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# A Maximal Inequality for Martingale Local Times

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## 1. Introduction

Let M and N be continuous local martingales, let $\hat{M}$, $\hat{N}$ denote $M-M_0$ and $N-N_0$ respectively, and let $L_t^a(M)$, $L_t^a(N)$ denote the local times of M and N respectively.

It was shown in [3] that

$$K_p \|\sup_a \sup_t |L_t^a(M)-L_t^a(N)|\ \|_p \geq \|\langle\hat{M}-\hat{N}\rangle_\infty^{\frac{1}{2}}\|_p ,$$

or equivalently,

$$c_p \|\sup_a \sup_t |L_t^a(M)-L_t^a(N)|\ \| \geq \|\ (\hat{M}-\hat{N})_\infty^*\|_p \tag{1.1}$$

for all $p\in(0,\infty)$, whilst Barlow and Yor established in [2] that

$$\|\sup_a \sup_t |L_t^a(M)-L_t^a(N)|\ \|_p \leq$$

$$C_p \|(M-N)_\infty^*\|_p^{\frac{1}{2}}\ \|M_\infty^*+N_\infty^*\|_p^{\frac{1}{2}}\ (1 \vee \ln \left\{\frac{\|M_\infty^*+N_\infty^*\|_p}{\|(M-N)_\infty^*\|_p}\right\})^{\frac{1}{2}}.$$

In this note we prove the following:

<u>Theorem 1</u> <u>For all</u> $p\in(1,\infty)$ <u>there is a universal constant</u> $c_p$ <u>such that for all continuous martingales</u> $M,N\in H^1$

$$\| \sup_a \sup_t |L_t^a(M) - L_t^a(N)| \|_p \le C_p \| \sup_a |L_\infty^a(M) - L_\infty^a(N)| \|_p .$$

2. Some preliminaries. We recall some properties of local times.

For a continuous semi-martingale $(X_t;\ t\ge 0)$ we may define (c.f. [1]) its family of local times by means of Tanaka's formula:

$$|X_t - a| = |X_0 - a| + \int_{0+}^t \operatorname{sgn}(X_s)\,dX_s + L_t^a(X)$$

where

$$\operatorname{sgn}(x) = \begin{cases} 1 & : x>0 \\ -1 & : x\le 0 \end{cases}$$

Note that $L_t^a(X)$ is increasing in t and increases only on $\{t:\ X_t = a\}$ (c.f. [4]).

Furthermore it has been shown in [5] that if X is a continuous local martingale then $L_t^a(X)$ has a bi-continuous version and we shall assume, without loss of generality, that we are working with such a version.

To simplify notation we fix M and N, two continuous martingales, and their filtration $(F_t;\ t\ge 0)$ and define

$$U(a,t) = (L_t^a(M) - L_t^a(N))$$

$$A_t = \sup_a (L_t^a(M) - L_t^a(N)) = \sup_a U(a,t)$$

$$B_t = \sup_a (L_t^a(N) - L_t^a(M)) = -\inf_a U(a,t)$$

$$D_t = \sup_a |L_s^a(M) - L_s^a(N)|$$

and for any $(X_t;\ t\geq 0)$

$$X_t^* = \sup_{s\leq t} |X_s|,\ \hat{X}_t = X_t - X_0.$$

3. Proof of Theorem 1. The crucial result is contained in the following lemma:

Lemma 2 Define

$$\sigma_x = \inf\{t\geq 0:\ A_t \geq 2x\}$$

$$\tau_x = \inf\{t\geq \sigma_x :\ U(M_{\sigma_x}, t)\leq x\}$$

where, as is usual $\inf\phi$ is taken as $\infty$ : then, if M and N are in $H^1$

$$\mathbb{E}[(2(\hat{M}-\hat{N})_\infty^* + A_\infty) I_{(\sigma_x<\infty,\ \tau_x=\infty)}] \geq x\ \mathbb{P}(\sigma_x<\infty) \qquad (3.1)$$

Proof It was shown in [3] that $A_t$ is continuous, so on $(\sigma_x<\infty)$, $A_{\sigma_x} = 2x$. Now M and N are in $H^1$ so $M_\infty^*, N_\infty^* < \infty$ a.s., so a.s. $U(a,\sigma_x)$ is zero off a compact set (since $L_t^a(X)$ only increases when X is at a) and continuous and we may conclude that $\sup_a U(a,\sigma_x)$ is attained.

We may deduce that, on $(\sigma_x<\infty)$, $\sup_a U(a,\sigma_x)$ is attained at $a=M_{\sigma_x}$ for, suppose not, then $\exists\, b\neq M_{\sigma_x}$ s.t. $2x = U(b,\sigma_x) > U(b,t)$ for all $t<\sigma_x$ but, since $b \neq M_{\sigma_x}$, $\exists\, t<\sigma_x$ s.t. $L_t^b(M) = L_{\sigma_x}^b(M)$ whilst (since $L_s^b(N)$ is increasing in s) $L_t^b(N) \le L_{\sigma_x}^b(N)$ so that $U(b,t) \ge U(b,\sigma_x)$ which contradicts the definition of $\sigma_x$. We conclude that, on $(\sigma_x<\infty)$, $U(M_{\sigma_x},\sigma_x) = 2x$ whilst M is in $H^1$ so has a limit variable $M_\infty$ and so

$$\mathbb{E}[U(M_{\sigma_x},\sigma_x) - U(M_{\sigma_x},\tau_x)] = \mathbb{E}[(2x-U(M_{\sigma_x},\tau_x))I_{(\sigma_x<\infty)}] \tag{3.2}$$

(since $\tau_x \ge \sigma$ so, on $(\sigma_x=\infty)$, $\sigma_x=\tau_x=\infty$).

Similarly, we may see that, on $(\tau_x<\infty)$, $U(M_{\sigma_x},\tau_x)=x$ so that (3.2) is

$$\mathbb{E}[2xI_{(\sigma_x<\infty)} - xI_{(\tau_x<\infty)} - U(M_{\sigma_x},\tau_x)I_{(\sigma_x<\infty,\tau_x=\infty)}] \tag{3.3}$$

Conversely, (3.2) is

$$\mathbb{E}[(L_{\tau_x}^{M_{\sigma_x}}(N) - L_{\sigma_x}^{M_{\sigma_x}}(N)) - (L_{\tau_x}^{M_{\sigma_x}}(M) - L_{\sigma_x}^{M_{\sigma_x}}(M))] \tag{3.4}$$

Applying Tanaka's formula to the two $(F_{\sigma_x+t} : t\ge 0)$ martingales, $m_t = M_{\sigma_x+t}$ and $n_t = N_{\sigma_x+t}$, we obtain the formulae

$$L_{\tau_x}^{M_{\sigma_x}}(M) - L_{\sigma_x}^{M_{\sigma_x}}(M) = L_{\tau_x-\sigma_x}^{M_{\sigma_x}}(m)$$

$$= |M_{\tau_x} - M_{\sigma_x}| + \int_{\sigma_x}^{\tau_x} \operatorname{sgn}(M_s-M_{\sigma_x})dM_s \tag{3.5.i}$$

$$L_{\tau_x}^{M_{\sigma_x}}(N) - L_{\sigma_x}^{M_{\sigma_x}}(N) = L_{\tau_x-\sigma_x}^{M_{\sigma_x}}(n)$$

$$= |N_{\tau_x}-M_{\sigma_x}| - |N_{\sigma_x}-M_{\sigma_x}|$$

$$+ \int_{\sigma_x}^{\tau_x} \operatorname{sgn}(N_s-M_{\sigma_x})dN_s \tag{3.5.ii}$$

Now M and N are in $H^1$ and $|\mathrm{sgn}(x)| = 1$ so the two stochastic integrals in (3.5) are uniformly integrable and so we may apply the optional sampling theorem to obtain:

$$\mathbb{E}[L_{\tau_x}^{M_{\sigma_x}}(M) - L_{\sigma_x}^{M_{\sigma_x}}(M)] = \mathbb{E}|M_{\tau_x} - M_{\sigma_x}| \tag{3.6.i}$$

$$\mathbb{E}[L_{\tau_x}^{M_{\sigma_x}}(N) - L_{\sigma_x}^{M_{\sigma_x}}(N)] = \mathbb{E}(|N_{\tau_x} - M_{\sigma_x}| - |N_{\sigma_x} - M_{\sigma_x}|) \tag{3.6.ii}$$

Substituting equations (3.6) in (3.4), and equating (3.2), (3.3) and (3.4) we see that

$$\mathbb{E}[2xI_{(\sigma_x<\infty)} - xI_{(\tau_x<\infty)} - U(M_{\sigma_x},\tau_x)I_{(\sigma_x<\infty,\tau_x=\infty)}]$$

$$= \mathbb{E}[|N_{\tau_x} - M_{\sigma_x}| - |N_{\sigma_x} - M_{\sigma_x}| - |M_{\tau_x} - M_{\sigma_x}|] \tag{3.7}$$

Now, by a similar argument to that given above, we may see that, on $(\tau_x<\infty)$, $N_{\tau_x} = M_{\sigma_x}$, so on $(\tau_x<\infty)$ the term inside the expectation on the RHS of (3.7) is non-positive whilst on $(\sigma_x=\infty)$ it disappears so that the RHS is dominated by

$$\mathbb{E}[(|N_\infty - M_\infty| - |N_{\sigma_x} - M_{\sigma_x}|)I_{(\sigma_x<\infty,\tau_x=\infty)}]$$

Observing that $|X_\infty| - |X_{\sigma_x}| \le 2\hat{X}_\infty^*$ and rearranging terms in (3.7) we achieve the inequality:

$$\mathbb{E}[(U(M_{\sigma_x},\tau_x) + 2(\hat{M}-\hat{N})_\infty^*)I_{(\sigma_x<\infty,\tau_x=\infty)})$$

$$\ge 2x\,\mathbb{P}(\sigma_x<\infty) - x\,\mathbb{P}(\tau_x<\infty) \tag{3.8}$$

All that remains, to complete the proof, is to see that, since $\tau_x \geq \sigma_x$, $\mathbb{P}(\tau_x<\infty) \leq \mathbb{P}(\sigma_x<\infty)$, whilst on $(\tau_x=\infty)$ $U(M_{\sigma_x},\tau_x) = U(M_{\sigma_x},\infty) \leq A_\infty$. □

<u>Lemma 3</u> If M and N are martingales in $H^1$

$$\mathbb{E}(2(\hat{M}-\hat{N})^*_\infty + A_\infty)I_{(A_\infty \geq x)} \geq x\,\mathbb{P}(A^*_\infty \geq 2x) \tag{3.9}$$

<u>Proof</u> On $(\sigma_x<\infty,\tau_x=\infty)$, $A_\infty \geq x$ whilst $(\sigma_x<\infty)=(A^*_\infty \geq 2x)$ so (3.9) follows immediately from (3.1). □

We may now establish the theorem:

<u>Proof of the theorem</u>: multiplying both sides of (3.9) by $px^{p-2}$ and integrating with respect to x we obtain, by Fubini's theorem:

$$\frac{p}{p-1}\,\mathbb{E}(2(\hat{M}-\hat{N})^*_\infty + A_\infty)A^{p-1}_\infty \geq \mathbb{E}(A^*_\infty)^p/2^p \qquad (3.10)_A$$

whilst reversing the roles of M and N in (3.9) we obtain:

$$\frac{p}{p-1}\,\mathbb{E}(2(\hat{M}-\hat{N})^*_\infty + B_\infty)B^{p-1}_\infty \geq \mathbb{E}(B^*_\infty)^p/2^p \qquad (3.10)_B$$

Clearly $D_t = A_t \vee B_t$, so that, since $A_t$ and $B_t$ are non-negative,

$$2D^p_t \geq A^p_t + B^p_t \geq D^p_t.$$

Thus, adding $(3.10)_A$ and $(3.10)_B$,

$$\frac{2p}{(p-1)} \; \mathbb{E}[(2(M-N)^*_\infty + D_\infty)D_\infty^{p-1}] \geq \mathbb{E}(D^*_\infty)^p/2^p$$

Applying Holder's inequality to the first term on the left, we obtain,

$$\frac{2^{p+1}p}{(p-1)} \; (2||(\hat{M}-\hat{N})^*_\infty||_p \; (||D_\infty||_p)^{p-1} + \mathbb{E}\, D_\infty^p) \geq \mathbb{E}(D^*_\infty)^p \tag{3.11}$$

Now, by (1.1), $||(\hat{M}-\hat{N})^*_\infty||_p \leq c_p||D^*_\infty||_p$, so substituting this inequality in (3.11):

$$\frac{2^{p+1}p}{(p-1)} \; (||D_\infty||_p^p + 2c_p||D^*_\infty||_p||D_\infty||_p^{p-1}) \geq ||D^*_\infty||_p^p \; , \tag{3.12}$$

and dividing both sides of (3.12) by $||D_\infty||_p^p$ we obtain the result that

$$||D^*_\infty||_p \leq K_p||D_\infty||_p$$

where $K_p$ is the largest zero of

$$f_p(x) = x^p - \frac{2^{p+1}p}{(p-1)} \; (2c_p \; x+1) \qquad \square$$

<u>Corollary 4</u> If M is in $H^1$ then for all $p\in(1,\infty)$, $a\in \mathbb{R}$

$$||(M-M_0)^*_\infty||_p \leq \frac{K_p}{2} \inf_{x\in \mathbb{R}} \; ||\sup_a|L^a_\infty(M) - L^{x-a}_\infty(M)|||_p$$

This follows immediately from theorem 1 and (1.1) by setting N = x-M.

Remarks

(1) Theorem 8 of [1] enables us to extend the range of p in Theorem 1 to $(1,\infty]$.

(2) Corollary 4 is a specific case of the more general result that
$$||(\hat{M}-\hat{N})^*_\infty||_p \le K_p \inf_{x\in \mathbb{R}} ||\sup_a |L^a_\infty(M) - L^{a-x}_\infty(N)|||_p .$$

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