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# FUNCTIONALS ASSOCIATED WITH SELF-INTERSECTIONS OF THE PLANAR BROWNIAN MOTION[1]

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## ABSTRACT

For every k=1,2,3,...and for a wide class of measures $\lambda$,we construct a one-parameter family $\mathcal{T}_k(\lambda,u),u\geq 0$ of functionals of the planar Brownian motion $(X_t,P_\mu)$ related to its self-intersections of multiplicity k during the time interval [0,u]. We investigate various families of functionals which converge to $\mathcal{T}_k(\lambda,u)$ and we evaluate the moment functions $P_\mu[\mathcal{T}_{k_1}(\lambda_1,u)\ldots\mathcal{T}_{k_n}(\lambda_n,u)]$.

## 1.MAIN RESULTS

1.1. We denote by $(X_t,P_\mu)$ the Brownian motion in $R^2$ with the initial law $\mu$ (which can be any $\sigma$-finite measure on $R^2$). If $0<t_1<\ldots<t_n$,then the joint probability density for $X_{t_1},\ldots,X_{t_n}$ is given by the formula

$$(1.1)\quad p_\mu(t,x)=\int\mu(dx_0)p_{t_1}(x_1-x_0)p_{t_2-t_1}(x_2-x_1)\ldots p_{t_n-t_{n-1}}(x_n-x_{n-1}).$$

Here

$$(1.2)\qquad p_t(x)=t^{-1}p(x/\sqrt{t}),\quad p(z)=(2\pi)^{-1}e^{-|z|^2/2}.$$

Put

$$(1.3)\qquad G_r(x)=\int_0^\infty e^{-rt}p_t(x)dt,$$

$$g_r(x_0)=1,\quad g_r(x_0,x_1,\ldots,x_M)=G_r(x_1-x_0)\ldots G_r(x_M-x_{M-1})\text{ for }M\geq 1.$$

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We drop the subscript r if it is equal to 1.

We write $f \simeq g$ if $f(\epsilon)-g(\epsilon)=O(|\epsilon|^{\alpha})$ for every $0<\alpha<2$ as $\epsilon\downarrow 0$. (If $\epsilon$ is a vector $(\epsilon_1,\ldots,\epsilon_n)$, then $|\epsilon|=\max|\epsilon_i|$.) We also introduce an equivalence relation for Brownian functionals depending on parameters u and $\epsilon$: $Y\approx Z$ means that

$$\int_0^{\infty} du\ e^{-ru}\ P_{\mu}[|Y_{\epsilon u}-Z_{\epsilon u}|^p]\simeq 0$$

for every $r>0$ and every $p\geq 2$. (Of course this relation depends on $\mu$.)

A special role in our investigation is played by a function

$$h_{\epsilon}=\frac{1}{\pi}\ln\frac{1}{\epsilon} \tag{1.4}$$

and by a one-parameter group of fractional linear transformations in $\mathbb{R}$

$$\phi_h(w)=(w^{-1}+h)^{-1}=\frac{w}{1+hw}. \tag{1.5}$$

1.2. We say that a pair of measures $(\mu,\lambda)$ on $\mathbb{R}^2$ is *admissible* if:

(a) $\lambda$ has a bounded density;

(b) either $\mu$ is finite or $\lambda$ is finite and $\mu$ has a bounded Hölder continuous density.

We consider finite sequences $b=(b_1,\ldots,b_M)$ of elements taken from the set $\{1,\ldots,n\}$ subject to the condition: $b_j\neq b_{j+1}$ for $j=1,\ldots,M-1$. We call them *routes*. We note that if $(\mu,\lambda_i)$ is an admissible pair for $i=1,\ldots,n$, then:

1.2.A. For every route b and every $r>0$

$$g_{rb}(\mu,\lambda_1,\ldots,\lambda_n)=\int\mu(dx_0)\lambda_1(dx_1)\ldots\lambda_n(dx_n)g_r(x_0,x_{b_1},\ldots,x_{b_M})<\infty. \tag{1.6}$$

We put

$$G_{\mu r}(x)=\int G_r(y-x)\mu(dy). \tag{1.7}$$

1.3. We start from a probability density $q(z)$ on $\mathbb{R}^2$ such that

$$\int|\ln|x||^k q(x)dx<\infty \quad \text{for all } k>0, \tag{1.8}$$
$$\int e^{\beta|x|}q(x)dx<\infty \quad \text{for some } \beta>0.$$

Put

$$q^{\epsilon}(x)=\epsilon^{-2}q(x/\epsilon) \tag{1.9}$$

and consider a sequence of functionals

$$(1.10)\quad T_k(\epsilon,\lambda,u)=\int_{D_k(u)} dt_1\ldots dt_k\ \rho(X_{t_1})q^{\epsilon}(X_{t_2}-X_{t_1})\ldots q^{\epsilon}(X_{t_k}-X_{t_{k-1}}),$$

$$k=1,2,\ldots$$

Here $\lambda(dx)=\rho(x)dx$ and

$$(1.11)\qquad D_k(u)=\{0<t_1<\ldots,t_k<u\}.$$

**Theorem 1.1.** *Let* $(\mu,\lambda)$ *be an admissible pair of measures and let* q *satisfy condition* (1.8). *There exist functionals* $\mathcal{T}_k(\lambda,u)$ (*independent of* q) *such that*

$$(1.12)\qquad \mathcal{T}_k(\epsilon,\lambda,u)\approx\mathcal{T}_k(\lambda,u).$$

*Here*

$$(1.13)\qquad \mathcal{T}_k(\epsilon,\lambda,u)=\sum_{\ell=1}^{k}\begin{bmatrix}k-1\\ \ell-1\end{bmatrix}(\kappa-h_\epsilon)^{k-\ell}\ T_\ell(\epsilon,\lambda,u),$$

$$(1.13a)\qquad \kappa=\frac{1}{\pi}\int[C+\ln\frac{|y|}{\sqrt{2}}]\ q(y)\ dy,$$

C=.5772157... *is Euler's constant.*

1.4. Putting $\{F(T)\}=\sum_1^n a_kT_k$ for every polynomial $F(T)=\sum_1^n a_kT^k$, we rewrite formula (1.13) in a compact form

$$\mathcal{T}_k=\{T(T+\kappa-h_\epsilon)^{k-1}\}.$$

We note that

$$(1.14)\qquad \phi_h(w)^{\ell}=\sum_{k=\ell}^{\infty}\begin{bmatrix}k-1\\ \ell-1\end{bmatrix}w^k(-h)^{k-\ell}$$

and therefore we get from (1.13) the following equation for generating functions

$$(1.15)\qquad \sum_{k=1}^{\infty}\mathcal{T}_k(\epsilon,\lambda,u)w^k=\sum_{\ell=1}^{\infty}\phi_{h_\epsilon-\kappa}(w)^{\ell}T_\ell(\epsilon,\lambda,u)$$

or

$$(1.16)\qquad \sum_{\ell=1}^{\infty}\mathcal{T}_\ell(\epsilon,\lambda,u)\phi_{\kappa-h_\epsilon}(v)^{\ell}=\sum_{k=1}^{\infty}v^kT_k(\epsilon,\lambda,u).$$

By comparing coefficients at $v^k$ and then taking into account (1.12), we get

$$(1.17)\quad T_k(\epsilon,\lambda,u)=\sum_{\ell=1}^{\infty}\begin{bmatrix}k-1\\ \ell-1\end{bmatrix}(h_\epsilon-\kappa)^{k-\ell}\mathcal{T}_\ell(\epsilon,\lambda,u)\approx\sum_{k=1}^{\infty}\begin{bmatrix}k-1\\ \ell-1\end{bmatrix}(h_\epsilon-\kappa)^{k-\ell}\mathcal{T}_\ell(\lambda,u).$$

1.5. Let

$$(1.18)\qquad T(\epsilon,z,u)=\int_0^u q^\epsilon(X_t-z)\,dt$$

We consider a sequence

$$(1.19)\qquad T^k(\epsilon,\lambda,u)=\frac{1}{k!}\int\lambda(dz)T(\epsilon,z,u)^k$$

$$=\int\lambda(dz)\int_{D_k(u)}q^\epsilon(X_{t_1}-z)\dots q^\epsilon(X_{t_k}-z)\,dt_1\dots dt_k,\qquad k=1,2,\dots$$

and we renormalize it by the formula

$$(1.20)\qquad \mathcal{T}^k(\epsilon,\lambda,u)=\sum_{\ell=1}^{k}L_{k\ell}(h_\epsilon)\;T^\ell(\epsilon,\lambda,u),\qquad k=1,2,\dots$$

where $L_{k\ell}$ is a polynomial with the leading term $h^{k-\ell}$.

**Theorem 1.2.** *Suppose that* $(\mu,\lambda)$ *and* q *satisfy conditions of Theorem 1.1 and let* $\mathcal{T}_k(\lambda,u)$ *be the functionals described there. Polynomials* $L_{k\ell}$ *can be chosen in such a way that*

$$(1.21)\qquad \mathcal{T}^k(\epsilon,\lambda,u)\approx\mathcal{T}_k(\lambda,u).$$

*Namely,*

$$(1.22)\qquad \mathcal{Y}[\phi_h(w)]^\ell=\sum_{k=\ell}^{\infty}L_{k\ell}(h)w^k.$$

*To describe* $\mathcal{Y}$ *we consider independent random variables* $Y_1,\dots,Y_n,\dots$ *with the probability distribution* q(x)dx *and we put*

$$(1.23)\qquad \varphi_j=-\frac{1}{\pi}\,[C+\ln(|Y_j-Y_{j+1}|/\sqrt{2})],$$

$$(1.24)\qquad \mathcal{Q}(v)=v[1+\sum_{1}^{\infty}v^nE(\varphi_1\dots\varphi_n)].$$

*The power series* $\mathcal{Y}(w)$ *is uniquely determined by either of two conditions*

$$(1.25)\qquad \mathcal{Y}[\mathcal{Q}(w)]=w\quad or\quad \mathcal{Q}[\mathcal{Y}(v)]=v.$$

1.6. The same argument as in subsection 1.4 shows that

$$(1.26)\qquad \sum_{k=1}^{\infty}\mathcal{T}^k(\epsilon,\lambda,u)w^k=\sum_{\ell=1}^{\infty}\mathcal{Y}[\phi_{h_\epsilon}(w)]^\ell T^\ell(\epsilon,\lambda,u)$$

or

$$(1.27) \qquad \sum_{\ell=1}^{\infty} \mathcal{T}^{\ell}(\epsilon,\lambda,u)\phi_{-h_\epsilon}[\mathcal{Q}(v)]^{\ell} = \sum_{k=1}^{\infty} v^k T^k(\epsilon,\lambda,u).$$

We get from (1.27) the following asymptotic decomposition

$$(1.28) \qquad T^k(\epsilon,\lambda,u) \approx \sum_{\ell=1}^{k} M_{k\ell}(h_\epsilon)\ \mathcal{T}_{\ell}(\lambda,u)$$

where $M_{k\ell}$ are polynomials defined by the formula

$$(1.29) \qquad \sum_{k=\ell}^{\infty} M_{k\ell}(h)v^k = \phi_{-h}[\mathcal{Q}(v)]^{\ell}.$$

We note that for $n\geq k$, $M_{nk}$ is a polynomial of degree n-k with the leading term $h^{n-k}$(for $n<k, M_{nk}=0$).

1.7. We denote by $\ell_i=\ell_i(b)$ the number of elements equal to i in a route $b=(b_1,\ldots,b_M)$ and we denote by $\mathfrak{B}_k$ the set of all routes for which $1\leq\ell_i\leq k_i$, $i=1,\ldots,n$.

For every n=0,1,2,... there exists a unique polynomial $\mathcal{P}_n$ such that

$$(1.30) \qquad \int\mathcal{P}_n(\log t)\ e^{-rt}dt=\left[-\frac{\ln r}{2\pi}\right]^n r^{-1}.$$

**Theorem 1.3.** *For every* $k_1,\ldots,k_n\geq 1$

$$(1.31) \qquad P_{\mu}\left[\mathcal{T}_{k_1}(\lambda_1,u_1)\ldots\mathcal{T}_{k_n}(\lambda_n,u_n)\right]=m_k(\lambda,u)$$

*where*

$$(1.32) \qquad m_k(\lambda,u) =\sum_{b\in\mathfrak{B}_k} a(k,b)\int\lambda(dz)\int_{D_M(u)} p_{\mu}(t_1,z_{b_1};\ldots;t_M,z_{b_M})\mathcal{P}_{\nu}[\log(u-t_M)]dt$$

*with*

$$(1.33) \qquad a(k,b)=\prod_{j=1}^{n}\binom{k_j-1}{\ell_j-1}, \quad \nu=\sum_{1}^{n}(k_j-\ell_j);$$

$$(1.34) \qquad \lambda(dz)=\lambda_1(dz_1)\ldots\lambda_M(dz_M), \qquad dt=dt_1\ldots dt_M.$$

1.8. All the stated results follow from Theorem 1.4. In this theorem we deal simultaneously with several density functions q and, to avoid confusion, we write q as an extra argument for functions which depend on q.

**Theorem 1.4.** *Suppose that densities* $q_1,\dots,q_n$ *satisfy condition* (1.8) *and* $(\mu,\lambda_i)$ is an admissible pair of measures for $i=1,\dots,n$. *Let* $1\le m\le n$. *Put*

$$\mathcal{C}_i(h,v)=\phi_{\kappa(q_i)-h}(v) \qquad \text{for } i=1,\dots,m, \tag{1.35}$$

$$=\phi_{-h}[\mathcal{Q}(q_i,v)] \qquad \text{for } i=m+1,\dots,n;$$

$$T(i,\epsilon_i,u)=T_{k_i}(q_i,\epsilon_i,\lambda_i,u) \qquad \text{for } i=1,\dots,m, \tag{1.36}$$

$$=T^{k_i}(q_i,\epsilon_i,\lambda_i,u) \qquad \text{for } i=m+1,\dots,n.$$

*We have*

$$\int_0^\infty e^{-ru}du\; P_\mu\Big[\prod_{i=1}^n \mathcal{T}(i,\epsilon_i,u)\Big] \tag{1.37}$$

$$\simeq r^{-1}\sum_{b\in\mathfrak{B}_k} \mathbf{a}(\mathbf{k},\mathbf{b})\Big[-\frac{\ln r}{2\pi}\Big]^{\nu} g_{rb}(\mu,\lambda_1,\dots,\lambda_n)$$

*where* $\mathbf{a}(\mathbf{k},\mathbf{b})$ *and* $\nu$ *are defined by* (1.33).

1.9. Theorems 1.1 through 1.4 will be proved in Section 4 after we develop necessary tools in Sections 2 and 3. The relation of the paper to the previous work is discussed in Section 5.

We use the following notation: if $a_j$ is a real-valued function on a finite set J, then $a_J$ means the product of $a_j$ over all $j\in J$.

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## 2. SOME PROPERTIES OF GREEN'S FUNCTION

2.1. In this section we get some estimates and asymptotic formulae for Green's function $G_r(X)$ defined by (1.3).

It is well-known (see e.g. [IM],p.233) that

(2.1) $$G_r(x)=\frac{1}{\pi}\,K_0(\sqrt{2r}|x|)$$

where $K_0$ is a modified Bessel function which can be described (see [W],3.71.14,and 3.7.2) by the formula

(2.2) $$K_0(r)=-I_0(r)\ln\frac{r}{2}+B(r).$$

Here

(2.3) $$I_0(r)=\sum_0^\infty a_m\, r^{2m}/(2m)!,\quad a_m=\binom{2m}{m}2^{-2m};$$

(2.4) $$B(r)=-C+\sum_1^\infty a_m(1+\frac{1}{2}+\ldots+\frac{1}{m}-C)r^{2m}/(2m!).$$

It follows from (2.2) that

(2.5) $$\frac{1}{\pi}\,K_0(2\epsilon r)=h_\epsilon\varphi_\epsilon(r)+\psi_\epsilon(r),$$

with

$$\varphi_\epsilon(r)=I_0(2\epsilon r),\quad \psi_\epsilon(r)=\frac{1}{\pi}\,[B(2\epsilon r)-I_0(2\epsilon r)\ln r]$$

and $h_\epsilon$ given by (1.4).

Since $a_m\to 0$ and $a_m(1+\frac{1}{2}+\ldots\frac{1}{m}-C)\to 0$ as $m\to\infty$,there exist constants $\gamma_1,\gamma_2,\gamma_3$ such that

(2.6) $$\varphi_\epsilon(r)\le\gamma_1 e^{2\epsilon r},\quad |\psi_\epsilon(r)|\le(\gamma_2+\gamma_3|\ln r|)e^{2\epsilon r}\text{ for all } r>0.$$

2.2. Suppose that a random variable $Y$ has a probability density $q$ which satisfies condition (1.8) and put $N=|Y|/\sqrt{2}$.It follows from(2.1),(2.2) and (2.5) that

(2.7) $$G(\epsilon Y)=h_\epsilon\varphi_\epsilon(N)+\psi_\epsilon(N)\le(\gamma_1 h_\epsilon+\gamma_2+\gamma_3|\ln N|)e^{2\epsilon N}$$

and by (1.8) there exist constants $\beta_k$ such that

(2.8) $$E[G(\epsilon Y)^k]\le\beta_k|\ln\epsilon|^k$$

for all sufficiently small $\epsilon$.

We claim that

(2.9) $$E\int[G(z)-G(z-\epsilon Y)]^2dz\simeq 0.$$

Indeed,the left side is equal to $2E[F(0)-F(\epsilon Y)]$ where

$$F(y)=\int G(z)G(z-y)dz=\int_0^\infty e^{-t}tp_t(y)dt$$

and (2.9) follows from an estimate

$$0\le F(0)-F(y)=(2\pi)^{-1}\int_0^\infty (1-e^{-y^2/2t})e^{-t}dt\le \text{const}.y^2(1+\int_{y^2/2}^\infty dt\ e^{-t}/t).$$

By (2.7)

(2.10) $\qquad EG(\epsilon Y)=a(\epsilon)h_\epsilon+b(\epsilon),\quad a(\epsilon)=E\varphi_\epsilon(N),\quad b(\epsilon)=E\psi_\epsilon(N).$

The functions $a(\epsilon)$ and $b(\epsilon)$ are even and analytic in a neighbourhood of 0. Since $a(0)=1, b(0)=-\kappa$ (cf.(1.13a)),we have $a(\epsilon)=1+O(\epsilon^2)$, $b(\epsilon)=-\kappa+O(\epsilon^2)$ and

(2.11) $$EG(\epsilon Y)\simeq h_\epsilon-\kappa$$

2.3. Now we investigate the functions

(2.12) $$c_k(\epsilon)=Eg(\epsilon V_1,\ldots,\epsilon V_k),\quad k=1,2,\ldots$$

where $g$ is given by (1.3) (with $r=1$) and $V_1,\ldots,V_k$ are i.i.d.random variables with a probability density $q$ subject to the condition (1.8). By (2.1)

(2.13) $$c_k(\epsilon)=E\prod_{j\in J}[\frac{1}{\pi}K_0(2\epsilon R_j)]$$

where $J=\{1,2,\ldots,k-1\}$, $R_j=|V_j-V_{j+1}|/\sqrt{2}$.

By (2.1) and (2.5),

(2.14) $$c_k(\epsilon)=\sum h_\epsilon^{|\Lambda|}\ f_{\Lambda\Gamma}(\epsilon).$$

Here $f_{\Lambda\Gamma}(\epsilon)=E(\varphi_{\epsilon,\Lambda}\psi_{\epsilon,\Gamma})$ and the sum is taken over all partitions of $J$ into disjoint sets $\Gamma$ and $\Lambda$,$|\Lambda|$ meaning cardinality of $\Lambda$.

The functions $f_{\Lambda\Gamma}(\epsilon)$ have the same properties as $a(\epsilon)$ and $b(\epsilon)$,and $f_{\Lambda\Gamma}(0)=E\psi_\Gamma$ where $\psi_j$ are defined by (1.23).Therefore $f_{\Lambda\Gamma}(\epsilon)=E\psi_\Gamma+O(\epsilon^2)$. By (2.14)

$$c_k(\epsilon)\simeq\sum h_\epsilon^{|\Lambda|}E\psi_\Gamma.$$

2.4. Consider the set $J$ as a linear graph with bonds $(1,2),\ldots,(k-2,k-1)$. Denote the connected components of $\Gamma$ enumerated in the natural order by $\Gamma_1,\ldots,\Gamma_m$. The sets $\Gamma_j$ and $\Gamma_{j+1}$ are separated by a connected component $\Lambda_j$ of $\Lambda$. Besides $\Lambda_1,\ldots,\Lambda_{m-1}$ the set $\Lambda$ can have two extra components: $\Lambda_0$ - to the left of $\Gamma_1$, and $\Lambda_m$ - to the right of $\Gamma_m$.All numbers $k_j=|\Gamma_j|$ and $\ell_j=|\Lambda_j|$ are strictly positive except $\ell_0$ and

$\ell_m$ which can vanish. The case m=0 is exceptional. In this case $\Lambda=J$.

Since $\psi_{\Gamma_1},\dots,\psi_{\Gamma_m}$ are independent, $E\psi_\Gamma=a_1\dots a_m$ where $a_i=E(\psi_1\dots\psi_i)$. Therefore

(2.15) $$c_k(\epsilon)=h_\epsilon^{k-1}+\sum h_\epsilon^{\ell_0+\ell_1+\dots+\ell_m} a_{k_1}\dots a_{k_m},$$

the sum is taken over all $m\ge1$ and all representations

(2.16) $$k-1=\ell_0+k_1+\ell_1+\dots+\ell_{m-1}+k_m+\ell_m$$

such that $\ell_0,\ell_m\ge0$ and the rest of terms are strictly positive.

It follows from (2.16) that

(2.17) $$\mathcal{M}(\epsilon,v)=\sum_1^\infty c_k(\epsilon)v^k\simeq\phi_{-h_\epsilon}[\mathcal{Q}(v)]$$

where $\mathcal{Q}$ is defined by (1.24) and the equivalence relation $\simeq$ for power series should be interpreted as an analogous relation between the corresponding coefficients.

## 3. RANDOM FIELDS ON DIRECTED TREES

3.1. *A directed tree* S is a finite collection of sites connected by arrows in such a way that:

(a) every site is the end of at most one arrow;

(b) there are no loops $s_1\to s_2\to\dots\to s_m\to s_1$.

We say that a site s is *initial* if no arrow enters it. Every connected component of S contains exactly one initial site.

We consider a family of independent random variables $Z_s$ indexed by sites $s\in S$ and random variables $Y_{ss'}$ indexed by arrows ss' and we assume that, within every connected component $S_b$, all $Z_s$ are identically distributed with a law $\lambda_b$, and all $Y_{ss'}$ are identically distributed with a density $q_b$.

Let $\epsilon=\epsilon_s$ be a positive function on S constant on each connected component. Obviously there exists a unique solution $V_s$ of the equations:

(3.1) $$V_{s'}-V_s=\epsilon_s Y_{ss'} \text{ for every arrow } ss',$$
$$V_s=Z_s \text{ for every initial site } s.$$

We call it *a random field over S with parameters* $(\epsilon,\lambda,q)$.

3.2. Suppose that a directed tree is ordered and let $1,\ldots,k$ be its sites enumerated according to the ordering. We consider only orderings with the property: all arrows have the form $ij$ with $i<j$.

If a directed tree S is connected, then 1 is its only initial site. We note that the joint density for $V_1,\ldots,V_k$ is equal to

$$q_S(x_1,\ldots,x_k;\epsilon,\lambda) = \rho(x_1)\prod_{ij} q^{\epsilon}(x_j-x_i)$$

and the joint density for $V_2,\ldots,V_k$ is

$$\tilde{q}_S(x_2,\ldots,x_k;\epsilon,\lambda)=\int dx_1\; q(x_1,\ldots,x_k;\epsilon,\lambda)$$

where the product is taken over all arrows, $\lambda(dz)=\rho(z)dz$, and $q^{\epsilon}$ is defined by (1.9). Put

(3.2) $$T_S(q,\epsilon,\lambda,u)= \int_{D_k(u)} q_S(X_{t_1},\ldots,X_{t_k};\epsilon,\lambda)dt_1\ldots dt_k,$$
$$\tilde{T}_S(q,\epsilon,\lambda,u)= \int_{D_{k-1}(u)} \tilde{q}_S(X_{t_1},\ldots,X_{t_{k-1}};\epsilon,\lambda)dt_1\ldots dt_{k-1}.$$

(the domains $D_k(u)$ are defined by (1.11)). In particular, random variables $T_{L_k}$ corresponding to the ordered tree

(3.3) $$L_k:\; 1\to 2\to \ldots\to k$$

coincide with $T_k$ defined by (1.10), and the random variables $\tilde{T}_{L^k}$ corresponding to

(3.4) $$L^k:\quad \begin{array}{c} 3 \\ \uparrow \\ 2\leftarrow 1\rightarrow 4 \\ \downarrow \\ k+1 \end{array}$$

are identical to $T^k$ given by (1.19).

**Theorem 3.1.** *Consider a tree* S *with ordered connected components* $S_1,\ldots,S_n$ *and put*

$$T(b,u)=T_{S_b}(q_b,\epsilon_b,\lambda_b,u) \qquad \textit{for } b=1,\ldots,m;$$
$$T(b,u)=\tilde{T}_{S_b}(q_b,\epsilon_b,\lambda_b,u) \qquad \textit{for } b=m+1,\ldots,n.$$

*Let V be the random field over S with parameters $(\epsilon,\lambda,q)$ and let $S^*$ be the set of all the sites in S except the initial sites of the components $S_{m+1},\dots,S_n$. Consider all one-to-one mappings from the set $\{1,2,\dots,N\}$ onto S and put $a\in\mathcal{A}$ if the restriction of a to any component $S_b$ is monotone increasing relative to the ordering of $S_b$.*

*We have*

$$(3.5)\qquad \int_0^\infty e^{-ru}du\ P_\mu[\prod_{b=1}^{n} T(b,u)]=r^{-1}\sum_{a\in\mathcal{A}} Eg_{\mu r}(V_{a_1\epsilon},\dots,V_{a_N\epsilon})$$

*where V is the random field over S with parameters $(\epsilon,\lambda,q)$ and*

$$g_{\mu r}=\int\mu(dx_0)g_r(x_0,x_1,\dots,x_N).$$

**Proof.** We note that

$$P_\mu[\prod_{b=1}^{n}T(b,u)]$$

$$=\sum_{a\in\mathcal{A}}\int_{0<t_{a_1}<\dots<t_{a_N}<u} P_\mu f_{\epsilon a}(X_{t_{a_1}},\dots,X_{t_{a_N}})\ dt_1\dots dt_N$$

where $f_{\epsilon a}(x_1,\dots,x_N)$ is the joint density for $V_{a_1\epsilon},\dots,V_{a_N\epsilon}$. Since $p_\mu(t_{a_1},x_1;\dots;t_{a_N},x_N)$ given by (1.1) is the joint density for $X_{t_{a_1}},\dots,X_{t_{a_N}}$, we have

$$(3.6)\qquad P_\mu f_{\epsilon a}(X_{t_{a_1}},\dots,X_{t_{a_N}})$$

$$=\int p_\mu(t_1,x_1;\dots;t_N,x_N)f_{\epsilon a}(x_1,\dots,x_N)dx_1\dots dx_N$$

$$=Ep_\mu(V_{a_1\epsilon},\dots,V_{a_N\epsilon}).$$

Formula (3.5) follows from (3.6) if we take into account that

$$(3.7)\qquad \int_0^\infty e^{-ru}du\int_{D_N(u)} p_\mu(t_1,x_1;\dots;t_N,x_N)\ dt= r^{-1}g_{\mu r}(x_1,\dots,x_N).$$

3.3. **Theorem 3.2.** *Consider a tree S with ordered connected components*

$$(3.8)\qquad S_b=L_{k_b}\ \textit{for}\ b=1,\dots,m;$$
$$=L^{k_b}\ \textit{for}\ b=m+1,\dots,n$$

*and let* $\mathbf{a}=(a_1,\dots,a_N)\in\mathcal{A}$. *Suppose that the first* $\ell_1$ *elements in* $(a_1,\dots,a_N)$ *belong to* $S_{b_1}$*, the next* $\ell_2$ *elements belong to* $S_{b_2}$ *with* $b_2\neq b_1$ *etc. Elements* $b_1,b_2,\dots,b_M$ *form a route* $\mathbf{b}$ *in the sense of Subsection 1.2.*

*If* $(\mu,\lambda_1,\dots,\lambda_n)$ *and* $\mathbf{q}=(q_1,\dots,q_n)$ *satisfy the conditions of Theorem 1.4, then*

(3.9)
$$Eg_{\mu r}(V_{a_1\epsilon},\dots,V_{a_N\epsilon}) \simeq g_{r\mathbf{b}}(\mu,\lambda)\prod_{j=1}^{M} c_{\ell_j b_j}(\epsilon_{b_j}\sqrt{r})$$

*where* $g_{r\mathbf{b}}(\mu,\lambda)$ *is given by (1.6) and*

(3.10)
$$c_{\ell b}(\epsilon)=\left[\int G(\epsilon y)q_b(y)dy\right]^{\ell-1} \quad \textit{if } b\le m,$$
$$=\int g(\epsilon y_1,\dots,\epsilon y_\ell)\prod_{j=1}^{\ell} q_b(y_j)dy_j \quad \textit{if } b>m.$$

**Proof.** We have

(3.11)
$$g_{\mu r}(V_{a_1\epsilon},\dots,V_{a_N\epsilon})=A_J$$

where $J=\{1,2,\dots,N\}$,

(3.12)
$$A_1(\epsilon)=G_{\mu r}(V_{a_1\epsilon}),$$
$$A_j(\epsilon)=G_r(V_{a_j\epsilon}-V_{a_{j-1}\epsilon}) \text{ for } j=2,\dots,N.$$

Let $\sigma_b$ be the initial site in $S_b$,

$\Gamma=\{j: a_{j-1}$ and $a_j$ belong to different connected components of $S\}$,

$\Lambda=\{j: a_{j-1}$ and $a_j$ belong to the same connected component of $S\}$.

Note that $J=\{1\}\cup\Gamma\cup\Lambda$ and

(3.13)
$$A_1(\epsilon)=G_{\mu r}Z(a_1) \quad \text{if } a_1\in S_b, b\le m,$$
$$=G_{\mu r}[Z(\sigma_b)+\epsilon_{a_1}Y(\sigma_b,a_1)] \quad \text{if } a_1\in S_b, b>m;$$
$$A_j(0)=G_r[Z(a_j)-Z(a_{j-1})] \quad \text{if } j\in\Gamma;$$
$$A_j(\epsilon)=G_r[\epsilon_{a_j}Y(a_{j-1},a_j)] \quad \text{if } a_{j-1},a_j\in S_b,\ b\le m,$$
$$=G_r\{\epsilon_{a_j}[Y(\sigma_b,a_j)-Y(\sigma_b,a_{j-1})]\} \quad \text{if } a_{j-1},a_j\in S_b,\ b>m, j>1.$$

By (2.9),

(3.14) $$E[A_J(\epsilon)-A_J(0)]^2\simeq 0 \text{ for } j\epsilon\Gamma.$$

Taking into account (2.8), we get

(3.15) $$EA_J(\epsilon)\simeq E[A_\Gamma(0)A_\Lambda(\epsilon)A_1(\epsilon)].$$

Note that

(3.16) $$A_\Gamma(0)=g_r(Z_{s_1},\ldots,Z_{s_M})$$

where $s_1=a_1, s_2=a_{\ell_1+1},\ldots,s_M=a_{\ell_{M-1}+1}$. Since $A_\Lambda(\epsilon)$ is a function of the Y's, it is independent of (3.16) and, by (3.15)

(3.17) $$EA_J(\epsilon)\simeq E[A_1(\epsilon)g_r(Z_{s_1},\ldots,Z_{s_M})]\ EA_\Lambda(\epsilon).$$

We claim that

(3.18) $$E\{[A_1(\epsilon)-A_1(0)]g_r(Z_{s_1},\ldots,Z_{s_M})\}\simeq 0.$$

Indeed the function $F(x)=Eg_r(x,Z_{s_2},\ldots,Z_{s_M})$ is bounded and therefore it is sufficient to check that

(3.19) $$E[A_1(\epsilon)-A_1(0)]^2\simeq 0.$$

Suppose that $a_1\epsilon S_b$. If $b<m$, then $A_1(\epsilon)$ does not depend on $\epsilon$. If $b>m$, then

$$A_1(\epsilon)-A_1(0)=G_{\mu r}[Z(\sigma_b)+\epsilon_{a_1}Y(\sigma_b,a_1)]-G_{\mu r}[Z(\sigma_b)].$$

If $\mu$ is finite, then we get (3.19) from (2.9). If $\mu$ has a bounded Hölder continuous density, then $G_{\mu r}(x)$ and its gradient are bounded and, since $\lambda$ is finite, we get (3.19) from the inequality

$$|A_1(\epsilon)-A_1(0)|\le \text{const.}\,\epsilon_{a_1}|Y(\sigma_b,a_1)|.$$

The set $\Lambda$ is the union of $\Lambda_1=[2,\ell_1],\ldots,\Lambda_M=[\ell_{M-1}+2,\ell_M]$. By (2.1) $G_r(x)=G(\sqrt{r}x)$ and therefore

(3.20) $$EA_{\Lambda_j}=c_{\ell_j b_j}(\epsilon_{b_j}\sqrt{r}).$$

We note that $A_{\Lambda_1},\ldots,A_{\Lambda_M}$ are independent and formula (3.9) follows from (1.6), (3.17) (3.18) and (3.20).

## 4. PROOFS OF MAIN RESULTS

**4.1. Proof of Theorem 1.4.** Let

$$\tilde{m}_k(\epsilon,\lambda,r)=\int_0^\infty e^{-ru}du\; P_\mu\Big[\prod_{i=1}^{n} T(i,\epsilon_i,u)\Big]. \tag{4.1}$$

It follows from Theorems 3.1 and 3.2 that

$$\tilde{m}_k(\epsilon,\lambda,r)\simeq r^{-1}\sum_{b\in\mathfrak{B}^k} g_{rb}(\mu,\lambda)\prod_{j=1}^{M} c_{\ell_j b_j}(\epsilon_{b_j}\sqrt{r}) \tag{4.2}$$

where $\mathfrak{B}^k$ is the set of all routes $b=(b_1,\dots,b_M)$ in $\{1,\dots,n\}$ which contain $k_1$ elements equal to $1,\dots,k_n$ elements equal to $n$.

We introduce generating functions

$$\mathcal{M}_b(\epsilon,v)=\sum_{\ell=1}^{\infty} c_{\ell b}(\epsilon)v^\ell. \tag{4.3}$$

By (3.10),(2.11),(2.17) and (1.35),

$$\mathcal{M}_b(\epsilon\sqrt{r},v)\simeq\mathcal{C}_b(h_{\epsilon\sqrt{r}},v)=\mathcal{C}_b(h_\epsilon-\rho,v) \quad \text{with } \rho=\frac{1}{\pi}\ln r. \tag{4.4}$$

Since $\ell_1+\dots+\ell_M=k_1+\dots+k_n$, we get from (4.3) and (4.2) that

$$\sum_{k_1,\dots,k_n\ge 1}\tilde{m}_k(\epsilon,\lambda;r)\; v_1^{k_1}\dots v_n^{k_n}\simeq r^{-1}\sum_b g_{rb}(\mu,\lambda)\prod_{j=1}^{M}\mathcal{C}_{b_j}(h_{\epsilon_{b_j}}-\rho,v_{b_j}), \tag{4.5}$$

the sum is taken over all routes b in the space $\{1,\dots,n\}$ which pass through every point.

We note that,if $w=\mathcal{C}_i(h,v)$,then $v=\mathcal{D}_i(h,w)$ where

$$\begin{aligned}\mathcal{D}_i(h,w)&=\phi_{h-\kappa(q_i)}(w) && \text{for } i\le m,\\ &=\mathcal{Y}[\phi_h(w)] && \text{for } i>m.\end{aligned} \tag{4.6}$$

In both cases,for every $\rho$,

$$\mathcal{C}_i[h-\rho,\mathcal{D}_i(h,w)]=\phi_\rho(w) \tag{4.7}$$

We rewrite (4.5) in the form

$$\sum_{k_1,\dots,k_n\ge 1}\tilde{m}_k(\epsilon,\lambda;r)\;\mathcal{D}_1(w_1)^{k_1}\dots\mathcal{D}_n(w_n)^{k_n}\simeq r^{-1}\sum_b g_{rb}(\mu,\lambda)\prod_{j=1}^{M}\phi_\rho(w_{b_j}) \tag{4.8}$$

It follows from (1.15) (1.26) and (4.6) that

$$(4.9)\qquad \sum_{k=1}^{\infty}\mathcal{T}(i,k,\epsilon_i,u)w^k=\sum_{\ell=1}^{\infty}\mathcal{D}_i(h_{\epsilon_i},w)^{\ell}\ \ T(i,\ell,\epsilon_i,u).$$

By comparing (4.8) and (4.9),we see that the right side in (1.37) is equal to the coefficient at $w_1^{k_1}\ldots w_n^{k_n}$ in the right side of (4.8).

If $\ell_i$ is the number elements in $(b_1,\ldots,b_M)$ which are equal to i,then by (1.14)

$$(4.10)\qquad \prod_{j=1}^{M}\phi_\rho(w_{b_j})=\prod_{i=1}^{n}\phi_\rho(w_i)^{\ell_i} =\prod_{i=1}^{n}\{\binom{k_i-1}{\ell_i-1}w_i^{k_i}(-\rho)^{k_i-\ell_i}\}$$

The coefficient at $w^{k_1}\ldots w^{k_n}$ in (4.10) is $a(\mathbf{k},\mathbf{b})\rho^{\nu}$ with $a(\mathbf{k},\mathbf{b})$ and $\nu$ defined by (1.33).This implies (1.37).

**4.2. Proof of Theorems 1.1 and 1.2.** The integral in formula (1.32) is the convolution of functions $1_{t>0}\int\mu(dz_0)p_t(z_{b_1}-z_0)$, $p_t(z_{b_j},z_{b_{j+1}})1_{t>0}$ for j=1,...,M-1 and $\mathcal{P}_\nu(\log t)1_{t>0}$. Therefore

$$(4.11)\qquad \int_0^\infty e^{-ru}du\ m_{\mathbf{k}}(\lambda,u)=r^{-1}\sum_{b\in\mathcal{B}_{\mathbf{k}}}a(\mathbf{k},\mathbf{b})g_{r\mathbf{b}}(\mu,\lambda)\left[-\frac{\ln r}{2\pi}\right]^{\nu}.$$

We compare this expression with (1.37) and we get

$$(4.12)\qquad \int_0^\infty e^{-ru}du\ P_\mu[\mathcal{T}_{k_1}(\epsilon_1,\lambda_1,u)\ldots\mathcal{T}_{k_n}(\epsilon_n,\lambda_n,u)]\simeq\int_0^\infty e^{-ru}du\ m_{\mathbf{k}}(\lambda,u).$$

To every r>0 there corresponds a measure $M_r(du,d\omega)=e^{-ru}du\ P(d\omega)$ on $\mathbb{R}_+\times\Omega$.It follows from (4.12) that $\|\mathcal{T}_{\mathbf{k}}(\epsilon,\lambda,u)-\mathcal{T}_{\mathbf{k}}(\epsilon',\lambda,u)\|_{r,p}\simeq0$ where $\|\cdot\|_{r,p}$ means the $L^{2p}(M_r)$-norm.Thus there exists an $L^{2p}(M_r)$-limit

$$(4.13)\qquad \mathcal{T}_{\mathbf{k}}(\lambda,u)=\lim_{\epsilon\downarrow0}\mathcal{T}_{\mathbf{k}}(\epsilon,\lambda,u)$$

and

$$(4.14)\qquad \mathcal{T}_{\mathbf{k}}(\epsilon,\lambda,u)\approx\mathcal{T}_{\mathbf{k}}(\lambda,u).$$

We conclude from (1.37) that $\mathcal{T}_{\mathbf{k}}(q,\epsilon,\lambda,u)\approx\mathcal{T}_{\mathbf{k}}(\tilde{q},\epsilon,\lambda,u)$. Hence $\mathcal{T}_{\mathbf{k}}(\lambda,u)$ does not depend on the choice of q.Theorem 1.1 is proved.

The same arguments prove Theorem 1.2.

**4.3. Proof of Theorem 1.3.** By (4.14),(1.37) and (4.11)

$$\int_0^\infty e^{-ru}du\ P_\mu\{\prod_{i=1}^{n}\mathcal{T}_{k_i}(\lambda_i,u)\}$$
$$= \lim_{\epsilon\downarrow 0}\int_0^\infty e^{-ru}du\ P\{\prod_{i=1}^{n}\mathcal{T}_{k_i}(\epsilon_i,\lambda_i,u)\}$$
$$=\int_0^\infty e^{-ru}du\ m_k(\lambda,u)$$

which implies (1.31).

## 5. BIBLIOGRAPHICAL NOTES

5.1. Interest in the self-intersections of the Brownian motion has increased significantly in connection with Symanzik's ideas in quantum field theory. The functional $\mathcal{T}_2(m,1)$ where m is the Lebesgue measure has been introduced in a pioneering work [V] by Varadhan which has appeared as an Appendix to Symanzik's memoir.For k>2,the functionals $\mathcal{T}_k(\lambda)$ have appeared first in [D1] and [D2] as a tool for a probabilistic representation of $P(\varphi)_2$ fields.

In [D2] we considered polynomials of the field

$$T_{\epsilon z}(\varsigma)= \int_0^\varsigma p_\epsilon(z,X_t)dt \tag{5.1}$$

where p is a symmetric transition density,$X_t$ is the corresponding Markov process and $\varsigma$ is an exponential killing time independent of X. Assuming that Green's function

$$G_r(x,y)=\int_0^\infty e^{-rt}\ p_t(x,y)dt \tag{5.2}$$

has singularity of the same kind as Green's function of the planar Brownian motion, we defined functions $B_{k\ell}(\epsilon,z)$ such that there exists an $L^p$-limit

$$\vdots T^k\vdots_\lambda= \lim_{\epsilon\downarrow 0}\int\lambda(dz)\sum_{\ell=0}^{n}B_{k\ell}(\epsilon,z)T^\ell_{\epsilon z}(\varsigma). \tag{5.3}$$

for all $p\geq 2$ and for a wide class of measures $\lambda$. *In our present notations* $\vdots T^k\vdots_\lambda=\mathcal{T}_k(\lambda,\varsigma)$.

The random fields (5.3) are closely related to Wick's powers $:\varphi^{2n}:_\lambda$ of the free Gaussian field associated with X. In fact,we have arrived at our renormalization by using this relation.

The direct construction of the fields $\mathcal{T}_k$ given in the present paper for the case of the Brownian motion on $\mathbb{R}^2$ has a number of advantages:

(i) Computations are much simpler than in [D2] and we get fields $\mathcal{T}_k(\lambda,u)$ defined for each u (not only $\mathcal{T}_k(\lambda,\varsigma)$).

(ii) We prove that $\mathcal{T}_k(\lambda,u)$ is the limit of fields $\mathcal{T}^k(\epsilon,\lambda,u)$ corresponding to a rather general density function q not just to the transition density p.

(iii) We get an explicit expression for the coefficients $B_{k\ell}(\epsilon)$ as polynomials in $\ln\epsilon$ (because of translation invariance of the Brownian motion,$B_{k\ell}$ do not depend on z).

(iv) We show that the functionals $T_k$ given by (1.10) also can be renormalized to converge to $\mathcal{T}_k$.Moreover the renormalization is much simpler than in the case of $T^k$.

The case k=2 has been studied also in [D3] and [D4].In [D3],the existence of $L^p$-limits

(5.4)
$$\varphi_\lambda(f) = \lim_{\epsilon\downarrow 0}\int\lambda(dz)\iint_{0<s<t} ds\,dt\, f(s,t)\left[p_\epsilon(z,X_s)p_\epsilon(z,X_t) - \frac{p_\epsilon(z,X_s)}{2\pi(t-s)+2\epsilon}\right]$$

has been proved for all sufficiently smooth functions f with compact support.In [D4] the functional $\varphi_\lambda(f)$ has been expressed in terms of stochastic integrals.The method is due to Rosen who used it in [R1] to get a simple proof of Varadhan's result.

5.2. Various results about the functional $\mathcal{T}_2(m,u)$ are contained in [Y1],[Y2],[Y3] and [R1],[R2] and [L1]. In particular in [L1], a relation between this functional and the measure of the Brownian sausage has been established. A renormalization for $T_3(m,u)$ is given in [Y4] (it has been discovered independently by J.Rosen).

5.3. Recently Rosen [R3] proved that for every bounded Borel set $B\subset\{0<t_1<\ldots<t_k\}$ there exists an $L^2$-limit

$$I^k(B)=\lim_{\epsilon\downarrow 0}\int_B\{p_\epsilon(X_{t_1},X_{t_2})\}\ldots\{p_\epsilon(X_{t_{k-1}},X_{t_k})\}dt_1\ldots dt_k$$

where $\{Y\}=Y-EY$. An interesting open problem is to express $I^k(D_k(u))$ through $\mathcal{T}_\ell(m,u)$. Such an expression is known only for $k\le 3$.

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