SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

KALYANAPURAM RANGACHARI PARTHASARATHY

# A remark on the paper "Une martingale d'opérateurs bornés, non représentable en intégrale stochastique", by J.L. Journé and P.A. Meyer

*Séminaire de probabilités (Strasbourg)*, tome 20 (1986), p. 317-320
<http://www.numdam.org/item?id=SPS_1986__20__317_0>

A remark on the paper " Une martingale d'opérateurs bornés, non représentable en intégrale stochastique ", by J.L. Journé and P.A. Meyer

by

K.R. Parthasarathy

Indian Statistical Institute, Delhi Centre
7, S.J.S. Sansanwal Marg, New Delhi 110016

Recently, J.L. Journé has constructed a remarkable example of a bounded operator valued quantum martingale $X$ in the usual filtration of the boson Fock space $\mathcal{F}_+(L_2(0,\infty))$ which does not admit the representation

$$dX = E dA^\dagger + F d\Lambda + G dA \tag{1}$$

in the sense of [1] over the domain $\mathcal{E}$, where $E,F,G$ are adapted operator processes, $A^\dagger,\Lambda,A$ are respectively the creation, conservation ( gauge ) and annihilation martingales, and $\mathcal{E}$ is the linear manifold generated by exponential vectors. In [3], a necessary and sufficient regularity condition was found for a bounded operator valued martingale $X$ to satisfy (1) with $E,F,G$ being bounded operator valued processes satisfying the condition

$$\int_0^t ( \|E(s)\|^2 + \|G(s)\|^2 )\, ds < \infty \quad \text{for all } t \ .$$

The purpose of this note is to indicate the possibility of achieving the representation (1) for the Journé example provided $\mathcal{E}$ is suitably restricted and the strictness of the definition of an adapted process is relaxed.

We begin with a class of examples of quantum martingales determined by second quantization of integral operators. Let $K(.,.)$ be a complex valued continuous function on $[0,\infty)\times[0,\infty)$. For every $t>0$ define the bounded operator $K_t$ on $L_2([0,\infty))$ by

$$(K_t f)(s) = \chi_{[0,t]}(s) \int_0^t K(s,\tau) f(\tau)\, d\tau + \chi_{(t,\infty)}(s) f(s) \tag{2}$$

where $\chi_C$ denotes the indicator function of the set C. Let $K_\infty$ denote the integral operator defined by

$$(K_\infty f)(s) = \int_0^\infty K(s,\tau) f(\tau)\, d\tau \ . \tag{3}$$

If $K_\infty$ is a bounded operator on $L_2[0,\infty)$ then $\|K_t\| \leqq \max(1, \|K_\infty\|)$ for all $t$ . Let $X_K(t)$ denote the operator on Fock space defined by the relations

$$X_K(t)\psi(f) = \psi(K_t f) \quad \text{for} \quad f \in L_2[0,\infty) \tag{4}$$

where $\psi(f)$ is the exponential vector corresponding to $f$. Then $X_K(t)$, the second quantization of $K_t$, is defined on the domain $\mathcal{E}$ and an easy computation shows that

$$< \psi(f\chi_{[0,a]}),\ X_K(t)\psi(g\chi_{[0,a]}) > = < \psi(f\chi_{[0,a]}), X_K(a)\psi(g\chi_{[0,a]}) >$$

for all $t \geqq a$. In other words, $X_K = \{X_K(t),\ t \geqq 0\}$ is a quantum martingale with domain $\mathcal{E}$, and $X_K(0) = $ identity. If $K_\infty$ is a contraction then $K_t$ is a contraction for every $t$ and hence $X_K(t)$ can be extended to a contraction on Fock space. In other words, $X_K$ becomes a contraction valued operator martingale.

By straightforward differentiation we obtain the relation

$$\frac{d}{dt} < \psi(f),\ X_K(t)\psi(g) > = \tag{5}$$
$$<\psi(f), X_K(t)\psi(g)>\{-\bar{f}(t)g(t) + \bar{f}(t)\int_0^t K(t,s)g(s)ds + g(t)\int_0^t K(s,t)\bar{f}(s)ds\}$$

in the generalized sense of absolute continuity. In the language of [1], (5) is equivalent to saying that $X_K$ obeys the quantum stochastic differential equation

$$dX_K = X_K L dA^\dagger - X_K d\Lambda + M X_K dA \tag{6}$$

where L and M are adapted processes of operators defined on the domain $\mathcal{E}$ by the relations

$$\left.\begin{aligned} L(t) &= a(\chi_{[0,t]}\overline{K(t,\cdot)}) \\ M(t) &= a^\dagger(\chi_{[0,t]}\overline{K(\cdot,t)}) \end{aligned}\right\} \tag{7}$$

and $a, a^\dagger$ are the usual annihilation and creation fields over $L_2[0,\infty)$. Since $\mathcal{E}$ is left invariant by the operators $X_K(t)$, $a(h)$ for all $t \geqq 0$, $h \in L_2[0,\infty)$, it follows that the coefficients $X_K L$, $-X_K$, $MX_K$ of $dA^\dagger$, $d\Lambda$ and $dA$ respectively in (6) are all well defined adapted processes on the domain $\mathcal{E}$ satisfying the inequalities

$$\int_0^t \{\|X_K(s)L(s)\psi(f)\|^2 + |f(s)|^2\|X_K(s)\psi(f)\|^2 + \|M(s)X_K(s)\psi(f)\|^2\}\, ds < \infty \tag{8}$$

for all t. We remark that the finiteness of the third integral in (8) follows from the canonical commutation rules.

We now try to relax the conditions on the kernel $K$. As long as $K_T$ is a bounded operator for each $t$ and

$$\int_0^t (|K(s,t)|^2 + |K(t,s)|^2)\, ds < \infty \quad \text{for each } t \tag{9}$$

it is clear that L(t) and M(t) are well defined on $\mathcal{E}$ by (7) and condition (8) obtains. This implies (6). In the example of Journé, $K(s,t) = (s-t)^{-1}$ and (9) breaks down. Then the definition of $L(t)$ and $M(t)$ by (7) does not make any sense. To face this situation we have to interpret equation (6) in a weak sense and we proceed as follows.

Let $\mathcal{L} \subset L_2[0,\infty)$ denote the linear manifold of functions $f$ which satisfy the local Lipschitz's condition

$$\|f\|_{\ell,t} = \sup_{0\leqq x,y\leqq t} \left|\frac{f(x)-f(y)}{x-y}\right| < \infty \quad \text{for all } t .$$

Denote by $\mathcal{E}(\mathcal{L})$ the linear manifold generated by the set $\{\psi(f),\ f\in\mathcal{L}\}$. For the kernel $K(s,t) = (s-t)^{-1}$ we define the martingale $X_K$ as before and observe that for $f,g\in\mathcal{L}$

$$\frac{d}{dt} < \psi(f),\ X_K(t)\psi(g) > =$$
$$< \psi(f),\ X_K(t)\psi(g) > \{ -\bar{f}(t)g(t) + \int_0^t \frac{\bar{f}(t)g(s)-\bar{f}(s)g(t)}{t-s}\, ds \} \quad (10)$$

The last integral in the above equation can be written as

$$\bar{f}(t) \int_0^t \frac{g(s)-g(t)}{t-s}\, ds + g(t) \int_0^t \frac{\bar{f}(t)-\bar{f}(s)}{t-s}\, ds .$$

This suggests the introduction of the Schwartz-Fock space $\mathcal{F}_S$ of sequences of distributions in the following sense. Any element of $\mathcal{F}_S$ is of the form $\lambda=(c,\lambda_1,\lambda_2,\ldots)$ where $\lambda_n$ is a symmetric distribution ( in the sense of Schwartz ) in the space $\mathbb{R}^n$ for $n=1,2,\ldots$, and $c\in\mathbb{C}$. Let $\mathfrak{M}$ be an arbitrary dense linear manifold in $L_2[0,\infty)$ and let $\mathcal{E}(\mathfrak{M})$ be the linear manifold in Fock space generated by $\{ \psi(f),\ f\in\mathfrak{M} \}$. Let $E = \{ E(t),\ t\geqq 0 \}$ be a family of linear maps $E(t) : \mathcal{E}(\mathfrak{M}) \to \mathcal{F}_S$ satisfying the following conditions :

1) For any $C_c^\infty$ function $f$ on $[0,\infty)$, the series

$$< \psi(f),\ E(t)\psi(g) > = 1 + \Sigma_{n=1}^{\infty}\ n!^{-1/2} (E(t)\psi(g))_n(\bar{f}^{\otimes n}) \quad (11)$$

converges absolutely, where $(E(t)\psi(g))_n$ denotes the n-th term of the sequence $E(t)\psi(g)$ ( a symmetric distribution on $\mathbb{R}^n$ ).

2) The scalar quantity

$$< \psi(f),\ E(t)\psi(g) > \exp(- \int_t^\infty \bar{f}(s)g(s)ds )$$

depends only on the values of $f$ and $g$ on the interval $[0,t]$ and is a Borel function of $t$.

Then we say that $E$ is a <u>generalized adapted process</u> with domain $\mathcal{E}(\mathfrak{M})$. If $X,E,F,G,H$ are five generalized adapted processes over $\mathcal{E}(\mathfrak{M})$ such that

$$< \psi(f),\ X(t_2)\psi(g)-X(t_1)\psi(g) > = \int_{t_1}^{t_2} < \psi(f),\ \bar{f}(s)E(s)\psi(g) +$$
$$+ \bar{f}(s)g(s)F(s)\psi(g) + H(s)\psi(g) > ds \quad \text{for all } f\in C_c^\infty,\ g\in\mathfrak{M},\ t_1<t_2$$

we say that

$$dX = E dA^{\dagger} + F d\Lambda + G dA + H dt$$

on the domain $\mathcal{E}(\mathfrak{M})$. With these conventions it is straightforward to verify that the Journé martingale $X_K$ obeys the generalized quantum stochastic differential equation

$$dX_K = E dA^{\dagger} + F d\Lambda + G dA$$

where E,F,G are generalized adapted processes over the domain $\mathcal{E}(\mathcal{L})$, defined by

$$E(t)\psi(g) = \left(\int_0^t \frac{g(s)-g(t)}{t-s}\, ds\right)\psi(K_t g)\ ,$$

$$F(t)\psi(g) = -\psi(K_t g),$$

$$G(t)\psi(g) = (1,\lambda_1,\lambda_2,\dots) \in \mathcal{F}_S\ ,$$

where for any test function $\phi$ on $\mathbb{R}^n$

$$\lambda_n(\phi) = n!^{-1/2} \Sigma_{j=1}^n \int_{\mathbb{R}^n} \chi_{[0,t]}(x_j)h(x_1)\dots\hat{h}(x_j)\dots h(x_n) \times \frac{\phi(x_1,\dots x_{j-1},t,x_{j+1}\dots x_n)-\phi(x_1,\dots,x_n)}{t-x_j}\, dx_1\dots dx_n\ ,$$

here $h=K_t g$, and the symbol ^ over a term implies its omission.

Just as the derivative of a bounded function on the line could be a distribution, it seems that the 'partial derivatives' of a bounded operator valued quantum martingale X in Fock space, with respect to the fundamental creation, conservation and annihilation martingales could be generalized adapted processes determining X .

I wish to thank P.A. Meyer for presenting me a detailed account of the example in [2], W. von Waldenfels and M. Schurmann for their patient hearing of several preliminary accounts of the contents of the present exposition.

REFERENCES

[1]. R.L. Hudson and K.R. Parthasarathy. Quantum Ito's formula and stochastic evolutions. Comm. Math. Phys. 93, 301-323 ( 1984 ).

[2]. J.L. Journé and P.A. Meyer. Une martingale d'opérateurs bornés, non représentable en intégrale stochastique. This volume.

[3]. K.R. Parthasarathy and K.B. Sinha. Stochastic integral representation of bounded quantum martingales in Fock space. Preprint, Indian Statistical Institute, Delhi.

Note. The norm conventions on Fock space are those of [1], and are slightly different from those used elsewhere in this volume ( these would require a factor $n!^{-1}$ instead of $n!^{-1/2}$ in formula (11) ).