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Ultimateness and the Azéma-Yor stopping time

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The purpose of this note is to give a correct proof of a result of Meilijson [3,p394], which was originally based on an identity proved wrong by Neil Falkner* (theorem 2). Our proof uses a special property of the Azéma-Yor stopping time (theorem 1 and lemma 1).

Let $(B_t)_{t\geq 0}$ denote standard Brownian Motion (started at zero) and for any stopping time τ define

$$M_\tau := \sup_{0\leq t\leq \tau} B_t .$$

A stopping time τ is called *standard*, if whenever σ_1 and σ_2 are stopping times with $\sigma_1 \leq \sigma_2 \leq \tau$, then

$$E|B_{\sigma_i}| < \infty , \; i=1,2, \text{ and}$$

$$E|B_{\sigma_1} - x| \leq E|B_{\sigma_2} - x| \quad \text{for all } x \in \mathbb{R}.$$

(As N. Falkner [2,p.386] showed, a stopping time τ is standard if and only if the process $(B_{t\wedge\tau})$ is uniformly integrable.)

Let X be a random variable with EX = 0 and define the function g_X on $\mathbb{R}$ by

$$g_X(x) := \begin{cases} E(X|X\geq x) & \text{if } P(X\geq x) > 0, \\ x & \text{otherwise.} \end{cases}$$

Azéma and Yor [1,p.95,p.625] showed that the stopping time T defined by

$$T := \inf\{t: M_t \geq g_X(B_t)\}$$

embeds (the distribution of) X, i.e. $B_T \overset{D}{=} X$, and is standard. We will refer to it as the A-Y stopping time (embedding X in (B_t)). It is also known that for any standard stopping time τ, that embeds X in (B_t),

$$(1) \qquad P(M_\tau \geq g_X(x)) \leq P(M_T \geq g_X(x)) = P(B_T \geq x) = P(X\geq x)$$

for $x \in \mathbb{R}$.

For the inequality we refer to Azéma and Yor [1,p.632].

The first equality is easily seen from the definition of T, while the second holds, because T embeds X.

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*I. Meilijson communicated this to me by letter.

Theorem 1.

Of all standard stopping times τ that embed X, the A-Y stopping time T is essentially* the only one with

(2) $$P(M_\tau \geq g_X(x)) = P(X \geq x) \quad , \; x \in \mathbb{R}. \qquad \square$$

A standard stopping time τ is called *ultimate*, whenever Y is a random variable with $E|Y-x| \leq E|B_\tau - x|$ for all $x \in \mathbb{R}$, then there exists a stopping time $\sigma \leq \tau$, that embeds Y.

Theorem 2. (I. Meilijson [3,p.394])

Assume τ is a standard stopping time embedding X. If τ is ultimate, then there are $a \leq 0 \leq b$ with $P(X \in \{a,b\}) = 1$. $\square$

Proof of Theorem 1.

We write g for g_X.

Let τ be a standard stopping time embedding X such that (2) holds.
Define the stopping time H_x by $H_x := \inf\{t: B_t \geq g(x)\}$ and put $\tau_x := \tau \wedge H_x$. Then $\{M_\tau \geq g(x)\} = \{H_x \leq \tau\}$.

For $z \leq x$

$$E|B_\tau - z| \geq E|B_{\tau_x} - z| =$$

$$(g(x) - z)P(H_x \leq \tau) + E|B_\tau - z| \, 1_{\{\tau < H_x\}} =$$

$$E(X - z) \, 1_{\{X \geq x\}} + E|B_\tau - z| \, 1_{\{\tau < H_x\}} =$$

$$E|B_\tau - z| + E|B_\tau - z| (1_{\{B_\tau \geq x, \tau < H_x\}} - 1_{\{B_\tau < x, \tau \geq H_x\}}).$$

So

(3) $$E|B_\tau - z| \, 1_{\{B_\tau \geq x, \tau < H_x\}} \leq E|B_\tau - z| \, 1_{\{B_\tau < x, \tau \geq H_x\}} \, , \; z \leq x.$$

Now using (2)

$$P(B_\tau \geq x, \; \tau < H_x) =$$

$$P(B_\tau \geq x) - P(B_\tau \geq x, \; \tau \geq H_x) =$$

$$P(X \geq x) - P(\tau \geq H_x) + P(B_\tau < x, \tau \geq H_x) =$$

$$P(B_\tau < x, \tau \geq H_x),$$

whence with $z \to -\infty$ in (3) it follows that

$$P(B_\tau \geq x, \tau < H_x) = P(B_\tau < x, \tau \geq H_x) = 0.$$

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*apart from disagreement on a null set.

Therefore

$$\{B_\tau \ge x\} = \{M_\tau \ge g(x)\} \text{ for all } x \in \mathbb{Q} \text{ (= the rational numbers) a.s..}$$

As for all $x \in \mathbb{R}$ we can find a sequence (x_n) in $\mathbb{Q}$ increasing to x and g is left-continuous, we get

$$\{B_\tau \ge x\} = \{M_\tau \ge g(x)\} \text{ for all } x \in \mathbb{R} \text{ a.s.,}$$

whence

$$M_\tau \ge g(B_\tau) \text{ a.s..}$$

(Simply observe that

$$B_\tau \in [x, x+\tfrac{1}{n}) \Longleftrightarrow M_\tau \in [g(x), g(x+\tfrac{1}{n}))$$

for all $x \in \mathbb{R}$ and all $n \in \mathbb{N}$ a.s..)

Now $t < T$ implies $M_t < g(B_t)$ and therefore $\tau \ge T$ a.s.. As τ is standard, it follows that for any stopping time σ with $T \le \sigma \le \tau$ a.s..

$$E|B_\sigma - x| = E|X - x| \quad \text{for all } x \in \mathbb{R},$$

which can only happen if $T = \tau$ a.s.. □

Let T- be the A-Y stopping time embedding -X in $(-B_t)$, then with $m_t = \inf_{0 \le s \le t} B_s$,

$$T- = \inf\{t: m_t \le -g_{-X}(-B_t)\}$$

and

$$B_{T-} \overset{D}{=} X.$$

<u>Lemma 1.</u>

If T = T- a.s., then there are $a \le 0 \le b$ with $P(X \in \{a,b\}) = 1$.

<u>Proof.</u>

First observe that

$$-g_{-X}(-x) \le x \le g_X(x) \qquad (x \in \mathbb{R})$$

Now for a path (of (B_t)) with T = T- and $B_T = B_{T-} = x$ we have

$$M_T \ge g_X(x) \quad (\ge x) \text{ , and}$$

$$m_T \le -g_{-x}(-x) \quad (\le x).$$

That implies however that

(4) $$-g_{-x}(-x) = x \text{ or } g_X(x) = x.$$

[If such a path first reaches level M_T and then level m_T it is forced to cross level x in between (continuity of paths) and 'T stops to soon', unless $-g_{-X}(-x) = x$; conversely if level m_T is reached before level M_T,'T- stops to soon', unless $g_X(x) = x$.]

Now (4) implies $x \leq$ es inf $X =: a(\leq 0)$, or $x \geq$ es sup $X =: b(\geq 0)$.
As $T = T-$ a.s., we can conclude

$$B_T \leq a \quad \text{or} \quad B_T \geq b \quad \text{a.s.}.$$

As $X \stackrel{D}{=} B_T$, it follows that $P(X \notin (a,b)) = 1$.

By definition of a and b $P(X \in [a,b]) = 1$.

It follows that a and b are finite and $P(X \in \{a,b\}) = 1$. □

Proof of theorem 2.

By lemma 1 it is enough to prove $\tau = T$ a.s. and $\tau = T-$ a.s..
As $T-$ is the A-Y stopping time embedding $-X$ in $(-B_t)$, it is sufficient to prove, that an ultimate stopping time is equal to the A-Y stopping time a.s., i.e. $\tau = T$ a.s..
With H_x as in the proof of theorem 1 we have for all $x \in \mathbb{R}$ by (1)

$$P(\tau \geq H_x) \leq P(T \geq H_x) = P(X \geq x).$$

As τ is ultimate and T is standard, there is a stopping time $\sigma_x \leq \tau$ with $B_{\sigma_x} \stackrel{D}{=} B_{T\wedge H_x}$. But then

$$P(M_\tau \geq g_X(x)) \geq P(B_{\sigma_x} \geq g_X(x)) = P(B_{T\wedge H_x} \geq g_X(x)) = P(T \geq H_x),$$

and so

$$P(M_\tau \geq g_X(x)) = P(X \geq x).$$

By theorem 1 it follows that $\tau = T$ a.s.. □

References.

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