_2 + \dots + X_N$ where N has geometric distribution with parameter $p\mu\{0\}$. Thus the distribution of the length of I_1 is exponential with parameter $\lambda(1-p\mu\{0\})$. The distribution of the length the interval J_1 which is contingent to I_1 in M from the right (i.e., $\inf J_1 = \sup I_1$) has distribution $\mu' + (\mu\{0\}/\Pi(\mathbb{R}_+))\Pi$ (note that $(\Pi(\mathbb{R}_+))^{-1}\Pi$ is the distribution of the jumps of the process y). Likewise for any other interval I_k in L and contingent to I_k interval J_k. The distribution of any interval contingent to M which does not coincide with any of J_k is

equal to the distribution of jumps of y, i.e. to $(\Pi(\mathbb{R}_+))^{-1}\Pi$. From the above it is easy to show that if M is a (Π,α,λ,μ)-set and (Π,α,λ,μ) is equivalent to $(\Pi_1,\alpha_1,\lambda_1,\mu_1)$ then there exists a $(\Pi_1,\alpha_1,\lambda_1,\mu_1)$-set whose distribution is the same as that of M.

DEFINITION. A random set M is called (Π,α,λ,μ)-generated if for each t there exists a random variable ϕ_t, such that $\phi_t \geq t$ a.s. and $M \cap [\phi_t,\infty[$ has the same distribution as a (Π,α,λ,μ)-set. In this case the quadruple (Π,α,λ,μ) is called the generator of the set M.

The next two theorems give the main result of this paper.

(2.12) **THEOREM.** Every stationary r.M. set M of the third type is (Π,α,λ,μ)-generated. The generator of M is unique up to equivalency and is subject to

$$\int_0^\infty x\Pi(dx) < \infty, \tag{2.13}$$

$$\int_0^\infty x\mu(dx) < \infty. \tag{2.14}$$

Each quadruple (Π,α,λ,μ) subject to (2.13) and (2.14) is a generator of a unique stationary right Markov set.

Let δ_a denote a unit measure concentrated at point a.

(2.15) **THEOREM.** A stationary r.M. set M of the third type is left Markov iff its generator (Π,α,λ,μ) is equivalent to $(\Pi,\alpha,\lambda,\delta_0)$. In this case the set $-M$ has the same distribution as M.

In the diffusion example presented above the set of visiting times of 0 becomes a left Markov set when the diffusion process is made continuous. That can be done if at the time $\tau + X$ the particle is moved on the other side of the elastic screen and starts again moving according to the original reflected diffusion law. In the operations research/reliability example of the introduction, the set of times when the servicemen is on site becomes left Markov if $F_1 = F_2$, that is if the distribution of the time of the first after a repair check up is the same as the distribution of the time between successive calls.

3. General properties of stationary Markov sets.

Here and in the sequel we will deal only with those stationary Markov sets which are a.s. nonempty. This is equivalent to

$$P\{D_t < \infty\} = 1 \qquad \text{for all } t \in \mathbb{R}. \tag{3.1}$$

The following proposition was proved in [12] (see Lemma 7.3).

(3.2) **PROPOSITION.** <u>If</u> M <u>is stationary Markov set then for each function</u> $f \in bG^o$ <u>there exist two constants</u> a <u>and</u> b <u>such that for each</u> t

$$P\{f \circ M^t | G_t\} = a\, 1_{D_t > t} + b\, 1_{D_t = t}\ .$$

For brevity we will denote indicator functions of $]-\infty,t[$, $]-\infty,t]$, $[t,\infty[$, $]t,\infty[$ by $1_{<t}$, $1_{\le t}$, $1_{\ge t}$ and $1_{>t}$ respectively.

The following corollary is a simple consequence of Propostion (3.2).

(3.3) **COROLLARY.** <u>If</u> M <u>is a stationary Markov set then there exist two measures</u> P_0 <u>and</u> P_1 <u>on</u> (Ω^o, G^o) <u>such that for each</u> $f \in bG^o$

$$P\{f \circ M^t | G_t\} = 1_{>t}(D_t)P_0\{f\} + 1_t(D_t)P_1\{f\}. \tag{3.4}$$

Let $\hat{M}$ denote the set of all points of M which belong to M with its right neighborhood.

(3.5) **PROPOSITION.** <u>For each</u> $f \in bG^o$ <u>and any stopping time</u> T <u>with respect to the filtration</u> G_{t+}

$$P\{f \circ M^T | G_{T+}\} = 1_{T \bar{\in} \hat{M}} P_0\{f\} + 1_T(\hat{M})\ P_1\{f\}. \tag{3.6}$$

Proof. Usual arguments show that Proposition (3.3) remains true if t in (3.4) is replaced by any stopping time with respect to G_t, taking finite or countable number of values.

It is sufficient to prove (3.6) for f of the form

$$f = g(r_{s_1}, r_{s_2}, \dots, r_{s_k})$$

where g is a bounded continuous function of k variables. For such f the function $f \circ M^t$ is continuous in t and

$$P\{f \circ M^T | G_{T+}\} = \lim_{n\to\infty} P\{f \circ M^{T_n} | G_{T_n}\} =$$
$$= \lim_{n\to\infty} [1_{>T_n}(D_{T_n}) P_0\{f\} + 1_{T_n}(D_{T_n}) P_1\{f\}], \tag{3.7}$$

where T_n is any sequence of stopping times, taking on finite or countable number of values and such that $T_n \downarrow T$.

Put

$$æ_n(x) \triangleq k\ 2^{-n}, \quad \text{if } (k-1)2^{-n} \le x < k\ 2^{-n} \tag{3.8}$$

and let (assuming $\inf \emptyset = +\infty$)

$$T_n' = \inf\{æ_n(s) : s \ge T,\ u \in M \ \text{ for all } s < u \le æ_n(s)\}.$$

The random variable T_n' is a stopping time (see [2], Ch VI) and so is

$$T_n = 1_T(\hat{M})\ æ_n(T) + 1_{T \bar{\in} \hat{M}} T_n' . \tag{3.9}$$

Each T_n given by (3.9) takes at most a countable number of values and $T_n \downarrow T$. By the construction $D_{T_n} > T_n$ on the set $\{T \bar{\in} \hat{M}\}$ and $\{T_n = D_{T_n}\}$ converges to the set $\{T \in \hat{M}\}$. Hence we can pass to a limit in (3.7) and obtain (3.6).

(3.10) **PROPOSITION.** For each $f \in bG^o$ and each stopping time T with respect to G^o_{t+} and each $i = 0, 1$,

$$P_i\{f \circ \tau_{d_T} | G^o_{T+}\} = 1_{T \bar{\epsilon} \hat{\omega}^o} P_0\{f\} + 1_T(\hat{\omega}^o) P_1\{f\}. \tag{3.11}$$

The proof is similar to the proof of previous proposition.

From now on we will consider only stationary sets of the third type, for which

$$P\{D_t = t\} > 0. \tag{3.12}$$

(Theorem 2 of [12] shows that failure of (3.12) implies that M is regenerative.)

(3.13) **PROPOSITION.** For each t

$$P\{D_t = t, \ t \bar{\epsilon} \hat{M}\} = 0. \tag{3.14}$$

Proof. Suppose the left hand side of (3.14) is equal to $\varepsilon > 0$. By virtue of Proposition (3.5)

$$P\{f \circ M^t | G_{t+}\} = P_0\{f\} \quad \text{on} \quad \{D_t = t, \ t \bar{\epsilon} \hat{M}\}. \tag{3.15}$$

On the other hand, using sequentially (3.4) and (3.15)

$$P\{f \circ M^t | G_t\} = P_1\{f\} = [\varepsilon P_0\{f\} + P\{D_t = t, t \epsilon \hat{M}\} P_1\{f\}] / P\{D_t = t\} \quad \text{on } \{D_t = t\}. \tag{3.16}$$

Equality (3.16), which is true for each f, shows $P_0 = P_1$, which contradicts the assumption that M is the set of the third type.

(3.17) **COROLLARY.** $P_1\{0 \in \hat{\omega}_0\} = 1$.

Proof. By proposition (3.13) the sets $\{D_t = t\}$ and $\{t \in \hat{M}\}$ are indistinguishable. Using (3.4),

$$P\{D_t = t\} = P\{D_t = t, \ 0 \in \hat{M}^t\} = P\{D_t = t\} P_1\{0 \in \hat{\omega}_0\}.$$

Thus, the statement follows from (3.12).

(3.18) **PROPOSITION.** <u>For any functions</u> $f \in bG^o$ <u>and</u> $g \in bG^o_{t+}$ <u>such that</u> $g = 0$ <u>on</u> $\{d_t = \infty\}$ <u>and each</u> $i = 0, 1$

$$P_i\{f \circ \tau_{d_t} g\} = P_i\{g; d_t > t\} P_0\{f\} + P_i\{g; d_t = t\} P_1\{f\}. \tag{3.19}$$

Proof. For $i = 1$. Put $T = t + s$. By (3.4) and (3.12)

$$P_1\{f \circ \tau_{d_t} g\} = P\{f \circ M^T g \circ M^s \mid D_s = s\}/P\{D_s = s\}.$$

Taking first conditional expectation with respect to G_{s+t}, we get

$$P_1\{f \circ \tau_{d_t} g\} = P\{g \circ M^s 1_{<t+s}(D_{t+s}) P_0\{f\} + g \circ M^s 1_{t+s}(D_{t+s}) P_1\{f\} \mid D_s = s\}/P\{D_s = s\},$$

which is equivalent to (3.19).

Let

$$\begin{aligned}
&\tilde{\eta}_t \triangleq \inf\{s > t:\ s \in \hat{\omega}^o\}, && \eta_t \triangleq \tilde{\eta}_t \circ M, \\
&\tilde{\gamma}_t \triangleq \inf\{s > \tilde{\eta}_t:\ s \bar{\in} \hat{\omega}^o\}, && \gamma_t \triangleq \tilde{\gamma} \circ M, \\
&\tilde{\nu}_t \triangleq \inf\{s > \tilde{\gamma}_t,\ s \in \omega^o\}, && \nu_t \triangleq \tilde{\nu}_t \circ M, \\
&\tilde{\eta} \triangleq \tilde{\eta}_0,\ \tilde{\gamma} \triangleq \tilde{\gamma}_0,\ \tilde{\nu} \triangleq \tilde{\nu}_0, && \eta \triangleq \eta_0,\ \gamma \triangleq \gamma_0,\ \nu \triangleq \nu_0.
\end{aligned} \tag{3.20}$$

(3.21) **PROPOSITION.** <u>For</u> $\tilde{\gamma}$ <u>and</u> $\tilde{\eta}$ <u>defined by</u> (3.20)

$$P_1\{\tilde{\eta} = 0\} = 1. \tag{3.22}$$

<u>and there exists a constant</u> $0 < \lambda < \infty$ <u>such that for each a</u>

$$P_1\{\tilde{\gamma} > a\} = e^{-\lambda a}. \tag{3.23}$$

Proof. (3.22) follows from (3.17). Let a, b > 0. Applying Proposition (3.18),

$$P_1\{\tilde{\gamma}>a+b\}=P_1\{\tilde{\gamma}>a,\tilde{\gamma}\circ\tau_a>b\}=P_1\{\tilde{\gamma}>a,d_a=a\}P_1\{\tilde{\gamma}>b\} + P_1\{\tilde{\gamma}>a,d_a>a\}P_0\{\tilde{\gamma}>b\} \quad (3.24)$$

If $\tilde{\gamma} > a$ then $a \in \hat{\omega}^o$ and $d_a = a$. Thus $P_1\{\tilde{\gamma} > a,\ d_a > a\} = 0$ and (3.24) equals to $P_1\{\tilde{\gamma} > a\}\ P_1\{\tilde{\gamma} > b\}$ whereas (3.23) follows.

Suppose (3.23) equals 1. Then $P_1\{\mathbb{R}_+ \subset \omega^o\} = 1$. The latter would imply that M is a mixture of a real line $\mathbb{R}$ and a regenerative set with the law of regeneration P_0. This contradicts the assumption that M is a set of the third type. Likewise, if (3.23) equals 0, then this would imply that $P\{d_a = a\} = 0$. The latter is with a contradiction to (3.12).

Let $\tilde{\eta}_t$, $\tilde{\gamma}_t$, ... etc. be given by (3.20). Define

$$\tilde{\eta}(0,t) \triangleq \tilde{\gamma}(0,t) \triangleq \tilde{\nu}(0,t) \triangleq t \ , \quad (3.25)$$
$$\tilde{\eta}(k,t) \triangleq \tilde{\eta}_{\tilde{\nu}(k-1,t)} \ ,$$
$$\tilde{\gamma}(k,t) \triangleq \tilde{\gamma}_{\tilde{\nu}(k-1,t)} \ ,$$
$$\tilde{\nu}(k,t) \triangleq \tilde{\nu}_{\tilde{\nu}(k-1,t)}, \quad k = 1, 2, \ldots ,$$
$$\eta(k,t) \triangleq \tilde{\eta}(k,t) \circ M,$$
$$\gamma(k,t) \triangleq \tilde{\gamma}(k,t) \circ M,$$
$$\nu(k,t) \triangleq \tilde{\nu}(k,t) \circ M.$$

When t is fixed we will write for brevity $\tilde{\eta}(k)$, $\tilde{\gamma}(k)$, $\gamma(k)$, etc. instead of $\tilde{\eta}(k,t)$, $\tilde{\gamma}(k,t)$, $\gamma(k,t)$, etc.

The points $\eta(k)$ and $\gamma(k)$ mark the beginnings and the ends of the intervals which $\hat{M} \cap [t,\infty[$ is composed of.

(3.26) **PROPOSITION.** <u>The sequence</u> $\{(\gamma(k) - \eta(k),\ \nu(k) - \gamma(k),\ \eta(k+1) - \gamma(k))\}$ <u>is a sequence of iid three-dimensionals vectors on</u> (Ω, F, P). <u>The sequences</u> $\{\gamma(k) - \eta(k)\}$ <u>and</u> $\nu(k) - \gamma(k)\}$ <u>are independent and for any</u> $a > 0$

$$P\{\gamma(k) - \eta(k) > a\} = e^{-\lambda a}, \tag{3.27}$$

where λ is the same as in Proposition (3.21).

The sequence $\{(\tilde\gamma(k) - \tilde\eta(k),\ \tilde\nu(k) - \tilde\gamma(k),\ \tilde\eta(k+1) - \tilde\nu(k))\}$ is a sequence of iid three-dimensional vectors on (Ω^o, G^o, P_i), $i = 0, 1$. The sequences $\{\tilde\gamma(k) - \tilde\eta(k)\}$ and $\{\tilde\nu(k) - \tilde\gamma(k)\}$ are independent and for any $a > 0$ and any $i = 0,1$

$$P_i\{\tilde\gamma(k) - \tilde\eta(k) > a\} = e^{-\lambda a}.$$

Proof. It follows from [2] Ch. VI that for each k the random variables $\eta(k)$, $\gamma(k)$ and $\nu(k)$ are stopping times and if $k > j$ then

$$\nu(j) \le \eta(k) < \gamma(k) \le \nu(k).$$

Let h be any bounded function of three variables. Since $\eta(k) \in \hat M$, using Proposition (3.5),

$$P\{h(\gamma(k)-\eta(k), \nu(k)-\gamma(k), \eta(k+1)-\nu(k)) \mid G_{\eta(k)+}\} = P_1\{h(\tilde\gamma, \tilde\nu-\tilde\gamma, \tilde\eta(2,0)-\tilde\nu)\}.$$

The above shows independence of $(\gamma(k) - \eta(k),\ \nu(k) - \gamma(k),\ \eta(k+1) - \nu(k))$ from the sequence $\{\gamma(j) - \eta(j), \nu(j) - \gamma(j),\ \eta(j+1) - \nu(j))\}$, $j = 1, 2, \ldots, k-1$.

Let g be a bounded function of one variable. Put $f(\omega^o) = g(\tilde\nu - \tilde\gamma)$. Then using Proposition (3.5)

$$\begin{aligned} &P\{g(\nu(k) - \gamma(k))\, 1_{>b}(\gamma(k) - \eta(k))\} \\ &= P\{1_{>b}(\gamma(k) - \eta(k)) f\circ\tau_{\eta(k)+b}\circ M\} \\ &= P\{1_{>b}(\gamma(k) - \eta(k))\}P_1\{f\}. \end{aligned} \tag{3.28}$$

The last equality in (3.28) is due to (3.4) and

$$\{\gamma(k) - \eta(k) > b\} \subset \{\eta(k) + b \in \hat M\}.$$

Likewise, setting $h(\omega^o) = 1_{>b}(\tilde\gamma)$

$$P\{\gamma(k) - \eta(k) > b\} = P\{h\circ\tau_{\eta(k)}\circ M\} = P_1\{h\} = P_1\{\tilde\gamma > b\}$$

and (3.27) follows from (3.23).

The proof of the second part of the proposition is done in a similar manner.

4. Deletion Operation and its Properties.

In this section we define an operator which removes parts of the set M in such a way that M becomes a regenerative set. Define $K:\Omega^o \to \Omega^o$ as

$$K(\omega^o) \triangleq \overline{\mathcal{K}(\omega^o)}, \tag{4.1}$$

where

$$\mathcal{K}(\omega^o) \triangleq \omega^o \backslash \text{closure}(\hat{\omega}^o) = \lim_{t\to-\infty} \omega^o \backslash \bigcup_{k=1}^{\infty} [\tilde{\eta}(k,t),\ \tilde{\gamma}(k,t)].$$

The operator $\mathcal{K}$ removes closure of the interior of ω^o, and the remaining set has no interior. Thus

$$\widehat{K(\omega^o)} = \emptyset\ .$$

(4.2) **PROPOSITION.** <u>For any</u> ω^o <u>and any</u> t

$$d_t \circ K(\omega^o) \bar{\in} \hat{\omega}^o\ . \tag{4.3}$$

Proof. Suppose (4.3) is wrong, then for some $k \geq 1$

$$d_t \circ K(\omega^o) \in [\tilde{\eta}(k,t),\ \tilde{\gamma}(k,t)[\ . \tag{4.4}$$

Since $d_t \circ K(\omega^o) \in K(\omega^o)$, the only way that (4.4) can be true is

$$d_t \circ K(\omega^o) = \tilde{\eta}(k,t)\ . \tag{4.5}$$

If $\tilde{\eta}(k,t) = t$ then (4.5) fails because in this case $d_t \circ K(\omega^o) \geq \tilde{\gamma}(k,t) > \tilde{\eta}(k,t)$. If $t < \tilde{\eta}(k,t)$ then (4.5) implies $]t,\tilde{\eta}(k,t)[\ \bar{\in}\ \mathcal{K}(\omega^o)$. Thus $\tilde{\eta}(k,t) \bar{\in} K(\omega^o)$ which contradicts (4.5).

(4.6) **THEOREM.** <u>The set</u> $K \circ M$ <u>is a stationary regenerative set.</u>

Proof. From a trivial relation

$$\widehat{\omega^o + s} = \hat{\omega}^o + s$$

it follows that

$$K \circ M + s = K(M+s). \tag{4.7}$$

Likewise

$$\tau_t \circ K \circ M = K \circ \tau_t \circ M \quad . \tag{4.8}$$

Relation (4.7) shows that stationarity of M implies stationarity of $K \circ M$.

Put $D_t' = d_t \circ K \circ M$. Then D_t' is a stopping time. By virtue of Proposition (3.5), Proposition (4.2) and (4.8), for any function $f \in bG^o$

$$P\{f \circ \tau_{D_t'} \circ K \circ M | G_{D_t'+}\} = P_0\{f \circ K \circ \tau_{D_t'} \circ M | G_{D_t'+}\} = P_0\{f \circ K\} \cdot \tag{4.9}$$

This proves that $K \circ M$ is regenerative with the law of regeneration

$$P = P_0 \circ K^{-1} \quad . \tag{4.10}$$

(4.11) **REMARK.** The proof of Theorem (4.6) shows that $K \circ M$ is regenerative with respect to the filtration $G_t' \overset{\Delta}{=} G_{D_t'+}$ which is larger than the natural filtration generated by $K \circ M$. It can be also shown that

$$P_0\{f \circ \tau_{d_t} \circ K | G^o_{d_t+}\} = P\{f \circ \tau_{d_t} | G^o_{d_t+}\} = P\{f\} \ . \tag{4.12}$$

We will call $K \circ M$ the regenerative part of the set M. By [6] the set $K \circ M$ is either perfect or discrete.

According to [7] and [12] their exists a process z_t with independent increments such that $K(\omega^o) = \overline{z_{\mathbb{R}_+}}$ for P_0 a.a. ω^o and such that the local time

$$\theta_s = (z^{-1})_s \overset{\Delta}{=} \inf\{t: z_t \geq s\} \tag{4.13}$$

is a continuous process adapted to the σ-field G^o_{t+} and for any $u \in z_{\mathbb{R}_+}$

$$\theta_{u+s} = \theta_u + \theta_s \circ \tau_u \ . \tag{4.14}$$

(4.15) **PROPOSITION.** <u>If</u> M <u>is a stationary Markov set with a perfect regenerative part then</u>

$$P_0\{0 \in \hat{\omega}^o\} = 0 \ .$$

Proof. Put

$$T_n'(\omega^o) = \inf\{æ_n(s):\ s \geq 0,\ u \bar{\epsilon}\, \omega^o \ \text{ for all } \ s < u \leq æ_n(s)\},$$

where $æ_n(s)$ is given by (3.8). Let

$$T_n(\omega^o) = \begin{cases} 0 \text{ if } 0 \in \hat{\omega}^o, \\ T_n'(\omega^o) \ \text{ otherwise.} \end{cases}$$

Then T_n is a sequence of stopping times such that

$$T_n = D_{T_n} = 0 \quad \text{on} \quad \{0 \in \hat{\omega}^o\}\,. \tag{4.16}$$

From [6] and [8] it follows that for any perfect regenerative set with the law of regeneration P

$$P\{0 \text{ is an isolated point in } \omega^o\} = 0\,. \tag{4.17}$$

On the other hand $K(\omega^o)$ and ω^o coincide in a neighborhood of 0 on $\{0 \bar{\epsilon}\, \hat{\omega}^o\}$. From (4.10) and (4.17) follows

$$P_0\{0 \bar{\epsilon}\, \hat{\omega}^o,\ \ 0 \text{ is an isolated point in } \omega^o\} = 0. \tag{4.18}$$

Combining (4.18) with (4.16) we get

$$d_{T_n} \downarrow 0 \quad \text{a.s. } P_0\,. \tag{4.19}$$

Take $f = g(r_{s_1}, r_{s_2}, \ldots, r_{s_k})$, where g is a positive bounded continuous function of k variables. By virtue of (4.19)

$$P_0\{f\} = \lim_n P_0\{f \circ \tau_{d_{T_n}}\} = \lim_n\{1_{T_n}(\hat{\omega}^o)\ P_1\{f\} + 1_{T_n \bar{\epsilon} \hat{\omega}^o}\ P_0\{f\}\}. \tag{4.20}$$

Suppose $P_0\{0 \bar{\epsilon}\, \hat{\omega}^o\} = \varepsilon > 0$. Then the right hand side of (4.20) converges to

$$\varepsilon P_1\{f\} + (1-\varepsilon) P_0\{f\},$$

which implies $P_0 = P_1$. The latter implies M is a regenerative set, and this is in contradiction with our assumption that M is the set of the third type.

Let $b(\omega^o)$ be the set of accumulation from the left points of ω^o, i.e. $x \in b(\omega^o)$ iff there exists a sequence $\{x_n\}$ such that $x_n < x$, $x_n \in \omega^o$ and $x_n \uparrow x$.

(4.21) **PROPOSITION.** If M has a perfect regenerative part then for each k and t

$$P\{\eta(k,t) \in b(M)\} = 1. \tag{4.22}$$

Proof. Suppose (4.22) fails. Then with a positive probability there exists an interval contiguous to M whose right end coincide with $\eta(k,t)$. Fubini's theorem implies an existance of u for which

$$P\{D_u = \eta(k,t),\ D_u > u\} > 0 . \tag{4.23}$$

Applying (3.4) to $f = 1_0(\hat{\omega}^o)$ and using (4.23), we get

$$P_0\{0 \in \hat{\omega}^o\} > 0,$$

which is in contradiction with proposition (4.15).

Put

$$\zeta_0 \overset{\Delta}{=} 0,$$

$$\zeta_k \overset{\Delta}{=} \theta_{\tilde{\eta}(k)} \equiv \theta_{\tilde{\gamma}(k)} \equiv \theta_{\tilde{\nu}(k)} ,$$

where θ_s is given by (4.13) and $\tilde{\eta}(k)$, $\tilde{\gamma}(k)$ and $\tilde{\nu}(k)$ stand for $\tilde{\eta}(k,0)$, $\tilde{\gamma}(k,0)$, and $\tilde{\nu}(k,0)$, given by (3.25).

(4.24) **PROPOSITION.** If M has a perfect regenerative part then $\zeta_k - \zeta_{k-1}$, $k = 1, 2, \ldots$ are exponential iid on $\{\Omega^o, G^o, P_0\}$.

Proof. Let $\tilde{\eta} \equiv \tilde{\eta}_0 \equiv \tilde{\eta}(1,0)$. Consider

$$P_0\{\zeta_1 > a + b\} = P_0\{\theta_{\tilde{\eta}} > a + b\} = P_0\{\theta_{\tilde{\eta}} > a,\ \theta_{\tilde{\eta}} - a > b\} .$$

Let

$$\sigma = \inf\{s: \theta_s \geq a\}. \tag{4.26}$$

Then σ is a stopping time with respect to G^o_{t+} and $\theta_\sigma = a$. Thus the right-hand side of (4.25) can be written as

$$P_0\{\theta_{\tilde{\eta}} > a,\ \theta_{\tilde{\eta}} - \theta_\sigma > b\} = P_0\{\theta_{\tilde{\eta}} > a,\ \theta_{\tilde{\eta}-\sigma} \circ \tau_\sigma > b\} \tag{4.27}$$

$$= P_0\{\theta_{\tilde{\eta}} > a,\ \theta_{\tilde{\eta}\circ\tau_\sigma} \circ \tau_\sigma > b\}$$

$$= P_0\{P_0\{\theta_{\tilde{\eta}\circ\tau_\sigma} \circ \tau_\sigma > b | G^o_{\sigma+}\};\ \theta_{\tilde{\eta}} > a\}$$

$$= P_0\{\theta_{\tilde{\eta}} > a\}\ P_0\{\theta_{\tilde{\eta}} > b\}\ .$$

The first equality in (4.27) is due to (4.14). The second equality holds because $\sigma > \tilde{\eta}$ and for any s $\tilde{\eta}\circ\tau_s = \tilde{\eta} - s$ on the set $\{s < \tilde{\eta}\}$. The last equality in (4.27) is a consequence of Proposition (3.10) and the equality

$$d_\sigma = \sigma$$

which is true for any perfect regenerative set and any σ given by (4.26). Equally (4.27) shows that ζ_1 has exponential distribution.

Since for any k

$$]\tilde{\eta}(k),\ \tilde{\nu}(k)[\ \bar{\in}\ K(\omega^o),$$

the quantities $\theta_{\tilde{\eta}(k)}$ and $\theta_{\tilde{\nu}(k)}$ coincide. Thus, in a way similar to the one in which (4.27) was obtained,

$$P_0\{\theta_{\tilde{\eta}(k+1)} - \theta_{\tilde{\eta}(k)} > a \mid G^o_{\tilde{\nu}(k)+}\}$$

$$= P_0\{\theta_{\tilde{\eta}(k+1)} - \theta_{\tilde{\nu}(k)} > a \mid G^o_{\tilde{\nu}(k)+}\}$$

$$= P_0\{\theta_{\tilde{\eta}(k+1)-\tilde{\nu}(k)} \circ \tau_{\tilde{\nu}(k)} > a \mid G^o_{\tilde{\nu}(k)+}\}$$

$$= P_0\{\theta_{\tilde{\eta}\circ\tau_{\tilde{\nu}(k)}} \circ \tau_{\tilde{\nu}(k)} > a \mid G^o_{\tilde{\nu}(k)+}\}$$

$$= P_0\{\theta_{\tilde{\eta}} > a\}\ .$$

The above equality shows that $\zeta_{k+1} - \zeta_k$ is independent of $\zeta_n - \zeta_{n-1}$, $n = 1, 2, \ldots, k$ and have the same distribution as ζ_1.

(4.28) **REMARK.** The proof of Proposition (4.24) also shows that $\zeta_k - \zeta_{k-1}$ is independent of the sequence of random vectors $(\tilde{\nu}(n) - \tilde{\gamma}(n), \tilde{\gamma}(n) - \tilde{\eta}(n))$, $n = 1, 2, \ldots$.

5. Structure of Stationary Markov Set.

In this section we will show that each stationary set M is $(\Pi, \alpha, \lambda, \mu)$-generated. This will be done separately for the case in which M has a perfect regenerative part and in the case in which M has a discrete regenerative part.

Suppose M is a set with a perfect regenerative part and $P_0 \cdot K^{-1}$ is its law of regeneration. Consider the process V_t on (Ω^o, G^o, P_0)

$$V_t = \sum_{k:\zeta_k < t} (\tilde{\nu}(k) - \tilde{\eta}(k)),$$

where $\zeta_k = \theta_{\tilde{\eta}(k)}$ with θ given by (4.13). The process V_t is of a pure jump type. In view of Proposition (4.24) $\zeta_k - \zeta_{k-1}$ are exponential iid. By virtue of the proposition (3.26) and Remark (4.28) the random variables $(\tilde{\nu}(k) - \tilde{\eta}(k))$ are iid independent of the point process ζ_k. Therefore, the process V_t is a process with independent increments.

Proposition (4.21) and Proposition (3.10) together with (4.18) show that $\tilde{\eta}(k)$ and $\tilde{\nu}(k)$ are points of accumulation of $K(\omega^o)$ a.s. P_0. This implies

$$z_{\zeta_k} = \tilde{\nu}(k), \quad z_{\zeta_k -} = \tilde{\eta}(k), \tag{5.1}$$

where z_t is the process whose image is equal to $K(\omega^o) \cap [0,\infty[$. From (5.1) and the definition of V_t follows

$$V_{\zeta_k} - V_{\zeta_k -} = z_{\zeta_k} - z_{\zeta_k -} \quad . \tag{5.2}$$

Put

$$W_t \overset{\Delta}{=} z_t - V_t \ .$$

Since both z_t and V_t are processes with independent increments, so is W_t. The set $K(\omega^o)$ has Lebesgue measure zero, therefore the process z_t has translation constant equal to zero and is of a pure jump type. In view of (5.2), W_t is an increasing process of a pure jump type such that

$$W_{\zeta_k -} = W_{\zeta_k} \ . \tag{5.3}$$

Relation (5.3) implies that $V_\bullet$ and $W_\bullet$ have no common points of discontinuity. Accordingly $V_\bullet$ and $W_\bullet$ are independent (see [11]).

(5.4) **THEOREM.** A stationary Markov set M with a perfect regenerative part is $(\Pi, \alpha, \lambda, \mu)$-generated.

Proof. For each t we need to find ϕ_t such that $M \cap [\phi_t, \infty[$ is a $(\Pi, \alpha, \lambda, \mu)$-set. In view of stationarity it is sufficient to consider only $t = 0$. Put $\phi = \nu$, where ν is given by (3.20). Let

$$\begin{aligned} X_k &\overset{\Delta}{=} \tilde{\gamma}(k) \circ \tau_\nu \circ M - \tilde{\eta}(k) \circ \tau_\nu \circ M \equiv \gamma(k+1,0) - \eta(k+1,0) , \\ Y_k &\overset{\Delta}{=} \tilde{\nu}(k) \circ \tau_\nu \circ M - \tilde{\gamma}(k) \circ \tau_\nu \circ M \equiv \nu(k+1,0) - \gamma(k+1,0) , \\ x_t &\overset{\Delta}{=} V_t \circ \tau_\nu + \phi , \\ y_t &\overset{\Delta}{=} W_t \circ \tau_\nu + \phi . \end{aligned} \tag{5.5}$$

Then for Z_t given by (2.7) we have

$$Z_t = z_t \circ \tau_\nu \circ M + \phi \ .$$

$$\sigma_k = \zeta_k \circ \tau_\nu \circ M$$

Let Π be the Levi's measure of the subordinator W. Let α be the parameter of the exponential distribution of $\zeta_k - \zeta_{k-1}$, λ be the parameter of the exponential distribution of $\gamma(k) - \eta(k)$ (see (3.27)) and

$$\mu(\Gamma) \overset{\Delta}{=} P\{\nu(k) - \gamma(k) \in \Gamma\} = P_0\{\tilde{\nu}(k) - \tilde{\gamma}(k) \in \Gamma\}.$$

We would like to show that the set $M \cap [\phi,\infty[$ is a (Π,λ,α,μ)-set as defined by (2.6) - (2.8). Since $\nu = D_\gamma$ and $\gamma \bar{\in} \hat{M}$, we can apply Proposition (3.5) and get

$$P\{f \circ M^\gamma | G_{\gamma+}\} = P\{f \circ \tau_\nu \circ M | G_{\gamma+}\} = P_0\{f\} . \qquad (5.6)$$

In particular, (5.6) shows that the law of $(V_\cdot \circ \tau_\nu \circ M,\ W_\cdot \circ \tau_\nu \circ M)$ on (Ω,F,P) is the same as the law of $(V_\cdot, W_\cdot)$ on (Ω^o,G^o,P^o). It also shows independence of ν and $(V_\cdot \circ \tau_\nu \circ M,\ W_\cdot \circ \tau_\nu \circ M)$.

For $M_1 \overset{\Delta}{=} \tau_\nu \circ M$ and for $z_t = V_t + W_t$, one has

$$K \circ M_1 = \overline{z_{\mathbb{R}_+} \circ M_1} .$$

Thus

$$\overline{z_{\mathbb{R}_+} \circ M_1 + \phi} = K \circ M \cap [\phi,\infty[.$$

The construction of the process V_t (and x_t by (5.5)) shows that L given by (2.7) coincides with the closure of $\hat{M} \cap [\phi,\infty[$. Since $M = K \circ M \cup \hat{M}$, we got the representation (2.8) with $x_\cdot$ and $y_\cdot$ given by (5.5). Proposition (4.24) shows that $\sigma_k = \zeta_k \circ \tau_\nu \circ M$ forms a Poisson point process. Proposition (3.26) and Remark (4.28) show the required independence of $\{X_k\}$, $\{Y_k\}$ and $y_\cdot$ as well as independence of $x_\cdot$ and $y_\cdot$ given by (5.5). This concludes the proof that $M \cap [\phi,\infty[$ is a (Π,α,λ,μ)-set.

A stationary Markov set with a discrete regenerative part cannot be treated in the same manner because Propositions (4.15) and (4.21) are no longer true in this case. As a result (5.1) and (5.2) as well as (5.3) might fail. The failure of (5.1) - (5.3) might result in dependence of the processes $V_\cdot$ and $W_\cdot$.

However, the case of a set with a discrete regenerative part can be treated "from scratch". The analysis of this case is rather simple, so we will only outline the main points without going into details.

Put

$$p = P_0\{0 \in \hat{\omega}^o\}. \tag{5.7}$$

It is easy to show that if M has a discrete regenerative part then $0 < p < 1$. Let λ be given by (3.23) and

$$\mu(\Gamma) = P_0\{\tilde{\nu} - \tilde{\gamma} \in \Gamma\} \tag{5.8}$$

$$\Pi(\Gamma) = P_0\{d_0 \in \Gamma \mid 0 \bar{\in} \hat{\omega}^o\} \tag{5.9}$$

Proposition (3.5) shows that the right endpoint of each interval contiguous to M belongs to $\hat{M}$ with probability p independently of the length of this interval. Thus $\tau_\nu \circ M$ can be described by means of a Markov renewal process $U(t)$ (see [1] Chapter 10) with three states. The holding time in the first state is exponential with parameter λ, the holding time in the second and third states have distribution μ and Π respectively. The transition matrix of the imbedded discrete Markov chain is

$$\begin{pmatrix} 0 & 1 & 0 \\ p & 0 & 1-p \\ p & 0 & 1-p \end{pmatrix}$$

The set of times when $U(t)$ undergoes transitions from one state to another or $U(t)$ is in the first state corresponds to $\tau_\nu \circ M$.

It is easy to verify that the set $M \cap [\nu,\infty[\ \equiv \nu + \tau_\nu \circ M$ is a (Π,α,λ,μ)-set where α is such that

$$p = \frac{\alpha}{\alpha+1}$$

(Note that if y_t is a $(0,\Pi)$-process and x_t is the process defined by (2.6), then $\alpha/(\alpha+1) = P\{\sigma_1 < \inf\{t: y_t \neq y_{t-}\}\}$).

(5.9) **PROPOSITION.** If M is a stationary (Π,α,λ,μ)-generated set then Π and μ satisfy (2.13), (2.14).

Proof. (For M with a perfect regenerative part, for M with a discrete regenerative part the proof is similar). Let x_t and y_t be the subordinators which generate (Π,α,λ,μ)-set (see (2.6)-(2.8)). Then

$$K\circ(\overline{Z_{\mathbb{R}_+}\cup L}) = \overline{Z}_{\mathbb{R}_+} .$$

The latter shows that the process $Z = x + y$ generates stationary regenerative set $K\circ M$. If Π' is the Levi's measure of Z then from [8] and [12]

$$\int_0^\infty x\,\Pi'(dx) < \infty \quad . \tag{5.10}$$

On the other hand it is known (see [11]) that

$$P\{Z_t - Z_0\} = t\int_0^\infty x\ \Pi'(dx) \tag{5.11}$$

The left hand side of (5.11) can be rewritten as

$$P\{y_t - y_0\} + P\{x_t - x_0\} = t\int_0^\infty x\ \Pi(dx) + t\,\alpha^{-1}[\lambda^{-1} + \int_0^\infty x\,\mu(dx)] \tag{5.12}$$

Relations (5.10), (5.11) (5.12) imply (2.13) and (2.14).

(5.13) **PROPOSITION.** If M is a (Π,α,λ,μ)-generated set, then the quadruple (Π,α,λ,μ) is determined by M uniquely up to equivalency.

Proof. The compliment of M consists of a union of open intervals (a,b). Since $M\cap[\phi_0,\infty[$ is a (Π,α,λ,μ)-set we can write (recalling representation (2.6) - (2.8)).

$$\begin{aligned} &P\{\sum_{\nu(1)\le a,\, b\le\eta(2)} f(b-a)\} \\ &= \lim_{k\to\infty} P\{\sum_{\nu(k)\le a,\, b\le \eta(k+1)} f(b-a)\} \end{aligned} \tag{5.14}$$

$$= Q\{ \sum_{\sigma_k < t < \sigma_{k+1}} f(Z_t - Z_{t-}) 1_{Z_t \neq Z_{t-}} \} \tag{5.14}$$

$$= Q\{ \sum_{\sigma_k < t < \sigma_{k+1}} f(y_t - y_{t-}) 1_{y_t \neq y_{t-}} \}$$

$$= \Pi\{f\} Q\{\sigma_{k+1} - \sigma_k\} = \alpha^{-1} \Pi\{f\} .$$

Here Q is the probability measure associated with the process x, y and Z in (2.6) - (2.8). Formula (5.14) shows that (Π, α) is determined by M uniquely up to proportionality.

On the other hand, direct computations show that for any (Π,α,λ,μ)-set

$$P\{\gamma - \eta\} = [\lambda(1-\alpha\,\mu\{0\} / (\alpha + \Pi(\mathbb{R}_+))]^{-1} \tag{5.15}$$

and for $\Gamma \subset]0,\infty[$

$$P\{\nu - \gamma \in \Gamma\} = \begin{cases} \mu\{\Gamma\} & \text{if } \Pi(\mathbb{R}_+) = \infty , \\ \mu'\{\Gamma\} + \mu\{0\}\,\Pi(\mathbb{R}_+)^{-1}\,\Pi(\Gamma), & \text{if } \Pi(\mathbb{R}_+) < \infty \end{cases} \tag{5.16}$$

(Here μ' is the restriction of μ on $]0,\infty[$)

Equalities (5.15) and (5.16) complete the proof of the proposition.

6. Markov Properties of the Residual Life Process

Consider the "residual life" process

$$R_t = \inf\{s - t: \ s > t, \ s \in M\} \tag{6.1}$$

associated with the stationary Markov set M. Markov property of M implies that R_t is a Markov (but not necessarily a strong Markov) process.

Consider a (Π,α,λ,μ)-set given by (2.7), (2.8) and the processes y., x. and Z. which generate it. (For definiteness we always choose a quadruple (Π,α,λ,μ) such that $\mu\{0\}=0$ if $\Pi(\mathbb{R}_+)<\infty$). Let

$$c_a \triangleq \inf\{s \geq 0: \ Z_s \geq a\}, \tag{6.2}$$

$$F_a = Z_{c_a} ,$$
$$H_a = Z_{c_a -} . \tag{6.3}$$

Let N be the union of σ_k given by (2.6). Then R_t given by (6.1) can be represented as

$$R_t = \begin{cases} 0 & \text{if } c_t \in N \text{ and } t \in L , \\ F_t - t & \text{otherwise} . \end{cases} \tag{6.4}$$

Let Q be the law of the subordinators $x_\circ$ and $y_\circ$ of (2.6) - (2.8). The transition function of R_t associated with a stationary (Π,α,λ,μ)-generated set is the same as transition function of R_t associated with any (Π,α,λ,μ)-set. Hence we can assume $Q\{y_0=0\}$ in (2.6) - (2.8). Then the transition function of the process R_t given by (6.4) is

$$\begin{aligned} p(t,x;\Gamma) &= 1_\Gamma(x) \quad \text{if } x<t, \\ p(t,x;\Gamma) &= Q\{F_t - t\in\Gamma, c_t \bar{\in} N\} + \sum_{k=1}^{\infty} Q\{F_t - t\in\Gamma, c_t=\sigma_k, Z_{\sigma_k -}+X_k \le t\}, \quad x>0, \ 0 \bar{\in} \Gamma , \\ p(t,x;\{0\}) &= \sum_{k=1}^{\infty} Q\{c_t=\sigma_k, Z_{\sigma_k -}+X_k > t\}, \quad x>0 , \\ p(t,0;\Gamma) &= e^{-\lambda t} 1_0\{\Gamma\} + \int_0^t \lambda e^{-\lambda y} \mu(\Gamma + t - y)dy + \int_0^t \mu_1(dy) p(t-y,y;\Gamma) , \end{aligned} \tag{6.5}$$

where μ_1 is a distribution of the jumps of the process $x_\circ$ (i.e., μ_1 is a convolution of μ and an exponential distribution with parameter λ).

A (Π,α,λ,μ)-generated set is stationary iff

$$m_t(\Gamma) \triangleq P\{R_t \in \Gamma\}$$

does not depend on t

Inversely, if we are able to construct a probability measure m which is invariant with respect to $p(t,x;\Gamma)$ then the stationary Markov process R_t with the one dimensional distribution m and the transition function p will yield a stationary (Π,α,λ,μ)-generated set by the formula

$$M = \overline{D_{\mathbb{R}}} ,$$

where $D_t = t + R_t$.

(6.6) **THEOREM.** If Π and λ are subject to (2.13), (2.14) then there exists a unique stationary probability measure m for the Markov process R_t associated with a (Π,α,λ,μ)-set. The measure m is given by

$$m(f) = C[\lambda^{-1}f(0) + \int_0^\infty f(t)\mu(]t,\infty[)dt + \alpha^{-1}\int_0^\infty f(t)\Pi(]t,\infty[)dt] .$$

where C is a normalizing constant.

For the proof of this theorem we need the following propostion.

(6.7) **PROPOSITION.** Let (y_s,Q) be a $(0,\Pi)$-process and let

$$c_a = \inf\{s : y_s \geq a\}.$$

Let S be an exponential random variable with parameter α independent of the process y_t. Then

$$Q\{\int_0^{y_S} f(y_{c_u}-u)du\} = \alpha^{-1}\int_0^\infty f(t)\,\Pi(]t,\infty[)dt . \tag{6.8}$$

Proof. The right-hand side of (6.8) can be rewritten as

$$\begin{aligned} &Q\{\sum_{y_s\neq y_{s-},\, s\leq S}\int_{y_{s-}}^{y_s} f(y_s-u)du\} \\ &= Q\{\sum_{y_s\neq y_{s-},\, s\leq S}\int_{y_{s-}}^{y_s} f(u)du\} \\ &= Q\{\sum_{y_s\neq y_{s-}} 1_{s\leq S}\, g(y_s-y_{s-})\} , \end{aligned} \tag{6.9}$$

where $g(t) = \int_0^t f(u)du$. The right hand side of (6.9) is equal to

$$Q\{\Pi(g)S\} = \Pi(g)\,Q\{S\} = \alpha^{-1}\Pi(g)$$

(see [11] Section 3). By Fibini's theorem

$$\Pi(g) = \int_0^\infty g(x)\,\Pi(dx) = \int_0^\infty\{\int_0^x f(t)dt\}\,\Pi(dx) = \int_0^\infty f(t)\int_t^\infty \Pi(dx) = \int_0^\infty f(t)\,\Pi(]t,\infty[)\,dt$$

whereas (6.8) follows.

Proof of the Theorem (6.6). Let $x.$ and $y.$ be the processes (with $Q\{y_0=0\}=1$) which generate a (Π,α,λ,μ)-set by formulae (2.6) - (2.8). Then the process R_t associated with this set by formula (6.4) is a regenerative process (see [1] Ch. 9) with the moments of regeneration $\rho_1, \rho_2, \dots, \rho_k, \dots$

$$\rho_k \overset{\Delta}{=} Z_{\sigma_k} .$$

Really, by virtue of the strong Markov property

$$x_s^k \overset{\Delta}{=} x_{\sigma_k+s} - x_{\sigma_k}$$

and

$$y_s^k = y_{\sigma_k+s} - y_{\sigma_k}$$

have the same distribution as the processes x_s and y_s respectively and are independent of $\{x_s, y_s;\ s \le \sigma_k\}$. Since

$$R_{\rho_k+t} = R_t \circ (x_\bullet^k, y_\bullet^k)$$

the process R_{ρ_k+t} is independent of $\{R_s,\ s \le \rho_k\}$ and has the same distribution as R_t. The same argument shows that the sequence $\{\rho_k\}$ forms a renewal process.

Since

$$\rho_{k+1} - \rho_k \equiv Z_{\sigma_{k+1}} - Z_{\sigma_k} = (y_{\sigma_{k+1}} - y_{\sigma_k}) + X_k + Y_k$$

and since X_k has a continuous (exponential) distribution, $\rho_{k+1} - \rho_k$ has a continuous distribution as well. Thus the renewal process $\{\rho_k\}$ is aperiodic and

$$E\{\rho_{k+1} - \rho_k\} = E\{X_k\} + E\{Y_k\} + E\{y_{\sigma_{k+1}} - y_{\sigma_k}\} \tag{6.10}$$

$$= \lambda^{-1} + \int_0^\infty t\ \mu(dt) + \alpha^{-1} \int_0^\infty t\ \Pi(dt)\ .$$

The right-hand side of (6.10) is finite by virtue of (2.13) and (2.14).

According to Theorem (2.25) of Chapter 9 of [1] there exists a unique stationary measure m for the regenerative process $R_\bullet$, given by

$$m(\Gamma) = C\ Q\{\int_0^{\rho_1} 1_\Gamma(R_t)dt\}, \tag{6.11}$$

where C^{-1} is equal to (6.10) (The expression in (6.11) is equivalent to the one given in Ch. 9, Theorem (2.25) of [1]). Since the process x_t is equal to 0 on the interval $]0,\sigma_1[$ the process Z_s coincides with y_s on $]0,\sigma_1[$ and

$$R_t = y_{c_t} - t \quad \text{for} \quad t \le y_{\sigma_1}$$

$$R_t = 0 \quad \text{for} \quad y_{\sigma_1} \le t < y_{\sigma_1} + X_1$$

$$R_t = Z_{\sigma_1} - t \equiv y_{\sigma_1} + X_1 + Y_1 - t \quad \text{for} \quad y_{\sigma_1} + X_1 < t < Z_{\sigma_1}$$

(see (2.6), (2.8) and (6.4)). Thus

$$Q\{\int_0^{\rho_1} 1_\Gamma(R_t)dt\} = Q\{\int_0^{y_{\sigma_1}} 1_\Gamma(y_{c_t}-t)dt\} + Q\{\int_0^{X_1} 1_\Gamma(0)dt\} + Q\{\int_0^{y_1} 1_\Gamma(Y_1-t)dt\} \tag{6.12}$$

The first term in the right-hand side of (6.12) equals to

$$\alpha^{-1} \int_0^\infty 1_\Gamma(t)\ \Pi(]t,\infty[)dt \tag{6.13}$$

by virtue of Proposition (6.7). The second term in (6.12) equals

$$1_\Gamma(0)\ E\ \{X_1\} = 1_\Gamma(0)\lambda^{-1} \tag{6.14}$$

The last term in (6.12) equals

$$Q\{\int_0^{Y_1} 1_\Gamma(Y_1-t)dt\} = Q\{\int_0^{Y_1} 1_\Gamma(t)dt\} = Q\{\int_0^\infty 1_\Gamma(t)1_{t<Y_1}dt\} = \int_0^\infty 1_\Gamma(t)\mu(]t,\infty[)dt. \tag{6.15}$$

From (6.11)-(6.15) follows Theorem (6.6).

From existence of a stationary measure for the process R_t follows

(6.16) **COROLLARY.** For each $(\Pi, \alpha, \lambda, \mu)$ subject to (2.13), (2.14) there exists a stationary $(\Pi, \alpha, \lambda, \mu)$-generated set.

This completes the proof of the Theorem (2.12).

(6.17) **REMARK.** The proof of Theorem (6.6) shows that any $(\Pi, \alpha, \lambda, \mu)$-generated set M with

$$P\{\inf M = -\infty\} = 1$$

is stationary.

7. Reversability Properties of Stationary Markov Sets.

In this section we will prove Theorem (2.15). We consider a stationary $(\Pi, \alpha, \lambda, \delta_0)$-generated set with a perfect regenerative part. The proof of Theorem (2.15) for M with a discrete regenerative part is similar. The closure of $\hat{M}$ consists of a union of closed intervals of iid exponential length and by virtue of Proposition (4.15) and (4.21) the endpoints of these intervals are points of accumulation of $M - \hat{M}$. Therefore the endpoints of these intervals belong to $K \circ M$.

According to Theorem (4.6) the set $K \circ M$ is a stationary regenerative set with Lebesgue measure zero. By Theorem 1 of [12] there exists a $(0, \Pi')$-subordinator whose range coincides with $K \circ M \cap [0,\infty[$. If $\mu = \delta_0$, then (2.6) - (2.8) show that Π' is the Levi's measure of the process Z and

$$\Pi' = \Pi + \alpha G_\lambda , \tag{7.1}$$

where G_λ is an exponential distribution with parameter λ. In particular, M has a perfect regenerative part iff

$$\Pi(\mathbb{R}_+) = \infty \quad .$$

To show that $-M$ has the same distribution as M we have to consider the two dimensional process

$$(L_t, R_t) = (\ell_t, r_t) \circ M \,.$$

It follows from the Markov property of M that the process (L_t, R_t) is a stationary Markov process. If M is a $(\Pi,\alpha,\lambda,\delta_0)$-generated set then the transition function of (L_t, R_t) is

$$\begin{aligned}
&p(t,(u,v);\Gamma) = 1_\Gamma(u,v), \quad \text{if} \quad v>t, \ \Gamma\subset]-\infty,0]\times[0,\infty[\,, \\
&p(t,(u,v);\Gamma) = Q\{(H_{t-v},F_{t-v})\in\Gamma, \ \ c_{t-v}\bar{\in} N\}, \quad v<t, \ \Gamma\subset]-\infty,0[\times]0,\infty[\\
&p(t,(u,v);\,(0,0)) = Q\{c_{t-v}\in N\} \equiv \sum_k Q\{c_{t-v}=\sigma_k\}, \ v<t \ , \qquad (7.2) \\
&p(t,(0,0);\Gamma) = \int_0^t \lambda e^{-\lambda s}\, p(t,(0,s),\Gamma)\, ds, \ \ \Gamma\subset]-\infty,0]\times[0,\infty[\quad .
\end{aligned}$$

Here Q is the law of the processes x_s, y_s and Z_s, of (2.6) - (2.8), c_t is given by (6.2) and (H_t, F_t) are given by (6.3).

Note that when $\mu=\delta_0$, the length of each jump of the process $Z_\bullet$ caused by a discontinuity of $x_\bullet$ is exponentially distributed and the range of each jump belongs to the $(\Pi,\lambda,\alpha,\delta_0)$-generated set. This results in simplifications in (7.2) as compared to (6.5) .

Let $\Pi(x;\Gamma) = \Pi(\Gamma-x)$. The process (L_t, R_t) is stationary due to stationarity of M. Repeating the proof of Theorem (6.6) for (L_t, R_t), we can get that the one-dimensional distribution n of this process is given by

$$n(\Gamma\times\Delta) = c[\lambda^{-1}\, 1_{(0,0)}(\Gamma\times\Delta) + \alpha^{-1}\int_{-\infty}^{0} 1_s(\Gamma)\, \Pi(x;\Delta)ds], \ \Gamma\subset\mathbb{R}_-,\ \Delta\subset\mathbb{R}_+\,. \qquad (7.3)$$

Let $(u,v)^T = (v,u)$ and if $\Delta \subset \mathbb{R} \times \mathbb{R}$ then Δ^T should be understood similarly.

Let

$$\begin{aligned} y_t^* &\triangleq -y_t, \quad x_t^* \triangleq -x_t , \\ Z_t^* &\triangleq -Z_t , \\ \Pi^*(\Gamma) &\triangleq \Pi(-\Gamma) , \\ \Pi^*(x;\Gamma) &\triangleq \Pi^*(\Gamma - x) = \Pi(-x;-\Gamma) . \end{aligned} \tag{7.4}$$

Consider the set $-M$. The process

$$(L_t^*,R_t^*) \triangleq (\ell_t,r_t) \circ (-M) = (-R_t,-L_t)$$

is a Markov process with the one-dimensional distribution (obtained by change of variables in (7.3)).

$$n^*(\Gamma\times\Delta) = C[\lambda^{-1} 1_{(0,0)}((-\Delta)\times(-\Gamma)) + \alpha^{-1}\int_0^\infty 1_s(\Delta)\,\Pi^*(s;\Gamma)ds], \ \Gamma\subset\mathbb{R}_-, \ \Delta\subset\mathbb{R}_+ \tag{7.5}$$

and the backward transition function

$$p^*(s,(u,v);\Gamma) = p(s,(-v,-u);\Gamma^T), \ \Gamma\subset]-\infty,0]\times[0,\infty[, \quad u\le 0\le v . \tag{7.6}$$

Let

$$\begin{aligned} \Lambda(\Gamma) &\triangleq Q\{\int_0^\infty 1_\Gamma(Z_s)ds\} , \\ \Lambda^*(\Gamma) &\triangleq Q\{\int_0^\infty 1_\Gamma(Z_s^*)ds\} = \Lambda(-\Gamma) , \\ \Lambda_b(\Gamma) &= \Lambda(\Gamma - b) , \\ -\Lambda_b^*(\Gamma) &= \Lambda^*(\Gamma - b) , \end{aligned} \tag{7.7}$$

(7.8) **PROPOSITION.** <u>For any function</u> f <u>of two variables</u>

$$Q\{\sum_{Z_{t-}\neq Z_t} f(Z_{t-},Z_t)1_{t\in\bar{N}}\} = \int_0^\infty \Lambda(dx)\int_0^\infty f(x,y)\,\Pi(x;dy)$$

$$Q\{_{Z^*_{t-}\neq Z^*_t} f(Z_{t-},Z_t)1_{t\in \bar{N}}\} = \int_{-\infty}^{0}\Lambda^*(dx)\int_{-\infty}^{0} f(x,y)\,\Pi^*(x;dy)$$

The proof of this proposition is well known.

(7.9) **PROPOSITION.** For any two functions g and h on $\mathbb{R}$

$$\int_{-\infty}^{\infty} g(x)\,\Pi(x,f)\,dx = \int_{-\infty}^{\infty} f(x)\,\Pi^*(x;g)\,dx \tag{7.10}$$

$$\int_{-\infty}^{\infty} \Lambda_x(f)g(x)\,dx = \int_{-\infty}^{\infty} f(x)\,\Lambda_x^*(g)\,dx \tag{7.11}$$

For the proof of this proposition see [12] Lemma 6.4 .

(7.11) **PROPOSITION.** Measures n and n^* given by (7.3) and (7.5) respectively, coincide.

Proof. The first term in brackets in the right-hand side of (7.3) is equal to the first term in the right-hand side of (7.5). The integral term in the right-hand side of (7.3) equals to the corresponding term in (7.5) by virtue of (7.10).

(7.12) **PROPOSITION.** For any two sets $\Gamma,\ \Delta\subset \mathbb{R}_-\times\mathbb{R}_+$

$$\int_{\Gamma} n(du,dv)p(s,(u,v);\Delta) = \int_{\Delta} n^*(du,dv)p^*(s,(u,v);\Gamma) \tag{7.13}$$

Proof. Consider Δ and Γ of the form

$$\begin{aligned} \Delta &= \Delta'\times\Delta'', \qquad \Delta'<0, \qquad \Delta''>0 \\ \Gamma &= \Gamma_1\times\Gamma_2 \qquad \Gamma_1<0, \qquad \Gamma_2>0 \end{aligned} \tag{7.14}$$

Put $\Delta_1=\Delta'+s$ and $\Delta_2=\Delta''+s$ and assume

$$\Gamma_2 < \Delta_1 \tag{7.15}$$

(The inequality between two subsets of the real line means the corresponding inequality between any two points from the first and the second set respectively.)

For $v < \Delta_1$ we can write

$$p(s,(u,v),\Delta) = Q\{H_{s-v} \in \Delta' + s - v,\ F_{s-v} \in \Delta'' + s - v,\ c_{s-v} \bar{\in} N\}$$

$$= Q\{\sum_{Z_{t-} \neq Z_t} 1_{\Delta_1 - v}(Z_{t-})\, 1_{\Delta_2 - v}(Z_t)\, 1_{t \bar{\in} N}\}$$

$$= \int_0^\infty \Lambda(dy)\, 1_{\Delta_1 - v}(y)\, \Pi(y;\Delta_2 - v) \qquad (7.16)$$

$$= \int_v^\infty \Lambda_v(dy)\, 1_{\Delta_1}(y)\, \Pi(y;\Delta_2)\ .$$

We used Proposition (7.8) in the third equality in (7.16) and the identity $\Pi(y - v;\Delta_2 - v) = \Pi(y,\Delta_2)$ in the fourth equality in (7.16). Thus the left hand side of (7.13) can be written as

$$\int_{-\infty}^{0} 1_{\Gamma_1}(s)\,ds \int_s^\infty \Pi(s;dv)\, 1_{\Gamma_2}(v) \int_v^\infty \Lambda_v(dy)\, 1_{\Delta_1}(y)\, \Pi(y;\Delta_2). \qquad (7.17)$$

Applying successively (7.10) then (7.11) and then again (7.10) and (7.17) we get the following sequence of equalities:

$$\int_\Gamma n(du,dv)p(s,(u,v),\Delta) = \int_0^\infty \Pi^*(v,\Gamma_1) 1_{\Gamma_2}(v)\,dv \int_v^\infty \Lambda_v(dy) 1_{\Delta_1}(y)\, \Pi(y;\Delta_2)$$

$$= \int_{-\infty}^{s} \Pi(y;\Delta_2) 1_{\Delta_1}(y)\,dy \int_{-\infty}^{y} \Lambda^*_y(dv) 1_{\Gamma_2}(v)\, \Pi^*(v;\Gamma_1) \qquad (7.18)$$

$$= \int_s^\infty 1_{\Delta_2}(x)\, \Pi^*(x;dy)\, 1_{\Delta_1}(y) \int_{-\infty}^{y} \Lambda^*_y(dv) 1_{\Gamma_2}(v)\, \Pi^*(v;\Gamma_1)\ .$$

From (7.5) and the analog of (7.16) for $p^*(\cdot(\cdot,\cdot),-)$ we get that (7.18) equals to the right-hand side of (7.13).

The proof of (7.13) for arbitrary Γ and Δ is done in a similar way.

(7.19) **COROLLARY.** The probability law of the set $-M$ is the same as that of M. In particular M is left Markov.

Proof. From proposition (7.11) we see that the processes $(\ell_t, r_t) \circ M$ and $(\ell_t, r_t) \circ (-M)$ have the same one dimensional distribuitons. Proposition (7.12) shows that these two processes have the same backward transition function. Therefore, these two processes have the same law. The rest follows from representation

$$M = \overline{(r \circ M)_{\mathbb{R}}} \quad .$$

In the remainder of this section we show why $(\Pi, \alpha, \lambda, \mu)$-generated set is not left Markov if $(\Pi, \alpha, \lambda, \mu)$ is not equivalent to $(\Pi_1, \alpha_1, \lambda_1, \delta_0)$.

If M has a perfect regenerative part (i.e., $\Pi(\mathbb{R}_+) = \infty$) then from (2.6) - (2.8) we see that the distribution of $\nu(k) - \gamma(k)$ is equal to μ. If $\mu \neq \delta_0$ then with positive probability $\nu(k) - \gamma(k) > 0$. By Fubini's theorem there exists t such that

$$P\{L_t < t,\ L_t = \gamma(k)\} > 0 \ .$$

However, the latter contradicts to Proposition (4.21) (or, to be precise, to the analog of the Proposition (4.21) for left Markov sets.)

If M has a discreet regenerative part (i.e. $\Pi(\mathbb{R}_+) < \infty$) and $(\Pi, \alpha, \lambda, \mu)$ is not equivalent to $(\Pi_1, \alpha_1, \lambda_1, \delta_0)$ then

$$\mu \neq c_1 \delta_0 + c_2 \Pi \ . \qquad (7.20)$$

From (2.6) - (2.8) we see that the distribution of the length of the jumps of the process y is $\Pi(\mathbb{R}_+)^{-1}\Pi$. The distribution of $\nu(k) - \gamma(k)$ is

$$\mu' + (1 - \mu\{0\})\, \Pi(\mathbb{R}_+^{-1})\, \Pi \qquad (7.21)$$

where μ' is a restriction of μ on $]0,\infty[$. If (7.20) is true then (7.21) is not equal to $\Pi(\mathbb{R}_+)^{-1}\, \Pi$. Elementary calculations show that in this case the conditional distribution of $R_t - L_t$ given the event

$$A \stackrel{\Delta}{=} \{L_t \in \text{closure } \hat{M},\ L_t > t\}$$

is different from the conditional distribution of $R_t - L_t$ given the compliment of A. The latter contradicts to the "left Markov" analog of Corollary (3.3). This completes the proof of Theorem (2.15).

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