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A SIMPLE PROOF OF THE LOGARITHMIC SOBOLEV INEQUALITY ON THE CIRCLE

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The purpose of this short note is to provide a simple and direct proof of the logarithmic Sobolev inequality on the circle $\mathbb{R}/2\pi\mathbb{Z}$, which we denote here and henceforth by E. Recall that if E is equipped with the uniform law $dx/2\pi$ and if $\langle f\rangle = \int_E f(x)\,dx/2\pi$ denotes the expectation for this law, then the logarithmic Sobolev inequality states that

$$(*)\qquad \langle f^2 \log f^2\rangle \leq \langle f^2\rangle \log \langle f^2\rangle + 2\langle f'^2\rangle\ ;$$

this holds for all f such that it makes sense : f' is in $L^2(dx/2\pi)$ and must satisfy $\langle f'\rangle = 0$; f is continuous and determined up to a constant by f'.

This inequality was proved by Weissler [5] by a direct but somewhat complicated calculation, and also by Rothaus [3] by a variational method. By considering the Brownian motion semi-group on the circle we provide a very simple proof of (*). (We note that (*) is equivalent to the hypercontractivity property of the Brownian motion semi-group). For additional related work concerning logarithmic Sobolev inequalities, we refer the reader to Gross [2] and Rothaus [4].

Proof. We note that it suffices to prove (*) when f is a C^∞ function. Actually, we will show for f in the class C^∞ and strictly positive that

$$(**)\qquad \langle f\log f\rangle - \langle f\rangle\log\langle f\rangle \leq \frac{1}{2}\left\langle\frac{f'^2}{f}\right\rangle\ ;$$

replacing f by $f^2+\varepsilon$ and letting ε tend to zero yields the claimed result (*).

Recall that the Brownian motion semi-group $(P_t)_{t>0}$ on $C^\infty(E)$ can be defined by

$$P_t f(x) = \int_E f(x+y) \sum_{n\in\mathbb{Z}} (4\pi t)^{-1/2} e^{-(y+2\pi n)^2/4t}\,dy,\quad f\in C^\infty(E),$$

and has the properties

$$\frac{d}{dt}P_t f = (P_t f)'' = P_t(f''),$$

$$P_\infty(f) = \langle f\rangle, \text{ and}$$

$$P_0 f = f.$$

To prove (**), notice that for all $f > 0$

$$\frac{d}{dt}\left(e^{2t}\left\langle\frac{(P_t f')^2}{P_t f}\right\rangle\right) = e^{2t}\left\langle\frac{2g'^2}{g} + \frac{2g'''g'}{g} - \frac{g'^2 g''}{g^2}\right\rangle$$

$$= 4e^{2t}\left\langle h'^2 + h'h''' + \frac{2h'^2h''}{h} - \frac{h'^4}{h^2}\right\rangle,$$

where we have made the successive substitutions $g = P_t f$ and $h^2 = 2g$. Integration by parts shows that $\langle h'h'''\rangle = -\langle h''^2\rangle$ and $3\langle\frac{h'^2h''}{h}\rangle = -\langle h'^3(\frac{1}{h})'\rangle = \langle\frac{h'^4}{h^2}\rangle$, whence we deduce

$$\frac{d}{dt}\left(e^{2t}\left\langle\frac{(P_t f')^2}{P_t f}\right\rangle\right) = -4e^{2t}\left\langle h''^2 - h'^2 + \frac{h'^4}{3h^2}\right\rangle \le 0,$$

since $\langle h''^2\rangle \ge \langle h'^2\rangle$ (expand h into a Fourier series). Consequently

$$\left\langle\frac{(P_t f')^2}{P_t f}\right\rangle \le e^{-2t}\left\langle\frac{(P_0 f')^2}{P_0 f}\right\rangle = e^{-2t}\left\langle\frac{f'^2}{f}\right\rangle.$$

In accordance with the notation developed in [1], set $U(x) = x\log x$; using the methods of [1] as well as the above inequality we deduce

$$\langle U\circ f\rangle - U(\langle f\rangle) = \langle U\circ P_0 f\rangle - \langle U\circ P_\infty f\rangle$$

$$= -\int_0^\infty \frac{d}{dt}\langle U\circ P_t f\rangle dt$$

$$= -\int_0^\infty \langle (P_t f)'' U'\circ P_t f\rangle dt$$

$$= \int_0^\infty \langle (P_t f)'(U'\circ P_t f)'\rangle dt$$

$$= \int_0^\infty \langle (P_t f')^2 U''\circ P_t f\rangle dt$$

$$= \int_0^\infty \left\langle\frac{(P_t f')^2}{P_t f}\right\rangle dt$$

$$\leq \int_0^\infty e^{-2t}\left\langle\frac{f'^2}{f}\right\rangle dt$$

$$= \frac{1}{2}\left\langle\frac{f'^2}{f}\right\rangle .$$

This is the desired conclusion and the proof is complete.

We should remark that above method does not seem to generalize to higher dimensions, the main difficulty being the lack of a suitable substitute for the integration by parts formula

$$3\left\langle\frac{h'^2h''}{h}\right\rangle = \left\langle\frac{h'^4}{h^2}\right\rangle .$$

Finally, notice also that a computation quite similar to the first part of the proof shows (by letting $h = e^{f/2}$) that for all $f \in C^\infty$

$$\langle e^f(f''^2-f'^2)\rangle \geq 0;$$

this implies the logarithmic Sobolev inequality (see Corollary 1 in [1]), but we don't know if it is strictly stronger or equivalent to it.

References

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