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POISSON REPRESENTATION OF STRICT REGULAR STEP FILTRATIONS*

F. B. Knight

0. Introduction

This paper is an outgrowth of the ideas of a previous paper by the author [6]. It is therefore convenient to begin by summarizing the relevant hypotheses and conclusions from Section 2 of [6]. We assume that $(\Omega, \mathcal{F}, P)$ is a complete probability space on which is a filtration $\mathcal{F}_t^o$, $t \geq 0$, augmented in the usual way to right-continuous $\mathcal{F}_t \supset \mathcal{F}_{t+}^o$ and satisfying the three conditions:

1) $\mathcal{F}_{0+}^o \equiv (\phi, \Omega)$,
2) $L^2(\Omega, \mathcal{F}_\infty, P)$ is separable (it suffices here that each $\mathcal{F}_t^o$ be countably generated), and
3) all $\mathcal{F}_t$-martingales are strict in the sense of [8], or equivalently any martingale starting at 0 of the form $XI_{\{t \geq T\}}$ is indistinguishable from 0. (We always assume that martingales have right-continuous paths with left limits for $t > 0$, abbreviated r.c.ℓ.ℓ)

According to the result of [8, p. 220], 3) is equivalent to assuming $\mathcal{F}_{T-} = \mathcal{F}_T$ for all $\mathcal{F}_t$-optional T, and we have argued in [6] that 1) and 3) express the fact that there is randomness of time without randomness of place (in particular, since $X_T \in \mathcal{F}_{T-}$ for any martingale X_t, the "place" X_T is predetermined at the time T). Under these conditions, we obtain a representation of any $X \in L^2(\mathcal{F}_t, P)$ with $EX = 0$ in the form (Theorem 2.4 of [6])

$$(0.1) \qquad X = \sum_{i < n_c(t)+1} \int h_i^{(c)}(u)\,dB_i(u \wedge \langle M_i^c\rangle_t) + \sum_{j < n_p(t)+1} \int k_j^{(d)}(u)\,dP_j(u \wedge \langle M_j^d\rangle_t)$$

where (B_i, P_j) is a "halted $n_c(t) + n_p(t)$ - dimensional Lévy process with Brownian and Poisson components". The precise definition (Definition 2.2 of [6]) of "halted" need not be repeated here, since the verbal expression is both shorter and simpler. The meaning is simply

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that (B_i, P_j) becomes a vector of mutually independent Brownian motions and compensated Poisson processes when we prolong them indefinitely beyond the "halting times" $(\langle M_i^c\rangle_t, \langle M_j^d\rangle_t)$, by attaching independent continuations of the same type in a product probability space.

In the above representation, (B_i, P_j) are fixed, independently of t and $X \in L^2(F_t, P)$, while the halting times are free of X, so that only the integrands $h_i^{(c)}(u)$ and $k_j^{(d)}(u)$ depend on X.

The representation (0.1) is not basically a new result. Rather, it is mainly an application of a known change-of-variables formula in stochastic integrals and an argument used in a different setting by P. A. Meyer [11]. However, a serious deficiency of the representation is of course that these integrands are not, in general, measurable over the filtration generated by (B_i, P_j), so we cannot regard the theorem as giving a canonical reduction of F_t to the filtration of such a halted Brownian-and-Poisson process. In more detail, we define (B_i, P_j) by time changes $\tau_i^{(c)}(u)$ and $\tau_j^{(d)}(u)$ of corresponding martingales (M_i^c, M_j^d), where $\tau_i^{(c)}(u)$ (resp. $\tau_j^{(d)}(u)$) is the inverse of $\langle M_i^c\rangle_v$ (resp. $\langle M_j^d\rangle_v$), in such a way that $h_i^{(c)}(u) = h_i(\tau_i^{(c)}(u-))$ (resp. $h_j^{(d)}(u) = h_j(\tau_j^{(d)}(u-)))$ is a previsible process of the time-changed filtration $F_{\tau_i^{(c)}(t)}$ (resp. $F_{\tau_j^{(d)}(t)}$). At this point, one loses sight of the meaning of (0.1) in terms of (B_i, P_j) since the integrands introduce additional information.

Our objective in the present paper is to rectify this situation in a particular case, previously introduced by Lepingle, Meyer, and Yor [9] as "hypothesis (BO)". Our result here is perhaps not surprising, but it is our hope that the same prescription will work in greater generality. Indeed, there is no known counterexample to its working under 1)-3) alone, but it is clear that the method used under (BO), namely transfinite induction, is limited to that case. Here we will denote (BO) as:

4) There are no continuous martingales other than constants, and there is a single F_t-optional set D whose sections for each $w \in \Omega$ are well-ordered in t, and which contains the discontinuity times of any martingale up to a P-null set.

The essential meaning of 4) is given in [9] as follows (on p. 608, line -4, a T_α should be $T_{\alpha+1}$ for the proof). Let $T_0 = 0$, and for each ordinal α let $T_{\alpha+1}(w) = \inf\{t > T_\alpha(w): (w,t) \in D\}$, and for limit ordinals β let $T_\beta = \sup_{\alpha < \beta} T_\alpha$, where α and β exhaust the countable ordinals. Then the family (T_α) are stopping times which, for every square-integrable martingale M_t, contain a.s. all the discontinuity times of M_t. It is easy (and instructive) to connect this hypothesis with the quantities obtained in [6]. For example, under condition 2) above we obtained in [6, Lemma 2.5 and the Remark following its proof], a single square integrable martingale M_d whose times of discontinuity contain those of any other, P-a.s. Thus the second part of 4) simply means that the discontinuity times of M_d are a.s. well-ordered (since a subset of a well-ordered set is also well-ordered). For the initiated reader, a yet simpler description is available in terms of the author's prediction process construction [5, Essay I] which will be used again in the sequel. Here we transfer the filtration to a canonical space of sequences of r.c.ℓ.ℓ. paths, for example by using a sequence $M_n(t) = E(X_n | \mathcal{F}_t)$ where $\{X_n\}$ is linearly dense in $L_0^2(\Omega,\mathcal{F},P)$ (as in [6, Theorem 2.4], where the subscript 0 indicates $EX_n = 0$). Then the prediction process Z_t of $(\mathcal{F}_t,P)$ is well-defined, and its times of discontinuity contain those of any martingale a.s. (and conversely, they equal those of M_d a.s. when M_d is represented on the canonical space--this is really an extension of the representation theory of Doob [3, I, §6]). Thus our hypothesis is that the times of discontinuity of Z_t are well-ordered (we can redefine Z_t on a P-null set to ensure that this holds everywhere).

The basic consequence of 4), as derived in [9, 2.2)], may be interpreted as saying that under 4) $\mathcal{F}_t$ is generated by a step process (for the exact definition of which, see for example P. A. Meyer [10]). Thus, according to [9, 2.2)], if $\mathcal{F}_{T_\alpha}$ is the usual stopped σ-field of T_α, we have $\mathcal{F}_\infty = \bigvee_\alpha \mathcal{F}_{T_\alpha}$ and for any stopping time T,

$$\mathcal{F}_T \cap \{T_\alpha \le T < T_{\alpha+1}\} = \mathcal{F}_{T_\alpha} \cap \{T_\alpha \le T < T_{\alpha+1}\}$$

for each α. Now for any Borel set E in the state space of Z_t and $s > 0$, we have

$$\{Z_{T_\alpha+s} \in E\} \cap \{T_\alpha+s < T_{\alpha+1}\} = \{Z_{T_\alpha+s} \in E\} \cap \{Z_{T_\alpha+u} \text{ is continuous, } 0 < u \le s\}\ ,$$

so taking $T = T_\alpha + s$ it follows by the strong Markov property of Z_t at T_α that on $\{Z_{T_\alpha+u}$ is continuous, $0 < u \le s\}$ (which is an element of $\mathcal{F}_T$) we have

$$I_{\{Z_{T_\alpha+s} \in E\} \cap \{T_\alpha+s < T_{\alpha+1}\}} = P\{Z_{T_\alpha+s} \in E | F_{T_\alpha} \text{ and}$$

$$\{Z_{T_\alpha+u} \text{ is continuous}, \quad 0 < u \leq s\}\}$$

$$= P^{Z_{T_\alpha}}\{Z_s \in E | Z_u \text{ is continuous}, \quad 0 < u \leq s\}.$$

Consequently, on $\{T_\alpha+s < T_{\alpha+1}\}$, $Z_{T_\alpha+s} = f(Z_{T_\alpha}, s)$ where $f(z,s)$ is non-random, from which it follows that F_t is generated (up to P-null sets) by the step process $W_t = Z_{T_\alpha}$ on $\{T_\alpha \leq t < T_{\alpha+1}\}$, all α. It may be remarked that, besides the usual requirements for a step process, this W_t also has left limits (along with Z_t).

Having stated our hypotheses 1)-4), we turn to discussion of conclusions. Instead of halted Lévy processes as in [6], we will obtain stopped Lévy processes in the usual sense, but only after prolonging them beyond the natural time span $\lim_{t \to \infty} \langle M_j^d \rangle_t$.

<u>Definition 0.2</u>. Let $(Y_k(t),\ k < N+1)$, $N \leq \infty$, be processes defined on the same space. We say that (Y_k) is a <u>stopped</u> N-dimensional Lévy process if there are measurable $0 \leq T_k \leq \infty$ such that

a) $Y_k(t) = Y_k(t \wedge T_k)$, $k < N+1$, $0 \leq t$, and

b) there is a sequence $(W_k;\ W_k(0) = 0,\ k < N+1)$ of independent Lévy processes (processes with homogeneous, independent increments) on a disjoint space such that, if we construct the product probability space (Ω^*, F^*, P^*) and on it define $Y_k^*(t) = Y_k(t \wedge T_k) + W_k(t - (t \wedge T_k))$, $t \geq 0$, then (Y_k^*) is a sequence of independent Lévy processes, <u>and</u> $(T_k) \overset{\text{def}}{=} \underline{T}$ is a stopping vector of $(Y_k^*) \overset{\text{def}}{=} \underline{Y}^*$ with respect to the generated filtrations $F_{\underline{t}}^* \supset F_{\underline{t}+}^{o*}$, $\underline{t} = (t_k)$. In other words, for any $t_k \geq 0$, $\bigcap_k \{T_k \leq t_k\} \in \overline{\sigma}\{Y_k^*(s_k),\ s_k \leq t_k,\ k < N+1\}$ where, here and in the sequence, $\overline{\sigma}\{\cdot\}$ denotes the generated σ-field $\sigma\{\cdot\}$ augmented by all P-null sets.

<u>Remark</u>. That these last σ-fields contain $\mathcal{F}^{o*}_{\underline{t}+}$ follows as in the case $N = 1$. For a fairly general treatment of vector-valued stopping times, see [T. Kurtz, 7]. Of course, the above definition is a transparent extension of the case $N = 1$.

It is trivial that a stopped Lévy process is also a halted Lévy process in the sense of [6], so Theorem 2.3 of [6] implies that the laws of Y^*_k and W_k coincide unless $P\{T_k = 0\} = 1$ or $P\{T_k = \infty\} = 1$, when the question becomes mute. We will prove an extension of (0.1) in which $(P_j(u \wedge \langle M^d_j \rangle_t), \; j < n_p(t)+1)$ becomes, for each t, a stopped Poisson process in u. It is therefore important to understand how these processes are related for different t. Suppose, therefore, that $\underline{U} \leq \underline{T}$ are such that both $\underline{Y}(t) = \underline{Y}(t \wedge \underline{T})$ and $\underline{Y}(t \wedge \underline{U})$ are stopped Lévy processes. Even if U_k or T_k are permitted to be 0 or ∞ a.s. it is easy to see that we can use the same $\underline{W} = (W_k)$ in Definition 0.2 to extend either process. However, we can extend $\underline{Y}(t \wedge \underline{U})$ to a Lévy process in another way. Namely, let $\underline{Y}^*$ be the extension of $\underline{Y}(t \wedge \underline{T})$ using $\underline{W}$. Then we can recover $\underline{Y}$ from $\underline{Y}^*$ using the stopping vector $\underline{T}$, and therefore we recover W_k on $\{T_k < \infty\}$ for each k in such a way that W_k is independent of $\underline{Y}$. Since $\underline{Y}(t \wedge \underline{U})$ is also a stopped Lévy process, if we follow the same prescription to recover $\underline{U}$, <u>but</u> we apply it to $\underline{Y}^*$ instead of the continuation of $\underline{Y}(t \wedge \underline{U})$, we again recover a process with the same law as W_k on $\{U_k < \infty\}$ which is independent of $\underline{Y}(t \wedge \underline{U})$. Then it follows that $\underline{Y}^*$ is also a continuation of $\underline{Y}(t \wedge \underline{U})$ as prescribed by Definition 0.2. But this means that we actually recovered $\underline{Y}(t \wedge \underline{U})$ a.s. (not just a process having the same law). Therefore, we can use the same continuation $\underline{Y}^*$ to recover both processes. Similarly, if we have a continuous family $(\underline{T}(t), \; 0 \leq t \leq \infty)$ which is non-decreasing in t, and each $\underline{T}(t)$ makes $\underline{Y}$ a stopped Lévy process, then we can recover all the processes $\underline{Y}(u \wedge \underline{T}(t))$, up to a fixed P-null set, from the single process $\underline{Y}(u \wedge \underline{T}(\infty))$.

1. <u>The Representation Theorem</u>

We require here only the cases $N = 1$ or $N = \infty$ from Definition 0.2 (the general case being needed only if there are continuous martingales). Besides, the case $N = 1$ is probably well-known, but we present it first for simplicity.

<u>Theorem 1.1</u>. a). Suppose, beside 1)-4), that for every t we have $P\{$the number of times a discontinuity in $(0,t]$ is finite$\} = 1$. Then

there is a stopped Poisson process $P(u)$ on $(\Omega, \mathcal{F}, P)$, and a continuous family $T(t)$, $0 \le t$, of stopping times of $P^*(u)$, such that, for every t, $\mathcal{F}_t = \bar{\sigma}(P(u \wedge T(t)),\ 0 < u)$.

Theorem 1.1 b) is the converse and is stated following the proof.

Proof. We make use of the martingale $\hat{M}_d$ referred to above, whose times of discontinuity equal those of the entire filtration $\mathcal{F}_t$ in the sense explained. (This is not a deep result, and probably not new.) Moreover, under 1)-3) we know by Lemma 2.5 of [6] that $\hat{M}_d$ generates all the square-integrable martingales of mean 0, in the sense that for any such M we have $M(t) = \int_0^t h(s)d\hat{M}_d(s)$ for a previsible $h(s)$,

$$E \int_0^t h^2(s)d\langle \hat{M}_d \rangle_s < \infty .$$

To clarify the implications of 4) in this situation, we again view the problem on the canonical sequence space where the process Z_t is well-defined and generates $\mathcal{F}_t$. Indeed, let us go one step farther and view the problem as defined on the canonical "prediction" space of Z_t itself, as defined in [5, Essay I, Definition 2.1].* The advantage of this step is that the Lévy system of Z_t, used to construct M_d in [6], originally is defined on the canonical path space of a Ray compactification for Z_t, in accordance with [2]. Then, as explained in [5, Essay IV, Theorem 1.2], we can identify the Ray-left-limit process with Z_{t-}, $t > 0$, excepting a P-null set of paths if necessary, in order to transfer the Lévy system for fixed P to the path space of Z_t. Now the point here is that the Lévy system ([5, Essay IV, Theorem 1.2]) consists of $(N_Z, \bar{H}_Z)$ where $N_Z(z_1, dz_2)$ is a kernel in the usual sense and $\bar{H}_Z$ is an <u>additive functional</u> of Z_t (this is an advantage of using the canonical space of Z_t). On this space we have, just as in

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*It can be shown, although it is tedious and will be omitted here, that the collection of all P satisfying 1)-4) on the canonical prediction space of Z_t defines a complete Borel packet (stochastically closed set for (Z_{t-}, Z_t)). In order to avoid this argument, we simply consider Z_t on the Borel space of <u>all</u> r.c.ℓ.ℓ. paths with values in the prediction state space H_0 and <u>left</u> limits in H (= all probability measures on the sequence space). Then our particular P defines completed σ-fields $\mathcal{F}_t$ which suffice to prove Theorem 1.1, in view of [5, Essay I, Definition 2.1, 2)]. This is, again, simply the device of representing the problem on a more convenient probability space.

Lemma 2.5 of [6],

$$(1.1) \qquad \hat{M}_d(t) = \sum_{s \le t} f(Z_{s-}, Z_s) - \int_0^t d\bar{H}_Z(s) \int_{H_0} N_Z(Z_{s-}, dz) f(Z_{s-}, z) .$$

Then the assumption 4) and the strong Markov property of Z_t at times T_α imply, just as for Z_t itself, that the increment

$$(1.2) \qquad \bar{H}_Z(T_\alpha + s) - \bar{H}_Z(T_\alpha) = \bar{H}_Z(s) \circ \theta^Z_{T_\alpha}$$

is P-a.s. a fixed function of Z_{T_α} on $\{T_\alpha + s < T_{\alpha+1}\}$. (Here we have used the translation operators θ^Z_t of Z_t, which are not available on the original sequence space but only on the "prediction space" of Z_t). The same reasoning applies to any other additive functional of Z_t which obeys (1.2) at each T_α. In particular, this is true for $\hat{M}_d(t)$.

For the present theorem, we need still more, namely a martingale which generates the given σ-fields $\mathcal{F}_t$, in the sense that $\mathcal{F}_t = \bar{\sigma}(M_s, s \le t)$. It is possible to show that $\hat{M}_d$ does have this property, but the proof requires several results which at present have no very convenient source (originally they were proved under extraneous hypotheses such as "absolute continuity", whose availability under 1)-4) is not clear). We will sketch the argument, and then show how to avoid it by constructing a different martingale which makes the desired property obvious.

It is easy to choose a sequence $0 < f_n < 1$ such that $(R^Z_\lambda f_n(z), 0 < \lambda \text{ rational})$ generates the σ-field of H (for example, as in Lemma 2.5 of [6], where R^Z_λ is the resolvent of Z_t) and therefore

$$\mathcal{F}_t = \bar{\sigma}(R^Z_\lambda f_n(Z_s), s \le t, 0 < \lambda \text{ rational}).$$

Then the generating martingale additive functionals of Kunita-Watanabe

$$M_{f_n,\lambda}(t) = R^Z_\lambda f_n(Z_t) - R^Z_\lambda f_n(Z_0) + \int_0^t (f_n(Z_s) - \lambda R^Z_\lambda f_n(Z_s))ds$$

have the same discontinuous as $R^Z_\lambda f_n(Z_t)$, and it follows that

$$\mathcal{F}_t = \bar{\sigma}(M_{f_n,\lambda}(s),\ \ s \leq t,\ \ 0 < \lambda \text{ rational}).$$

This is clear because the right side contains the generated σ-field of the step process $W_t = Z_{t_\alpha}$ on $\{T_\alpha \leq t < T_{\alpha+1}\}$, in view of the quasi-left-continuity of Z_t at limit ordinals β (all the discontinuity times of Z_t are totally inaccessible when 3) is assumed). Thus to show that $\hat{M}_d$ generates $\mathcal{F}_t$ it suffices to show that each $M_{f_n,\lambda}$ is measurable over the generated σ-fields of $\hat{M}_d$. At this point we can invoke Motoo's Theorem for right processes ([2, (2.5)]) to project $M_{f_n,\lambda}$ onto the subspace of martingale additive functionals generated by $\hat{M}_d$, and since $\hat{M}_d$ generates all square-integrable martingales (using previsible integrands) it follows that it also generates the subset of all martingale additive functionals. Thus we obtain <u>functions</u> $g_{n,\lambda}(z)$ such that

$$M_{f_n,\lambda}(t) = \int_0^t g_{n,\lambda}(Z_{s-})d\hat{M}_d(s),$$

and it is clear by induction on α that the discontinuities

$$\Delta M_{f_n,\lambda}(T_\alpha) = g_{n,\lambda}(Z_{T_{\alpha-}})\Delta\hat{M}_d(T_\alpha)$$

are in the α-field generated by $(\hat{M}_d(t \wedge T_\alpha),\ 0 < t)$. Then it follows that $\hat{M}_d$ generates $\mathcal{F}_t$ in the required sense.

To avoid this argument, we can also directly construct a martingale $M^*(t)$ which obviously generates $\mathcal{F}_t$, as follows. It is well-known that there is a bounded, one-to-one, Borel function $f^*(x_1,x_2,\ldots)$: $X_1^\infty(0,1) \longleftrightarrow (0,1)$. (In fact, any two uncountable Lusin spaces are isomorphic [1, Appendix to Chap. III, Theoreme 80].) It follows that if we order the collection

$$(\lambda R_\lambda^Z f_n,\ 1 \leq n,\ \ 0 < \lambda \text{ rational}) = (g_1,g_2,\ldots)\ ,$$

then the process $h^*(Z_s) = f^*(g_1(Z_s),g_2(Z_s),\ldots)$ does generate $\mathcal{F}_t$, since it generates $\sigma(R_\lambda^Z f_n(Z_s))$ for each n and λ. Here again, since the process is a fixed function of Z_{T_α} on $\{T_\alpha \leq s < T_{\alpha+1}\}$, it suffices to generate its "discontinuities" $h^*(Z_s) - h^*(Z_{s-})$ at all times T_α. On the other hand, from (1.1) we know that for any $\varepsilon > 0$ and $k > 0$,

$$M^*_{\varepsilon,K}(t) = \sum_{s \leq t} (h^*(Z_s) - h^*(Z_{s-}) + K) I_{\{f(Z_{s-},Z_s) > \varepsilon\}}$$

$$- \int_0^t d\bar{H}_Z(s) \int_{\{f(Z_{s-},z) > \varepsilon\}} N_Z(Z_{s-},dz)(h^*(z) - h^*(Z_{s-}) + K)$$

is a square-integrable martingale additive functional with $E(M^*_{\varepsilon,K}(t))^2 \leq \left(\frac{1+K}{\varepsilon}\right)^2 E(\hat{M}_d(t))^2$. Now let $M^*_3(t) = M^*_{1/3,6}(t)$ and $M^*_n(t) = M^*_{n^{-1},2n}(t) - M^*_{(n-1)^{-1},2n}(t)$, $n > 3$. The martingales $M^*_n(t)$, $3 \leq n$, are orthogonal (having no jumps in common) and together they generated $\mathcal{F}_t$ in the sense required. Also, the jumps of $M^*_n(t)$ are of size $2n - 1 < \Delta M^*_n(t) < 2n + 1$. It is easy to check that the intervals $(2^{-n}(2n-1), 2^{-n}(2n+1))$, $n \geq 3$, are disjoint. Therefore, if we define

$$M^*(t) = \sum_{n=3}^{\infty} 2^{-n} M^*_n(t) ,$$

we obtain a square integrable martingale whose jumps determine uniquely those of all the M^*_n. Then $M^*(t)$ generates $\mathcal{F}_t$ as required. We note explicitly, for use in Theorem 1.2 a) below, that no use was made so far of the extra "finite number of jumps" assumption. Thus M^* generates $\mathcal{F}_t$ under 1)-4) above.

Under the extra assumption of finitely many jumps in finite time, we next replace M^* by a local martingale with unit jumps. We can write, for a certain $g(z_1,z_2)$ whose exact expression in terms of h^*, f, etc.

need not concern us,

$$(1.3)\qquad M^*(t) = \sum_{s \le t} g(Z_{s-}, Z_s) - \int_0^t d\bar{H}_Z(s) \int_{H_0} N_Z(Z_{s-}, dz) g(Z_{s-}, z) \; .$$

Now let $T_1, T_2, \ldots$ as before denote the successive jump times, so that $P\{\lim_{n \to \infty} T_n = \infty\} = 1$ in the present case (these jump times are the same for $M^*(t)$ as for Z_t, P-a.s.). It follows easily from the definition of a Lévy system and the optimal stopping theorem for martingales that for each n the expression

$$M_1^d(t \wedge T_n) = \sum_{s \le (t \wedge T_n)} I_{\{Z_{s-} \ne Z_s\}} - \int_0^{t \wedge T_n} d\bar{H}_Z(s) N_Z(Z_{s-}, H)$$

is a square integrable martingale with

$$E^2 M_1^d(T_n) = E \sum_{s \le T_n} I_{\{Z_{s-} \ne Z_s\}} \le n \; .$$

Thus if we define

$$M_1^d(t) = \sum_{s \le t} I_{\{Z_{s-} \ne Z_s\}} - \int_0^t d\bar{H}_Z(s) N_Z(Z_{s-}, H) \; ,$$

we obtain a locally square integrable local martingale. Now letting

$$h(Z_{v-}) = \left(\int N_Z(Z_{v-}, dz) g(Z_{v-}, z)\right) (N_Z(Z_{v-}, H))^{-1}$$

where $0/0 = 0$, it follows that for each n,

$$M^*(t \wedge T_n) = \int_0^{t \wedge T_n} h(Z_{v-}) dM_1^d(v) \; .$$

Indeed, both sides have the same continuous part, and an application of Schwartz' inequality (as in [6, (2.7)]) shows that the right side is square integrable. Then the difference is a pure-jump martingale, which must be 0 by 3). Consequently, we see easily by induction on n that $M^*(t \wedge T_n)$ and $M_1^d(t \wedge T_n)$ generate the same σ-fields, and hence letting $n \to \infty$ we obtain that likewise M_1^d generates F_t. In other words, F_t is generated by the process $\sum_{s \leq t} I_{\{Z_{s-} \neq Z_s\}}$ alone (since the compensator is a fixed function of Z_T in $\{T_n \leq t < T_{n+1}\}$, and it is clear by induction that $Z_{T_n} \in \sigma(T_1,\ldots,T_n)$).

Now we define, as in [6], $P_1(u \wedge \langle M_1^d\rangle_t) = M_1^d(\tau_1(u) \wedge t)$, where $\tau_1(u) = \inf\{s\colon \langle M_1^d\rangle_s > u\}$ is the inverse of

$$\langle M_1^d\rangle_t = \int_0^t d\bar{H}_Z(s)N_Z(Z_{s-},H) \;,$$

in such a way that

$$P_1(u \wedge \langle M_1^d\rangle_t) = M_1^d(t) \quad \text{for} \quad u \geq \langle M_1^d\rangle_t \;.$$

It follows immediately by a theorem of S. Watanabe [13], as in the case of [6, Theorem 2.4], that $P_1(u \wedge \langle M_1^d\rangle_t)$ is a halted compensated Poisson process for each t. We need to show that it is actually a stopped compensated Poisson process, whose generated σ-field equal F_t. We note that $P_1(u)$ is defined for all $u < \lim_{t \to \infty} \langle M_1^d\rangle_t = \langle M_1^d\rangle_\infty$. Now on $\{\langle M_1^d\rangle_\infty < \infty\}$, $P_1(u)$ obviously has a.s. only finitely many jumps in $(0,\langle M_1^d\rangle_\infty)$, and it is clear that if we define $P_1(\langle M_1^d\rangle_\infty) = \lim_{t \to \infty} P_1(\langle M_1^d\rangle_t)$ then $P_1(u \wedge \langle M_1^d\rangle_\infty)$ is again a halted compensated Poisson process (the joint distributions of its product space continuation for all u are the limits as $t \to \infty$ of those of the continuations of $P_1(u \wedge \langle M_1^d\rangle_t)$), hence the continuation is again a compensated Poisson process).

We will reconstruct M_1^d from $P_1(u \wedge \langle M_1^d\rangle_\infty)$. Recalling that, given Z_{T_n}, $\langle M_1^d\rangle_t$ is a.s. a fixed function of t in $\{T_n \leq t < T_{n+1}\}$, while $Z_{T_n} \in \sigma(T_1,T_2,\ldots,T_n)$, let $A_n(t;t_1,\ldots,t_n)$, $0 \leq n$, $t_1 < \ldots < t_n \leq t$, denote a choice continuous in t and measurable in $(t_1,\ldots,t_n)$, in such a way that if $T_1,T_2,\ldots$ are given then $\langle M_1^d\rangle_t = A_n(t;T_1,\ldots,T_n)$; $T_n \leq t \leq T_{n+1}$, defines $\langle M_1^d\rangle_t$ conditional on $\{T_1,T_2,\ldots\}$ (in case $n = 0$, we just have $\langle M_1^d\rangle_t = A_0(t)$). Such A_n are easy to construct by considering an arbitrary choice defined for rational t, since $\langle M_1^d\rangle_t$ is continuous in t. Then by definition of $P_1(u)$ we have a.s. $P_1(A_0(T_1)) = M_1^d(T_1)$, $P_1(A_1(T_2;T_1)) = M_1^d(T_2),\ldots,$ $P_1(A_n(T_{n+1};T_1,\ldots,T_n)) = M_1^d(T_{n+1})$ for all $T_{n+1} < \infty$ (we recall that there is zero probability that any T_n occurs interior to an interval in which $\langle M_1^d\rangle_t$ remains constant). Letting $S_1 < S_2 < \ldots$ denote the successive jump times of $P_1(u \wedge \langle M_1^d\rangle_\infty)$ it follows that $S_1 = A_0(T_1),\ldots,S_{n+1} = A_n(T_{n+1};T_1,\ldots,T_n)$ for all n with $T_{n+1} < \infty$. Thus to reconstruct M_1^d from $P_1(u \wedge \langle M_1^d\rangle_\infty)$ we need only define

$$T_1 = \inf\{t\colon S_1 = A_0(t)\},\ldots,T_{n+1} = \inf\{t > T_n : S_{n+1} = A_n(t;T_1,\ldots,T_n)\}$$

for all n with $S_{n+1} < \infty$, and then $M_1^d(t) = P_1(A_n(t;T_1,\ldots,T_n))$ on $\{T_n \le t \le T_{n+1}\}$ for all n, where $\inf\{\phi\} = \infty$ and $T_{n+1} = \infty$ when $S_{n+1} = \infty$ (on an exceptional set where the corresponding $\langle M_1^d\rangle_t$ is discontinuous, we take $M_1^d \equiv 0$).

It is immediate that, apart from the exceptional set, we have for $s \le t$, $M_1^d(s) \in \sigma(P_1(u \wedge \langle M_1^d\rangle_t),\ 0 \le u)$ since for $n \ge 0$ the right side contains $\{S_n I_{\{T_n \le t\}},\ 1 \le n\}$ and therefore also $\{T_n I_{\{T_n \le t\}},\ 1 \le n\}$. But this is equally true if we replace $P_1(u)$ by the ordinary Poisson process $P(u) = P_1(u) + u$, as in the statement of the theorem. Finally, to see that $\langle M_1^d\rangle_t$ is a stopping time of the continued process $P^*(u)$ (Definition 0.2), we can use a result of Pittenger [12] explained below, provided we show that (considered on the product space) we have $M_1^d(t) \in \overline{\sigma}(P^*(u),\ 0 \le u)$ for all t. For this it is enough that $T_n \in \overline{\sigma}(P^*(u),\ 0 \le u)$ for each n. Now let $S_1^* < S_2^* < \ldots$ denote the discontinuity times of P^*. Then

$$S_n = \begin{cases} S_n^* & \text{on } \{S_n < \infty\} \\ \infty & \text{on } \{S_n = \infty\}, \end{cases}$$

and

$$T_1 = \begin{cases} \inf\{t\colon S_1^* = A_0(t)\} & \text{on } \{S_1 < \infty\} \\ \infty & \text{on } \{S_1 = \infty\} \end{cases}.$$

But on $\{S_1 = \infty\}$ we have $S_1^* > \langle M_1^d \rangle_\infty = A_0(\infty)$, so $T_1 = \inf\{t\colon S_1^* = A_0(t)\}$ holds everywhere. Suppose for induction that $T_1, T_2, \ldots, T_n \in \overline{\sigma}(P^*(u),\ 0 \le u)$. Since

$$T_{n+1} = \begin{cases} \inf\{t > T_n\colon S_{n+1}^* = A_n(t;T_1,\ldots,T_n)\} & \text{on } \{S_{n+1} < \infty\} \\ \infty & \text{on } \{S_{n+1} = \infty\} \end{cases},$$

where $S_{n+1}^* > \langle M_1^d \rangle_\infty = A_n(\infty;T_1,\ldots,T_n)$ on $\{S_n < \infty = S_{n+1}\}$, it follows that (with $\inf\{\phi\} = \infty$) $T_{n+1} = \inf\{t > T_n\colon S_{n+1}^* = A_n(t;T_1,\ldots,T_n)\}$ holds everywhere on $\{S_n < \infty\} = \{T_n < \infty\}$. Then by induction $T_{n+1} \in \overline{\sigma}(P^*(u),\ 0 \le u)$, as required.

If we apply the same reasoning to the stopped martingale $M_1^d(t \wedge t_0)$ for fixed t_o, we get a different Poisson continuation $P^*(t;t_o)$ which coincides with the former up to time $\langle M_1^d \rangle_{t_o}$. The above argument now shows that $\langle M_1^d \rangle_{t_o}$ is in the completed σ-fields of the continuation of $P^*(t;t_o)$.

Once again, an argument of [A. O. Pittenger, 12, §6] shows immediately that $\langle M_1^d \rangle_{t_o}$ is a stopping time of the generated σ-fields $\mathcal{F}_t^*$ of $P^*(t;t_o)$. The proof of this result will be included in a more general case needed below for Theorem 1.2 a), so we postpone it here. Let us simply state the result we need as

<u>Lemma 1.2</u> (Pittenger). Let X_t be a Borel right process with semigroup P^x, and let $0 \le R \in \mathcal{F}_\infty$ (= the usual completed, generated σ-field of X_t). Then if, for a fixed P^μ, we have the strong Markov property $P^\mu\{\theta_R^{-1} S \mid \mathcal{F}_R\} = P^{X_R}(S)$ on $\{R < \infty\}$ for all $S \in \mathcal{F}_\infty^o$, where $\mathcal{F}_R = \sigma(X(t \wedge R),\ 0 \le t) \vee \sigma(R)$, then R is an $\mathcal{F}_t^\mu$-stopping time.

In our case, X_t is a canonical realization of the ordinary Poisson process $P^*(t;t_o)$, and $R \overset{a.s.}{=} \langle M_1^d \rangle_{t_o}$. The strong Markov property at R follows immediately from the definition of the halted Poisson process $P(u \wedge \langle M_1^d \rangle_{t_o})$.

Let us remark, finally, that the converse of Theorem 1.1 a) is also true (but we must be careful to include the fact that $\langle M\rangle_t$ is continuous).

Theorem 1.1. b). Let $P^*(u)$ be an ordinary Poisson process, and let $T(t)$ be a non-decreasing family of F_t^*-stopping times with $T(t)$ continuous in t and $T(o) = 0$. Then the family $F_t = \bar{\sigma}\{P^*(u \wedge T(t)), 0 < u\}$ satisfies 1)-4), and has finitely many times of discontinuity in finite time intervals.

Proof. It follows by the optional sampling theorem of Doob that for each t_o, $P^*(t_o \wedge T(t)) - (t_o \wedge T(t))$ is a martingale, hence $P^*(T(t)) - T(t)$ is a local martingale, and locally square integrable. Since $T(t)$ is continuous, it is clear that $\langle P^*(T(\cdot)) - T(\cdot)\rangle_t = T(t)$. Now we can use $P^*(T(t)) - T(t)$ the same way as $M_1^d(t)$ above to define for each t a stopped Poisson process, which is obviously the same as $P^*(u \wedge T(t))$. Therefore, $\bar{\sigma}(P^*(u \wedge (T(t))), 0 < u) \subset \bar{\sigma}(P^*(T(u)), u \leq t)$. According to the reconstruction of $\{P^*(T(u)), u \leq t\}$ from $\{P^*(u \wedge T(t)), 0 < u\}$ given in the preceeding proof, we also have $\bar{\sigma}(P^*(T(u)), u \leq t) \subset \bar{\sigma}(P^*(u \wedge T(t)), 0 < u)$, so that $F_t = \bar{\sigma}(P^*(T(u)), u \leq t)$. Now it is clear that the fields generated by $P^*(T(t))$ as on the right do satisfy 1)-4) and have finitely many times of discontinuity in finite times, concluding the proof (that there are no continuous martingales follows routinely because $T(t)$ is a fixed function of $(S_1,\ldots,S_n)$ on $\{S_n \leq T(t) < S_{n+1}\}$, where (S_n) denote the jump times of $P^*(t)$).

It is obvious that if the discontinuity times have a finite accumulation point with positive probability, the conclusion of a) does not hold. Indeed, there cannot be a finite N and a stopped N-dimensional Poisson process generating F_t. Nevertheless, we have

Theorem 1.2. a). Under 1)-4) above, there is a stopped Poisson process $(P_n(u), 1 \leq n) = \underline{P}(u)$, and a continuous family $(T_n(t)) = \underline{T}(t)$ of stopping vectors of $\underline{P}^*(u)$ (of dimension $N = \infty$) such that, for every t, $ET_n(t) < \infty$, $1 \leq n$, and $F_t = \bar{\sigma}(P_n(u \wedge T_n(t)), 1 \leq n, 0 < u)$.

<u>Proof</u>. The proof is essentially the same as for Theorem 1.1 a) except for two difficulties: first, the original order of the jump times T_α is not preserved in the combined set of jump times of (P_n), and second, we must use transfinite induction on α so the case of limit ordinals α is a new feature. Neither problem, however, causes any great difficulty, as we shall see.

We go back to expression (1.3) for the locally square integrable martingale $M^*(t)$ which generates $\mathcal{F}_t$, and for $1 \leq n$ set

$$(1.4) \qquad M_n^*(t) = \sum_{s \leq t} g(Z_{s-},Z_s) I_{\{|g(Z_{s-},Z_s)| > \frac{1}{n}\}}$$

$$- \int_0^t d\overline{H}_Z(s) \int_H N_Z(Z_{s-},dz) g(Z_{s-},z) I_{\{|g(Z_{s-},z| > \frac{1}{n}\}}.$$

Then the sequence M_1^*, $M_2^* - M_1^*$, ..., $M_{n+1}^* - M_n^*$, ... is orthogonal and together they generate $\mathcal{F}_t$. Moreover, each has only finitely many jumps in finite times. Then we can proceed exactly as for $M^*(t)$ in Theorem 1.1 to show that the sequence

$$(1.5) \qquad M_1^d(t) = (\sum_{s \leq t} I_{\{|g(Z_{s-},Z_s)| > 1\}})$$

$$- \int_0^t d\overline{H}_Z(s) \int_H N_Z(Z_{s-},dz) I_{\{|g(Z_{s-},z)| > 1\}} ,$$

$$M_{n+1}^d(t) = \sum_{s \leq t} I_{\{\frac{1}{n+1} < |g(Z_{s-},Z_s)| \leq \frac{1}{n}\}}$$

$$- \int_0^t d\overline{H}_Z(s) \int_H N_Z(Z_{s-},dz) I_{\{\frac{1}{n+1} < |g(Z_{s-},z)| \leq \frac{1}{n}\}}, \quad 1 \leq n ,$$

is locally square integrable, and

$$(1.6) \qquad M^d_{n+1}(t) = \int_0^t h_n(Z_{s-})\,d(M^*_{n+1} - M^*_n)(s)$$

for suitable $h_n(z)$, $1 \le n$, with an analogous expression for $M^d_1(t)$. Now it follows as before by induction on α that for each finite integer α, $M^*(t \wedge T_\alpha)$ and $(M^d_n(t \wedge T_\alpha), 1 \le n)$ generate the same σ-fields. Moreover, if this is true for α, then since Z_s is a fixed function of Z_α on $\{T_\alpha \le s < T_{\alpha+1}\}$, it is likewise true for $\alpha + 1$. Suppose finally that β is a limit ordinal. Then $\lim_{\alpha \uparrow \beta} T_\alpha = T_\beta$, and $T_\alpha < T_\beta$ on $\{T_\alpha < \infty\}$. Thus Z_t, which under 3) has only totally inaccessible times of discontinuity, is a.s. continuous at T_β on $\{T_\beta < \infty\}$. Therefore

$$\begin{aligned}\overline{\sigma}(Z_{s \wedge T_\beta}, s \le t) &= \bigvee_{\alpha < \beta} \overline{\sigma}(Z_{s \wedge T_\alpha}, s \le t)\\ &= \bigvee_{\alpha < \beta} \overline{\sigma}(M^d_n(s \wedge T_\alpha), 1 \le n, s \le t)\\ &= \overline{\sigma}(M^d_n(s \wedge T_\beta), 1 \le n, s \le t)\end{aligned}$$

which completes the induction step. It is well known that there exists a sequence $\alpha_k \uparrow$ of countable ordinals with $P\{T_{\alpha_k} \uparrow \infty\} = 1$ (we will review the proof of this just before Lemma 1.3 below). Consequently we obtain as required

$$\begin{aligned}(1.7) \qquad \mathcal{F}_t &= \lim_{k \to \infty} \overline{\sigma}(M^d_n(s \wedge T_{\alpha_k}), 1 \le n, s \le t)\\ &= \overline{\sigma}(M^d_n(s), 1 \le n, s \le t).\end{aligned}$$

At the same time, we note explicitly that, since each $M^d_n(t)$ is continuous at limit ordinals β (along with Z_t), (1.7) implies that $\mathcal{F}_t$

is generated entirely by the times of discontinuity $T_\alpha \le t$ of the combined sequence $(M_n^d,\ 0 \le n)$. In symbols, $\mathcal{F}_t = \overline{\sigma}\{T_\alpha I_{\{T_\alpha \le t\}},\ \alpha \in \chi_0\}$.

We now set $P_n(u \wedge \langle M_n^d \rangle_t) = M_n^d(\tau_n(u) \wedge t)$, $1 \le n$, $0 < u$ where $\tau_n(u) = \inf\{s: \langle M_n^d \rangle_s > u\}$. The definitions are obviously consistent in t, which defines $P_n(u \wedge \lim_{t \to \infty} \langle M_n^d \rangle_t)$ for all n and u. It follows by [13] or [11, Theorem 2'] that $(P_n(u \wedge \langle M_n^d \rangle_t),\ 1 \le n)$ is a halted compensated Poisson process for each t (as also in [6, Theorem 2.4, Case 2]), and letting $t \to \infty$ we obtain by convergence of distribution that $(P_n(u \wedge \langle M_n^d \rangle_\infty))$ is likewise (as in Theorem 1.1 a) above). Thus our problem is again to show that this is a stopped compensated Poisson process, and we will follow the same line of argument as in Theorem 1.1, by reconstructing (M_n^d) from (P_n). We know that the M_n^d have no jump times in common, so we introduce the notation (T_α, n_α) for the jump times and their associated processes, setting for completeness $n_\alpha = 0$ if α is a limit ordinal of if $T_\alpha = \infty$. We also know that in each $\{T_\alpha \le t < T_{\alpha+1}\}$, each $\langle M_n^d \rangle_t$ is a fixed function of t and $\{(T_\beta, n_\beta),\ \beta \le \alpha\}$. Since $\langle M_n^d \rangle_t$ is continuous, we can again introduce functions $A_{\alpha,n}$ such that $A_{\alpha,n}(t;(t_\beta,n_\beta),\ \beta \le \alpha) = \langle M_n^d \rangle_t$ on $\{T_\alpha \le t < T_{\alpha+1}\}$ given $(T_\beta, n_\beta) = (t_\beta, n_\beta)$, $\beta \le \alpha$, where each $A_{\alpha,n}$ is continuous in t for $t_\alpha \le t$, and measurable in $((t_\beta,n_\beta),\ \beta \le \alpha)$ over the product Borel field.

For $\alpha = 0$ we just write $A_{o,n}(t)$. Thus, apart from a fixed P-null set, we have

$$(1.8) \qquad P_n(A_{\alpha,n}(t;(T_\beta,n_\beta),\ \beta \le \alpha)) = M_n^d(t)$$

on $\{T_\alpha \le t \le T_{\alpha+1}\}$ for all n and α, where we again use the fact that each M_n^d is a.s. constant during the level stretches of $\langle M_n^d \rangle$ (easily seen by optional stopping of the martingale $(M_n^d)^2 - \langle M_n^d \rangle$ at times $T_r = \inf\{t > r: \langle M_n^d \rangle_t \ne \langle M_n^d \rangle_r\}$).

Next, suppose that $P_n(u \wedge \langle M_n^d\rangle_t)$, $0 < u$, $1 \leq n$, are given for fixed t, and let us reconstruct $(M_n^d(s), 0 < s \leq t)$ outside a P-null set as follows. Let $S(k,n)$ denote the k^{th} jump time of $P_n(u \wedge \langle M_n^d\rangle_t)$, $1 \leq k$, or ∞ if there are $< k$ jumps, and set

$$(1.9) \qquad T_1(t) = \inf\{s\colon S(1,n) = A_{o,n}(s) \quad \text{for some} \quad n\}.$$

We note that $T_1(t)$ coincides a.s. with the first time of discontinuity T_1 on $\{T_1 \leq t\}$, and in this case it occurs for a unique $n = n_1$. Indeed, we have $M_n^d(s) = P_n(A_{o,n}(s))$, $0 \leq s \leq T_1$, for all n, and since $\inf(\phi) = \infty$, $T_1(t) = \infty$ is equivalent to "$S(1,n) = \infty$ for all n". Thus we have determined $M_n^d(s \wedge t \wedge T_1)$ for all s and n. Assume now that for a countable ordinal α we determined $((T_\beta(t),n_\beta), \beta \leq \alpha)$ in such a way that a.s.

$$(1.10a) \qquad T_\beta(t) = \begin{cases} T_\beta & \text{on} \quad \{T_\beta \leq t\} \\ \infty & \text{elsewhere} \end{cases},$$

and consequently on $\{T_\beta \leq s \leq T_{\beta+1} \wedge t\}$, for all $n,s,\beta < \alpha$,

$$(1.10b) \qquad M_n^d(s \wedge t \wedge T_{\beta+1}) = P_n(A_{\beta,n}(s;(T_\gamma(t),n_\gamma), \gamma \leq \beta) \wedge t) \text{ a.s.}$$

Then by the inaccessibility of jumps, if α is a limit ordinal we have also determined

$$(1.11) \qquad M_n^d(t \wedge T_\alpha) = \lim_{\beta \uparrow \alpha} M_n^d(t \wedge T_\beta),$$

so in any case $M_n^d(s \wedge t \wedge T_\alpha)$ is determined for all s. Then we define

$$(1.12) \qquad T_{\alpha+1}(t) = \inf\{s > T_\alpha(t)\colon S(k,n) = A_{\alpha,n}(s;(T_\beta,n_\beta), \beta \leq \alpha)$$

$$> A_{\alpha,n}(T_\alpha;(T_\beta,n_\beta), \beta \leq \alpha) \text{ for some } n \text{ and } k\},$$

with $\inf \phi = \infty$. Since $A_{\alpha,n}(s;(T_\beta,n_\beta), \beta \le \alpha) \ge \langle M_n^d \rangle_{T_\alpha}$, it follows by (1.10b) and the continuity of $\langle M_n^d \rangle_s$ that for each n only one k is possible in (1.12) on the basis of $M_n^d(s \wedge t \wedge T_\alpha)$, namely the first k exceeding the number of jumps of $M_n^d(s \wedge t \wedge T_\alpha)$, $0 < s$. Moreover, it follows from 4) (well-ordering of jump times) that the n in (1.12) is uniquely determined on $\{T_{\alpha+1} \le t\}$, P-a.s., where it equals $n_{\alpha+1}$. Consequently, we see that a.s.

$$T_{\alpha+1}(t) = \begin{cases} T_{\alpha+1} & \text{on } \{T_{\alpha+1} \le t\} \\ \infty & \text{elsewhere} \end{cases},$$

as required, and this extends the determination (1.10b) of M_n^d to $M_n^d(s \wedge t \wedge T_{\alpha+1})$. Finally, if α is a limit ordinal and we have determined $((T_\beta(t),n_\beta), \beta < \alpha)$ satisfying (1.10a) and (1.10b) with T_β in place of T_α, then we need only set $T_\alpha(t) = \lim_{\beta \uparrow \alpha} T_\beta(t)$, $n_\alpha = 0$, and repeat (1.11) to extend the determination to $\beta = \alpha$. Thus by transfinite induction, applying (1.12) whenever n and k are uniquely determined, and setting $T_{\alpha+1}(t) = \infty$ otherwise, we determine $T_\alpha(t)$ for all α, P-a.s., from $(P_n(u \wedge \langle M_n^d \rangle_t), 1 \le n, 0 < u)$, which simultaneously determine $(M_n^d(s \wedge t))$ by (1.10b). It can be seen easily that these definitions are consistent in t, so that apart from a single P-null set we have determined T_α $(= \lim_{t \to \infty} T_\alpha(t))$ for all α, and for all t and n, from $(P^n(u \wedge \langle M_n^d \rangle_\infty), 1 \le n, 0 < u)$.

To show that the $(T_\alpha(t))$ thus determined are stopping vectors of the continuation $(P_n^*(u))$ (or more precisely that when we extend $(\Omega, \mathcal{F}, P)$ to the product space $(\Omega^*, \mathcal{F}^*, P^*)$ the $(T_\alpha(t))$, as functions of (w,w') depending only on $w \in \Omega$, are stopping vectors of the augmented generated σ-fields of $(P_n^*(u))$), it will be enough by Lemma 1.3 below to show that $(T_\alpha(t)) \in \bar{\sigma}(P_n^*(u), 1 \le n, 0 < u)$ for all α and t. This will then imply that $M_n^d(t)$ is also in $\bar{\sigma}(P_n^*(u), 1 \le n, 0 < u)$. As in Theorem 1.1 a) above, we can just as well treat the case of (T_α) and then specialize to $T_\alpha(t)$ by using the stopped processes $M_n^d(s \wedge t)$. The necessary induction on α is then quite analogous to that of Theorem 1.1. We let $S^*(k,n)$ denote the successive jump times of P_n^*, and $S(k,n)$ the jump times of $P_n(u \wedge \langle M_n^d\rangle_\infty)$ on Ω^*. Then $S^*(k,n) \leq S(k,n)$, and $S^*(k,n) = S(k,n)$ unless $S(k,n) = \infty$. We claim that (1.9) implies $T_1 = \inf\{s\colon S^*(1,n) = A_{o,n}(s)$ for some $n\}$, a.s. Clearly the above is $\leq T_1$. Suppose for contradiction that it is $< T_1$ at a certain sample point. Then there is an n_o with

$$\inf\{s\colon S^*(1,n_o) = A_{o,n_o}(s)\} < \inf\{s\colon S(1,n) = A_{o,n}(s)\} \quad \text{for all } n.$$

In particular, $S^*(1,n_o) < \infty$ and $S(1,n_o) = \infty$. But this implies $S^*(1,n_o) > \langle M_{n_o}^d\rangle_\infty = A_{o,n_o}(\infty)$, which is a contradiction.

Suppose next, for induction, that $T_\beta \in \overline{\sigma}(\underline{P}^*)$, $\beta \leq \alpha$. We want to show that we can replace $S(k,n)$ by $S^*(k,n)$ in (1.12), namely that

$$(1.13)\qquad T_{\alpha+1} = \inf\{s > T_\alpha\colon S^*(k,n) = A_{\alpha,n}(s;(T_\beta,n_\beta),\ \beta \leq \alpha)$$
$$> A_{\alpha,n}(T_\alpha;(T_\beta,n_\beta),\ \beta \leq \alpha \ \text{ for some } n \text{ and } k\}.$$

Obviously we may assume $T_\alpha < \infty$ and that the right side is also finite. Hence it this is false at a certain sample point there is an n_o and k_o with

$$(1.14)\qquad \inf\{s > T_\alpha\colon S^*(k_o,n_o) = A_{\alpha,n_o}(s;(T_\beta,n_\beta),\ \beta \leq \alpha)$$
$$> A_{\alpha,n_o}(T_\alpha;(T_\beta,n_\beta),\ \beta \leq \alpha)\}$$

$$< \inf\{s > T_\alpha : S(k,n) = A_{\alpha,n}(s;(T_\beta,n_\beta),\ \beta \le \alpha)$$

$$> A_{\alpha,n}(T_\alpha;(T_\beta,n_\beta),\ \beta \le \alpha)\},\ \text{all}\ \ k\ \ \text{and}\ \ n.$$

This implies (with $n = n_o$) that $S^*(k_o,n_o) < S(k_o,n_o)$ and $S(k_o,n_o) = \infty$, which last implies $S^*(k_o,n_o) > \langle M^d_{n_o}\rangle_\infty$. Now we distinguish two cases: a) $T_\alpha < T_{\alpha+1} = \infty$, and b) $T_\alpha < T_{\alpha+1} < \infty$. In case a) we have $S(k,n_o) \ne A_{\alpha,n_o}(s;(T_\beta,n_\beta),\ \beta \le \alpha)$ for all k and s when the right side exceeds its value at $s = T_\alpha$, whence $\langle M^d_{n_o}\rangle_\infty = A_{\alpha,n_o}(\infty;(T_\beta,n_\beta),\ \beta \le \alpha)$, which contradicts $S^*(k_o,n_o) > \langle M^d_{n_o}\rangle_\infty$. In case b) we have a unique $n_{\alpha+1}$ such that

$$T_{\alpha+1} = \inf\{s > T_\alpha : S(k,n_{\alpha+1}) = A_{\alpha,n}(s;(T_\beta,n_\beta),\ \beta \le \alpha)$$

$$> A_{\alpha,n}(T_\alpha;(T_\beta,n_\beta),\ \beta \le \alpha)\ \ \text{for some}\ \ k\}.$$

Now we observe that without loss of generality we can assume that $k_o - 1$ is the number of jumps of P_{n_o} by time $\langle M^d_{n_o}\rangle_{T_\alpha}$. Otherwise, since P_{n_o} and $P^*_{n_o}$ agree up to time $\langle M^d_{n_o}\rangle_{T_{\alpha+1}}$, we could reduce k_o in (1.14) and strengthen the inequality (it also is clear that no smaller k_o than this is possible when $T_\alpha < \infty$). But since

$$S^*(k_o,n_o) > \langle M^d_{n_o}\rangle_\infty \ge \langle M^d_{n_o}\rangle_t = A_{\alpha,n_o}(t;(T_\beta,n_\beta),\ \beta \le \alpha)$$

for $T_\alpha \le t \le T_{\alpha+1}$, the left side of (1.14) is not less than $T_{\alpha+1}$ if

$$A_{\alpha,n_o}(T_{\alpha+1};(T_\beta,n_\beta),\ \beta \le \alpha) > A_{\alpha,n_o}(T_\alpha;(T_\beta,n_\beta),\ \beta \le \alpha)\ ,$$

as follows by the definition of $T_{\alpha+1}$. This contradicts (1.14) with $n = n_{\alpha+1}$. On the other hand, if

$$A_{\alpha,n_o}(T_{\alpha+1};(T_\beta,n_\beta),\ \beta \leq \alpha) = A_{\alpha,n_o}(T_\alpha;(T_\beta,n_\beta),\ \beta \leq \alpha)$$

then the left side of (1.14) is still at least $T_{\alpha+1}$, and the same contradiction obtains, proving (1.13). Finally, if α is a limit ordinal and $T_\beta \in \overline{\sigma}(\underline{P}^*)$, $\beta < \alpha$, then clearly $T_\alpha = \lim_{\beta \to \alpha} T_\beta \in \overline{\sigma}(\underline{P}^*)$. Thus by induction we have shown that $T_\alpha \in \overline{\sigma}(\underline{P}^*)$ for all countable ordinals α.

Now to obtain $M_n^d(t) \in \overline{\sigma}(\underline{P}^*)$, we note first that $T_\alpha(t) \in \overline{\sigma}(\underline{P}^*)$ by applying the above proof to $(M_n^d(s \wedge t, 0 < s)$. Now let α_k be an increasing sequence of ordinals with $P\{\lim_{k\to\infty} T_{\alpha_k}(t) \geq t\} = 1.$** Then $M_n^d(t) = \lim_{k\to\infty} M_n^d(t \wedge T_{\alpha_k}(t)$ a.s., which is in $\overline{\sigma}(\underline{P}^*)$ as required.

We turn now to the demonstration that each $\langle M_n^d\rangle_t$ is a stopping vector of the continuation $(\underline{P}^*)$. This is an immediate consequence of the following lemma, which is easily generalized farther as indicated in the proof.

Lemma 1.3. Let $(B_i,P_j;i < m+1,\ j < n+1)$; $m,n \leq \infty$ be a halted Brownian-and-compensated-Poisson process, with halting vector $\underline{T} = (S_i,T_j;i < m+1,\ j < n+1)$ and product space continuation (B_i^*,P_j^*) so that $P_j^*(t)(= P_j(t \wedge T_{m+j}) + \hat{P}_j(t - (t \wedge T_{m+j}))$, with P_j and $\hat{P}_j$ independent) is a compensated Poisson process, $j < n+1$. Then in order that $(B_i,P_j;i < m+1,\ j < n+1)$ be a stopped Brownian-and-compensated-Poisson process (Def. 0.2) with stopping vector $\underline{T}$ is is necessary and sufficient that $\underline{T} \in \overline{\sigma}(B_i^*(s),\ P_j^*(s),\ 0 \leq s)$.

Proof. For notational convenience we take $m = 0$. The general case is treated by obvious modification. The necessity is also obvious, so we assume $\underline{T} \in \overline{\sigma}(\underline{P}^*)$. Replacing $\underline{T}$ by $(T_j \wedge t)$, and then letting $t \to \infty$, we may and do assume that all components are finite.

**The existence of such a sequence is easily shown. Consider $E(T_\alpha(t) \wedge t)$, which is non-decreasing in α, and strictly increasing unless $E(T_\alpha(t) \wedge t) = t$. Then clearly $(\sup_{\alpha \text{ countable}} E(T_\alpha(t) \wedge t)) \leq t$ and there exists a sequence α_k with $\lim_{k\to\infty} E(T_{\alpha_k}(t) \wedge t) = \sup_\alpha E(T_\alpha(t) \wedge t)$. If this were $< t$, then letting $\alpha_\infty = \lim_{k\to\infty} \alpha_k$, we would have $E(T_{\alpha_\infty+1}(t) \wedge t) > \sup_\alpha(T_\alpha(t) \wedge t)$, a contradiction. Therefore $E(T_{\alpha_\infty}(t)) = t$ as required.

We now take, without loss of generality, $\underline{P}^*(=(P_j^*))$ to be the coordinate process on the canonical space Ω^* of sequences (w_j) of r.c.ℓ.ℓ. paths, and for any $\underline{t} = (t_j)$ we define the translation operator $\theta_{\underline{t}}$ by $\theta_{\underline{t}}(w_j(s)) = (w_j(t_j + s))$. By definition of a halted Lévy process, $\underline{T}$ satisfies the strong Markov property

$$(1.15) \qquad P(\theta_{\underline{T}}^{-1} S \mid \mathcal{G}_{\underline{T}}) = P^{(\underline{P}^*(\underline{T}))}(S), \; S \in \sigma(\underline{P}^*).$$

Here, to avoid notational confusion, we write P rather than P^* for the probability, and $P^{\underline{x}}$ for the probability of an n-tuple of independent compensated Poisson processes starting at $\underline{x}$, and we write also $\mathcal{G}_{\underline{t}}$ for the uncompleted filtration, with $\mathcal{G}_{\underline{T}} = \sigma(P_j^*(t \wedge T_j), \; j < n+1, \; 0 \leq t)$.

It is important to note that $\underline{T} \in \mathcal{G}_{\underline{T}}$ (even without knowing $\underline{T}$ is a stopping vector) because the continuous part of $P_j^*(t \wedge T_j)$ is $-t$ for $t \leq T_j$ and $-T_j$ for $t \geq T_j$. The lemma actually can be generalized to an arbitrary right-continuous strong Markov process with parameter $\underline{t}$, simply by replacing $\mathcal{G}_{\underline{T}}$ by $\mathcal{G}_{\underline{T}} \vee \sigma(\underline{T})$ in (1.15) and thereafter.

We fix $\underline{t}$, and show that over the set $\{\underline{T} \leq \underline{t}\}$, where $\leq$ is taken component-wise, we have

$$(1.16) \qquad P(\theta_{\underline{t}}^{-1} S \mid \mathcal{G}_{\underline{t}} \vee \mathcal{G}_{\underline{T}}) = P^{\underline{P}^*(\underline{t})}(S), \; S \in \sigma(\underline{P}^*).$$

To this effect, we note first that

$$(1.17) \qquad \mathcal{G}_{\underline{t}} \vee \mathcal{G}_{\underline{T}} = \mathcal{G}_{\underline{T}} \vee \sigma(\underline{P}^*(\underline{T} + (\underline{u} \wedge (\underline{t} - \underline{T}) \vee \underline{0})); \underline{0} \leq \underline{u}).$$

Indeed, the right side is included in the left by composition of measurable functions, since each $(P_j^*(s), \; s \leq t_j)$ is measurable in $(s,(w_j))$ with respect to $\mathcal{B}[0,t_j] \times \mathcal{G}_{\underline{t}}$. Conversely, for $\underline{s} \leq \underline{t}$ and $\underline{A} = \times_{k=1}^{m} A_k$, with finite $m \leq n$ and Borel sets A_k, we can write

$$\{\underline{P}^*(\underline{s}) \in \underline{A}\} = \bigcup_{K \subset \{1,\ldots,m\}} \Big[\bigcap_{k \in K} (\{s_k \leq T_k\} \cap$$

$$\cap \{P_k^*(s_k \wedge T_k) \in A_k\}) \cap \bigcap_{\{k \notin K\}}(\{s_k > T_k\}$$

$$\cap \{P_k^*(T_k + (s_k - T_k) \wedge (t_k - T_k)) \in A_k\})] ,$$

where K ranges over all disjoint subsets. Then by filling in extra $A_k = R$ for the coordinates not included in $\{k \notin K\}$ it is easy to see that this set is in the right side of (1.17), as required.

Next, for $S_1 = \bigcap_{i=1}^{N} (\underline{P}^*(\underline{T} + (\underline{u}_i \wedge (\underline{t} - \underline{T}) \vee \underline{0})) \in \underline{A}_i)$, with $\underline{A}_i$ as above, $1 \leq i \leq N$, we will show that on $\{\underline{T} \leq \underline{t}\}$ we have

$$(1.18) \qquad E(I_{\theta_{\underline{t}}^{-1}(S) \cap S_1} | G_{\underline{T}}) = E(P^{\underline{P}^*(\underline{t})}(S) I_{S_1} | G_{\underline{T}}); \ S \in \sigma(\underline{P}^*) .$$

Indeed, using (1.15) and routine measurability argument the right side becomes

$$E^{\underline{P}^*(\underline{T})}(P^{\underline{P}^*(\underline{t}-\underline{T})}(S); \theta_{\underline{T}}(S_1))$$

on $\{\underline{T} \leq \underline{t}\}$, where we define

$$\theta_{\underline{T}} S_1(w) = \{w' \in \bigcap_{i=1}^{N} (\underline{P}^*(\underline{u}_i \wedge (\underline{t} - \underline{T}(w)), w') \in \underline{A}_i)\}.$$

But the left side of (1.18) becomes

$$E(I_{\theta_{\underline{T}}(\theta_{\underline{t}}^{-1}(S) \cap S_1)}) = E^{\underline{P}^*(\underline{T})}(I_{\theta_{\underline{t}-\underline{T}}^{-1}(S)} I_{\theta_{\underline{T}}(S_1)}) ,$$

where for fixed $\underline{T}$ we have $\theta_{\underline{T}}(S_1) \in G_{\underline{t}-\underline{T}}$. Thus by the (simple) Markov property of $\underline{P}^*$ at time $\underline{t} - \underline{T}$ this becomes the same as the right side, proving (1.18) for such S_1. Both sides being monotone in S_1, (1.18) follows. Then it follows that if $S_2 \in G_{\underline{T}}$ with $S_2 \subset \{\underline{T} \leq \underline{t}\}$ we have

$$P((\theta_{\underline{t}}^{-1}S) \cap S_1 \cap S_2) = E(P^{\underline{P}^*(\underline{t})}(S);\ S_1 \cap S_2)\ ,$$

and because finite unions of such $S_1 \cap S_2$ generate the trace of $\mathcal{G}_{\underline{t}} \vee \mathcal{G}_{\underline{T}}$ on $\{\underline{T} \leq \underline{t}\}$ by (1.17), this implies (1.16).

Now we can show that $\{\underline{T} \leq \underline{t}\}$ is in the augmentation of $\mathcal{G}_{\underline{t}}$, as required. First of all, changing $\underline{T}$ on a P-null set if necessary, we may and shall assume that $\underline{T}\ \varepsilon\ \sigma(\underline{P}^*)$ (the definition of a halted or stopped Lévy process is immune to such a change). It follows that there is a Borel function $f(\underline{x}_m;\ 1 \leq m)$ such that $I_{\{\underline{T} \leq \underline{t}\}}(w) = f(\underline{P}^*(\underline{t}_m, w);\ 1 \leq m)$ for some vector sequence $(\underline{t}_m)$. Moreover, replacing each $\underline{t}_m$ by the pair $\underline{t}_m \wedge \underline{t}$ and $\underline{t}_m \vee \underline{t}$, and for each j changing f to depend on $P_j^*(\underline{t}_m \wedge \underline{t})_j$ if this equals $P_j^*(\underline{t}_{m,j})$ but to depend on $P_j^*(\underline{t}_m \vee \underline{t})_j = P_j^*(\underline{t}_{m,j})$ otherwise, we can assume that each $\underline{t}_m$ is either $\leq \underline{t}$ or $\geq \underline{t}$. Next we introduce sets $C(w)$, $w\ \varepsilon\ \Omega^*$, given by

$$C(w) = \{w'\ \varepsilon\ \Omega^*:\ 1 = f(\underline{P}^*(\underline{t}_m, w),\ \underline{t}_m \leq \underline{t};\ \ \underline{P}^*(\underline{t}_m - \underline{t}, w'),\ \underline{t}_m \nleq \underline{t})\}.$$

In other words, we fix the coordinates $\underline{t}_m \leq \underline{t}$, and translate the rest of the coordinates by $\underline{t}$. Since f is Borel in any subset of coordinates, it is easy to see that $C(w)\ \varepsilon\ \mathcal{F}^*$ for each w, and that $P^{\underline{x}}(C(w))$ is measurable in $(\underline{x}, w)$ with respect to $\mathcal{B}^\infty \times \mathcal{F}^*$, $\mathcal{B}^\infty$ denoting the Borel field of R^∞. Indeed, this is trivially true if, for Borel f_1 and f_2,

$$I_{\{\underline{T} \leq \underline{t}\}} = f_1(\underline{P}^*(\underline{t}_m, w);\ \underline{t}_m \leq t) f_2(\underline{P}^*(\underline{t}_m, w);\ \underline{t}_m \nleq \underline{t})$$

and linear combinations of such products generate all bounded Borel f by monotone closure. Finally, by the same reasoning and the Markov property, we have

$$(1.19)\qquad P(\{\underline{T} \leq \underline{t}\} \,|\, \mathcal{G}_{\underline{t}}) = P^{\underline{P}^*(\underline{t}, w)}(C(w))\ ,$$

and in the same way

$$(1.20)\qquad P(\theta_{\underline{t}}^{-1}S \cap \{\underline{T} \leq \underline{t}\} \,|\, \mathcal{G}_{\underline{t}}) = P^{\underline{P}^*(\underline{t}, w)}(S \cap C(w)),\ S\ \varepsilon\ \sigma(\underline{P}^*).$$

Returning to (1.16) it follows that for $S_2 \in G_{\underline{t}}$ we have

$$P(\theta_{\underline{t}}^{-1}S \cap \{\underline{T} \le \underline{t}\} \cap S_2) = E(P^{\underline{P}^*(\underline{t})}(S); \{\underline{T} \le \underline{t}\} \cap S_2).$$

Since $P^{\underline{P}^*(\underline{t})}(S) \in G_{\underline{t}}$, (1.19) and (1.20) now imply

$$(1.21) \quad P(\theta_{\underline{t}}^{-1}(S) \cap \{\underline{T} \le \underline{t}\} \cap S_2) = E(P^{\underline{P}^*(\underline{t})}(S)P^{\underline{P}^*(\underline{t})}C(w); S_2)$$

$$= E(P^{\underline{P}^*(\underline{t})}(S \cap C(w)); S_2) ,$$

and consequently

$$P^{\underline{P}^*(\underline{t})}(S)P^{\underline{P}^*(\underline{t})}C(w) = P^{\underline{P}^*(\underline{t})}(S \cap C(w)), \text{ P-a.s.}$$

Since $\sigma(P^*)$ is separable, this identity holds P-a.s. for all $S \in \sigma(P^*)$, and taking $S = C(w)$ yields $P^{\underline{P}^*(\underline{t})}C(w) = 0$ or 1, P-a.s. But $\{w: P^{\underline{P}^*(\underline{t})}C(w) = 1\} \in G_{\underline{t}}$, and we see from (1.21) that this set is P-a.s. equal to $\{\underline{T} \le \underline{t}\}$, since finite unions of $\theta_{\underline{t}}^{-1}(S) \cap S_2$ generate $\sigma(P^*)$. Therefore $\{\underline{T} \le \underline{t}\} \in G_{\underline{t}}$ up to a P-null set, and hence $\underline{T}$ is a stopping vector of the augmented filtration.

Final Remark. The converse of Theorem 1.2 a), analogous to Theorem 1.1 b) proved above, is also true, but we omit the proof at present. It is easiest to assume $E\underline{T}(t) < \infty$ for all t, which was the case in Theorem 1.2 a) anyway. The main point is that any filtration

$$F_t = \sigma(B_i^*(s \wedge T_i^c(t)), i < m+1; P_j^*(s \wedge T_j^d(t)), j < n+1; 0 < s)$$

where $\underline{T}(t) = (T_i^c(t), T_j^d(t))$ is continuous in t, $\underline{T}(0) = \underline{0}$, $E\underline{T}(t) < \infty$, and each $\underline{T}(t)$ is a strict stopping vector, automatically satisfies 1)-3). To get the absence of continuous martingales, one then requires only absence of the B_i^* (Brownian) terms. Thus the well-ordering of discontinuity times is unnecessary for the converse, provided we also omit it from the conclusion. Since this argument has its natural setting in greater generality than the present paper, we will defer it to a later date.

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