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THE FIRST PASSAGE PROBLEM FOR GENERALIZED ORNSTEIN-UHLENBECK PROCESSES WITH NON-POSITIVE JUMPS *

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1. Introduction. Let (Ω,F,P) be a probability space. We consider a cadlag stationary random process $S_t, t\geq 0$, with independent increments and non-positive jumps $\Delta S_t = S_t - S_{t-} = S_t - \lim_{s\uparrow t} S_s \leq 0$, that is defined on this space and satisfies $S_0 = 0$. It is well known ([3]) that the characteristic function of S_t has the form

(1.1) $$E\exp(iuS_t) = \exp t\{ibu - cu^2 + \int_{(-\infty,0)} F(dx)(e^{iux} - 1 - iux.1_{\{x\geq -1\}})\},$$

where $-\infty < b < \infty$, $c \geq 0$, and the Lévy measure $F(.)$ satisfies

(1.2) $$\int_{(-\infty,0)} F(dx)\ 1\wedge x^2 < \infty .$$

Following Skorokhod ([8]) one can use the analytical continuation of (1.1) to the half-plane $\mathrm{Re}(iu) > 0$ and obtain the Laplace transform of S_t by substituting u instead of iu. Thus, we have

(1.3) $$E\exp(uS_t) = \exp t\psi(u),\ u \geq 0,$$

where

(1.4) $$\psi(u) = bu + cu^2 + \int_{(-\infty,0)} F(dx)(e^{ux} - 1 - ux.1_{\{x\geq -1\}}).$$

For arbitrary $\lambda>0$ and $-\infty < x < \infty$ we define the random process X_t, $t\geq 0$, by the formula

(1.5) $$X_t = e^{-\lambda t}(x + \int_{(0,t]} e^{\lambda v}\, dS_v),$$

the stochastic integral w.r.t. the semi-martingale S being understood in the usual sense.

Definition. The random process X will be called the starting at x generalized Ornstein-Uhlenbeck process with parameter $\lambda>0$.

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Certainly, the process X is characterized by the triplet $(b,c,F(.))$ as well. With $b = 0$, $c = {}^1/_2$ and $F(.) = 0$ our definition yields the standard Wiener process S and the usual Ornstein-Uhlenbeck process X.

Given a real number $\mu > x$, let us introduce the first passage time

(1.6) $$T_\mu(x) = \inf\{t \geqq 0 : X_t \geqq \mu\}.$$

As far as $\Delta X_t = \Delta S_t \leqq 0$, if $T_\mu(x) < \infty$ one gets immediately the equality $X_{T_\mu(x)} = \mu$.

The purpose of this paper is to determine the distribution of $T_\mu(x)$, $\mu > x$, by means of Laplace transform

(1.7) $$\gamma_\mu(\theta,x) = E\exp(-\theta T_\mu(x)), \quad \theta > 0.$$

It should be noted that generally speaking, we have no equation for the transition density of X and the usual Darling-Siegert approach to the first passage problem of diffusion processes ([2]) is not applicable in our case. Our approach is based on martingale techniques and depends essentially on the existence of suitable martingales on the process X (see Theorem 1 below). Besides the new generality of the explicit representation for $\gamma_\mu(\theta,x)$ (Section 4), this approach gives us in particular the possibility to obtain ones again and in a natural way the interesting result of Novikov ([6]) concerning the first passage times of a stable process S through one-sided non-linear boundaries. The basic tool in this special case is the suitable time-change (Section 6) that transfers the linear problems for X_t, $t \geqq 0$, into some non-linear problems for S_t, $t \geqq 0$, and conversely. We make use of the reconversion in order to give an example of optimal stopping problem that admits a solution in terms of $T_\mu(x)$.

2. The process X. For the next we need to calculate the conditional Laplace transforms of the process X that was defined in (1.5). Let us introduce the σ-algebras $F_t^X = \sigma(X_s, 0 \leqq s \leqq t)$; $t \geqq 0$, and the functions $L(u;t,s) = E\{\exp(uX_t)|F_s^X\}, s < t$, $u > 0$.

Since the stochastic integral in (1.5) might be looked at as an integral taken in the sense of convergence in probability ([4]), a simple argument leads to the following result.

Proposition 1. For any $0 \leq s < t$ and $u \geq 0$ one has

$$\text{(2.1)} \qquad L(u;t,s) = \exp\{e^{-\lambda(t-s)}X_s.u + \int_s^t \psi(u.e^{-\lambda(t-v)})\, dv\}.$$

Proof. With an arbitrary subdivision $s = t_o < t_1 < \ldots < t_n = t$, $\varepsilon = \max_{i\leq n}|t_i - t_{i-1}|$ and $Y_t = \int_{(0,t|} e^{\lambda v}\, dS_v$, we get

$$E\{\exp(uY_t)|F_s^X\} = \exp(uY_s).\, E\{\exp(u \int_{(s,t]} e^{\lambda v}\, dS_v)|F_s^X\}$$

$$= \exp(uY_s) \lim_{\varepsilon\downarrow 0} \prod_{i=1}^{n} E\exp(u.e^{\lambda t_{i-1}}.(S_{t_i} - S_{t_{i-1}}))$$

$$= \exp(uY_s) \lim_{\varepsilon\downarrow 0} \prod_{i=1}^{n} \exp\{\psi(u.e^{\lambda t_{i-1}})(t_i - t_{i-1})\}$$

$$= \exp(uY_s + \int_s^t \psi(u.e^{\lambda v})\, dv)$$

as a consequence of (1.3) and the independent increments property of S.

Now starting with (1.5) we have

$$L(u;t,s) = \exp(e^{-\lambda t}.xu)\; E\{\exp(u.e^{-\lambda t}.Y_t)|F_s^X\}$$

$$= \exp\{e^{-\lambda t}.xu + e^{-\lambda t}.Y_s u + \int_s^t \psi(u.e^{-\lambda(t-v)})\, dv\}$$

and the latter obviously implies (2.1).

Corollary 1. The Laplace transform of X_t has the form

$$E\exp(uX_t) = \exp\{e^{-\lambda t}.xu + \int_0^t \psi(u.e^{-\lambda(t-v)})\, dv\},\; u \geq 0.$$

Corollary 2. The process X is a cadlag Markov process. (Certainly, X has also the strong Markov property.)

3. The martingale M. We are going to introduce a martingale $M_t(\theta)$, $t \geq 0$, depending on the process X trajectories. To this end, one observes that because of (1.2) the quantity $F[-1,-z]$ is finite for every z, $0 < z \leq 1$. Thus, the measure

$$G(dz) = F[-1,-z]\, dz$$

on $(0, 1]$ is well defined. We need the following assumption.

Hypothesis G. Either $c > 0$ or the measure $G(.)$ satisfies the condition

$$\lim_{z\to 0+} z^{\kappa}.G(z,1] = C > 0 \tag{3.1}$$

for some constant κ, $0 < \kappa < 1$.

Next, one defines successively

$$g(y) = -\frac{1}{\lambda}\int_1^y \frac{\psi(u)}{u}\, du, \quad y > 0, \tag{3.2}$$

and

$$M_t(\theta) = e^{-\theta t}.\int_0^\infty y^{\frac{\theta}{\lambda}-1}.\exp\{X_t.y + g(y)\}\, dy, \quad t \geq 0. \tag{3.3}$$

The next statement is crucial because it permits an essential use of the martingale theory later on.

Theorem 1. Under the hypothesis G for any positive θ the random process $M_t(\theta)$, $t \geq 0$, is a martingale w.r.t. F_t^X, $t \geq 0$.

Proof. First, we observe that our hypothesis G implies the convergence of the integral in (3.3). In fact, we have

$$g(y) = -\frac{b}{\lambda}(y-1) - \frac{c}{2\lambda}(y^2-1) - \frac{1}{\lambda}g_1(y) - \frac{1}{\lambda}g_2(y),$$

where

$$g_1(y) = \int_1^y \frac{\psi_1(u)}{u}\, du \quad , \qquad g_2(y) = \int_1^y \frac{\psi_2(u)}{u}\, du$$

and

$$\psi_1(u) = \int_{(-\infty,-1)} F(dx)(e^{ux}-1), \quad \psi_2(u) = \int_{[-1,0)} F(dx)(e^{ux}-1-ux), \quad u \geq 0.$$

The convergence of the integral at $y = 0$ is obvious, because $\infty > \lim_{y\downarrow 0} g(y) \geq -\infty$.

Now let us denote $d_1 = \int_{(-\infty,-1)} F(dx) \geq 0$, $d_2 = \int_{[-1,0)} F(dx)\, x^2 \geq 0$. In consequence of (1.2) one gets $0 \leq d_1 + d_2 < \infty$. Our function ψ_1 satisfies $0 \geq \psi_1(u) \downarrow -d_1$ and $0 \geq \frac{\psi_1(u)}{u} \uparrow 0$ as $u\uparrow\infty$. This means that $|g_1(y)| \leq \int_1^y \left|\frac{\psi_1(u)}{u}\right| du \leq d_1 \ln y$. On the other hand $0 < e^{ux} - 1 - ux \leq \frac{u\, x}{2}$, $u > 0$, $-1 \leq x < 0$, and in this way one obtains the inequalities $0 \leq \frac{\psi_2(u)}{u} \leq \frac{u}{2}.\, d_2 < \infty$ and $0 \leq g_2(y) \leq \frac{d_2}{4}(y^2-1)$.

If $c > 0$, the corresponding term $-\frac{c}{2\lambda}(y^2 - 1)$ in $g(y)$ ensures the convergence. If $c = 0$, by the equality $\frac{\psi_2(u)}{u} = \int_0^1 (1 - e^{-uz})\, G(dz)$, where obviously $0 \leqq \int_0^1 z\, G(dz) = \frac{d_2}{2} < \infty$, the hypothesis (3.1) and the corollary of Theorem 4.15 in [1] one gets $\underline{\lim}_{u\to\infty} u^{-\kappa}.\frac{\psi_2(u)}{u} \geqq C.\Gamma(1-\kappa) > 0$. Consequently, $\frac{\psi_2(u)}{u} \geqq C_2.u^{\kappa}$ for any C_2 belonging to the interval $(0, C.\Gamma(1-\kappa))$ and $u \geqq u_2(C_2) > 0$ (sufficiently large). This implies $g_2(y) \geqq C_2.y^{1+\kappa} + C_1$, $y > u_2(C_2)$, and the convergence of our integral too.

Secondly, applying Fubini's lemma and (2.1) for $0 \leqq s \leqq t$ (and with $z = ye^{-\lambda(t-s)}$) we get

$$E\{M_t(\theta) \mid F_s^X\} = e^{-\theta t}. \int_0^\infty y^{\frac{\theta}{\lambda}-1} E\{\exp(X_t.y + g(y)) \mid F_s^X\}dy$$

$$= e^{-\theta s}. \int_0^\infty y^{\frac{\theta}{\lambda}-1} \exp\{g(y) - \theta(t-s) + e^{-\lambda(t-s)}y.X_s + \int_s^t \psi(ye^{-\lambda(t-v)})\, dv\}\, dy$$

$$= e^{-\theta s}. \int_0^\infty z^{\frac{\theta}{\lambda}-1} \exp\{zX_s + g(ze^{\lambda(t-s)}) + \int_0^{t-s} \psi(ze^{\lambda v})\, dv\}\, dz.$$

But the function $f(u,z) = g(ze^{\lambda u}) + \int_0^u \psi(ze^{\lambda v})\, dv$, $u \geqq 0$, satisfies the condition

$$\frac{\partial f(u,z)}{\partial u} = g'(ze^{\lambda u}).\, z\lambda e^{\lambda u} + \psi(ze^{\lambda u}) = g'(y).\lambda y + \psi(y) \equiv 0$$

with $y = ze^{\lambda u}$, in view of (3.2). Therefore,

$$f(u,z) = \text{const} = f(0,z) = g(z)$$

and we get $E\{M_t(\theta) \mid F_s^X\} = X_s$, that completes the proof.

Remark 1. We emphasize the fact that Theorem 1 is valid for every process X with S containing a Gaussian component $(c > 0)$. If the process S has no Gaussian component $(c = 0)$, the condition (3.1) is nevertheless fulfilled for a class of measures $F(.)$ that includes the stable processes S with parameter α satisfying $1 < \alpha < 2$. Because of its importance, we consider this special case in Section 5.

4. The Laplace transform of $T_\mu(x)$. Now we are in a position to derive an explicite expression for the Laplace transform $\gamma_\mu(\theta,x)$. Due to the particular structure of

the martingale $M(\theta)$ we have the following result.

Theorem 2. Under the hypothesis G the next equality holds:

(4.1) $$\gamma_\mu(\theta,x) = \frac{\int_0^\infty y^{\frac{\theta}{\lambda}-1} \exp(xy + g(y))\, dy}{\int_0^\infty y^{\frac{\theta}{\lambda}-1} \exp(\mu y + g(y))\, dy}, \quad \theta > 0.$$

Proof. We put $T_\mu(x)\wedge t$ instead of t in (3.3) and we make use of the well known martingale property that

$$E\, M_{T_\mu(x)\wedge t}(\theta) = E\, M_0(\theta) = \int_0^\infty y^{\frac{\theta}{\lambda}-1} \exp(xy + g(y))\, dy.$$

Next, one observes that

$$0 \leqq M_{T_\mu(x)\wedge t}(\theta) \leqq \int_0^\infty y^{\frac{\theta}{\lambda}-1} \exp(\mu y + g(y))\, dy$$

and, moreover, when $T_\mu(x) = \infty$ then

$$0 \leqq M_{T_\mu(x)\wedge t}(\theta) = M_t(\theta) \leqq e^{-\theta t} \int_0^\infty y^{\frac{\theta}{\lambda}-1} \exp(\mu y + g(y))\, dy$$

as well. Therefore,

$$\lim_{t\to\infty} E\, M_{T_\mu(x)\wedge t}(\theta) = E\, M_{T_\mu(x)} \cdot 1_{\{T_\mu(x) < \infty\}} = \int_0^\infty y^{\frac{\theta}{\lambda}-1} \exp(\mu y + g(y))dy.\ \gamma_\mu(\theta,x).$$

The right-hand sides of our equalities give directly (4.1).

Remark 2. For the validity of Theorem 2 we need not (and we did not use) any fact about the finiteness of $T_\mu(x)$. It is well known that $T_\mu(x) < \infty$ P-a.s. if and only if $\lim_{\theta\downarrow 0} \gamma_\mu(\theta,x) = 1$. The latter equality is easily verified when there exists $\lim_{y\downarrow 0} g(y) > -\infty$ or when $\int_{(-\infty,-1)} F(dx)\,|x| < \infty$.

5. The case of stable process S with parameter $1 < \alpha \leq 2$. Now we turn to the particular case when the following hypothesis is satisfied.

Hypothesis H_α. Either $F(.) = 0$ and $c > 0$ (we characterize this by posing $\alpha = 2$), or $c = 0$ and $F(dx) = \frac{\sigma.dx}{|x|^{\alpha+1}} 1_{\{x < 0\}}$ for some $\sigma > 0$ and $1 < \alpha < 2$.

Using standard arguments (see [8] , §25, Theorem 4) one obtains the equivalent form of H_α in the terms of our function ψ : H_α , $1 < \alpha \leqq 2$, means that

(5.1) $$\psi(u) = \overline{\psi}(u) = \bar{b}u + \bar{\sigma}u^\alpha$$

with some $\bar{b}$, $-\infty < \bar{b} < \infty$, and $\bar{\sigma} > 0$. In this situation by (3.2) we get

(5.2) $$g(y) = \bar{g}(y) = -\frac{\bar{b}}{\lambda}(y-1) - \frac{\bar{\sigma}}{\alpha\lambda}(y^\alpha - 1),$$

and the martingale $M(\theta)$ is well defined via (3.3).

Following Novikov we introduce the function

$$H(\nu,\alpha,x) = \frac{1}{\Gamma(-\alpha\nu)} \int_0^\infty y^{-\alpha\nu-1} \exp(xy - \frac{1}{\alpha}y^\alpha)\, dy,$$

which turns to be analytic in the half-plane $\mathrm{Re}\,\nu < 1$. All the essential properties of $H(\nu,\alpha,x)$ are collected in the supplement of [6].

Next we obtain a special case of Theorem 2.

Proposition 2. Under the hypothesis H_α, $1 < \alpha \leq 2$, the following equality holds for $\theta > 0$:

(5.3) $$\gamma_\mu(\theta,x) = \frac{H(-\frac{\theta}{\alpha\lambda}, \alpha, (\frac{\lambda}{\bar{\sigma}})^{\frac{1}{\alpha}}(x - \frac{\bar{b}}{\lambda}))}{H(-\frac{\theta}{\alpha\lambda}, \alpha, (\frac{\lambda}{\bar{\sigma}})^{\frac{1}{\alpha}}(\mu - \frac{\bar{b}}{\lambda}))}.$$

Moreover, this formula defines also an analytical continuation of the Laplace transform $\gamma_\mu(\theta,x)$ to the half-plane $\mathrm{Re}\,\theta > -\alpha\lambda.\nu_\alpha(\bar{\mu})$, where $\bar{\mu} = (\frac{\lambda}{\bar{\sigma}})^{\frac{1}{\alpha}}(\mu - \frac{\bar{b}}{\lambda})$ and $\nu_\alpha(z)$ is the smallest positive zero of $H(\nu,\alpha,z)$ with (α,z) fixed.

Proof. Applying the change of variables $y = (\frac{\lambda}{\bar{\sigma}})^{\frac{1}{\alpha}}z$ we see the formula (5.3) is another form of (4.1) for $\theta > 0$. As far as the right-hand side of (5.3) is analytic in θ in the half-plane $\mathrm{Re}\,\theta > -\alpha\lambda.\nu_\alpha(\bar{\mu})$ (see [6]), the left-hand side can be analytically continued in θ to this half-plane.

Corollary 3. Since $\lim_{\nu\to 0} H(\nu,\alpha,x) = 1$, $-\infty < x < \infty$, under the hypothesis H_α we get $\lim_{\theta\downarrow 0} \gamma_\mu(\theta,x) = 1$ and, consequently, $T_\mu(x) < \infty$ P-a.s.

6. The time change - two applications. Throughout this section we suppose the hypothesis H_α holds with some α, $1 < \alpha \leq 2$, and $\bar{b} = 0$ (see (5.1)). As a consequence we have

$$\psi(u) = \bar{\psi}(u) = \bar{\sigma}.u^\alpha, \quad 1 < \alpha \leq 2,$$

and the process X is stationary too (see (2.1)).

Let us introduce the real (increasing and continuous) function

$$\delta(t) = (\alpha\lambda)^{-1}(e^{\alpha\lambda t} - 1), \ t \geq 0,$$

which determines an one-to-one mapping of $[0,\infty)$ onto $[0,\infty)$, and the convers function

$$\rho(t) = (\alpha\lambda)^{-1}\ln(1 + \alpha\lambda t), \ t \geq 0.$$

Lemma 1. The distributions of S_t, $t \geq 0$, and of $\tilde{S}_t = \int_0^{\rho(t)} e^{\lambda v} dS_v$, $t \geq 0$, coinside.

Proof. As in Proposition 1 one calculates

$$E \exp(u\tilde{S}_t) = E \exp(uY_{\rho(t)}) = \exp\{\bar{\sigma}u^\alpha . \delta(\rho(t))\} = \exp(\bar{\sigma}u^\alpha t), \ u > 0.$$

But under the hypothesis stated (H_α and $\bar{b} = 0$) the latter term is just $E \exp(uS_t)$. The lemma is proved.

Now for any constants a, b and c such that $b \geq 0$ and $ab^{\frac{1}{\alpha}} + c > 0$, define the stopping time $\tau(a,b,c)$ w.r.t. F_t^S, $t \geq 0$, by the formula

$$\tau(a,b,c) = \inf \{t > 0 : S_t \geq a(t + b)^{\frac{1}{\alpha}} + c\} \tag{6.1}$$

and pose

$$\tau_\mu(x) = \tau(\mu(\alpha\lambda)^{\frac{1}{\alpha}}, (\alpha\lambda)^{-1}, -x), \ \mu > x. \tag{6.2}$$

The following simple fact is valid in our situation.

Theorem 3. The stopping time $T_\mu(x)$ has the same distribution as $\rho(\tau_\mu(x))$ does.

Proof. We define similarly $\tilde{\tau}(a,b,c)$ and $\tilde{\tau}_\mu(x)$ by replacing S_t by $\tilde{S}_t$ in (6.1) and (6.2). Next, starting with (1.6), we calculate

$$\begin{aligned} T_\mu(x) &= \inf \{t : x + Y_t \geq \mu e^{\lambda t}\} \\ &= \inf \{\rho(s) : Y_{\rho(s)} \geq \mu e^{\lambda\rho(s)} - x\} \\ &= \inf \{\rho(s) : \tilde{S}_s \geq \mu(1 + \alpha\lambda s)^{\frac{1}{\alpha}} - x\} = \rho(\tilde{\tau}_\mu(x)). \end{aligned}$$

The statement of the theorem follows from Lemma 1 which says the distribution of $\tilde{\tau}_\mu(x)$ coinsides with the distribution of $\tau_\mu(x)$.

From Theorem 3 and Proposition 2 we deduce the following result of A.Novikov (see [6], Theorem 1).

<u>Theorem 4</u>. For every a,b,c with $b \geqq 0$, $ab^{\frac{1}{\alpha}} + c > 0$, one has

(6.3) $$E\,(\tau(a,b,c) + b)^{\nu} = b^{\nu}.\frac{H(\nu,\alpha,-cb^{-\frac{1}{\alpha}}.d)}{H(\nu,\alpha,\, ad)}\quad, \text{ if } b > 0 \text{ and } \nu < \nu_{\alpha}(ad),$$

and

(6.4) $$E\,(\tau(a,b,c)^{\nu}) = \begin{cases} \dfrac{(cd)^{\alpha\nu}}{H(\nu,\alpha,\, ad)} & , \text{ if } \nu < \nu_{\alpha}(ad), \\ +\infty & , \text{ if } \nu \geqq \nu_{\alpha}(ad), \end{cases}$$

where $d = (\alpha\bar{\sigma})^{\frac{1}{\alpha}}$.

<u>Proof</u>. Assume $b > 0$ and put $x = -c$, $\lambda = (\alpha b)^{-1}$, $\mu = ab^{\frac{1}{\alpha}}$. Then

$$\mu - x = ab^{\frac{1}{\alpha}} + c > 0,\quad \bar{\mu} = (\frac{\lambda}{\sigma})^{\frac{1}{\alpha}}.\mu = ad$$

and by Proposition 2 with $\nu = -\frac{\theta}{\alpha\lambda}$ we get the equalities

$$E\,(\tau(a,b,c) + b)^{\nu} = E\,(\tilde{\tau}_{\mu}(x) + \frac{1}{\alpha\lambda})^{\nu}$$
$$= b^{\nu}.E\,(\alpha\lambda\tilde{\tau}_{\mu}(x) + 1)^{\nu} = b^{\nu}.E\,\exp\{\nu\ln(1 + \alpha\lambda\tilde{\tau}_{\mu}(x))\}$$
$$= b^{\nu}.E\,\exp\{-\,\theta\rho(\tilde{\tau}_{\mu}(x))\} = b^{\nu}.\frac{H(\nu,\alpha,-cb^{-\frac{1}{\alpha}}.d)}{H(\nu,\alpha,\, ad)},$$

provided that $\theta > -\alpha\lambda\nu_{\alpha}(ad)$ (or $\nu < \nu_{\alpha}(ad)$). The rest statements of the theorem follow from the properties of $H(\nu,\alpha,x)$, the case $b = 0$ being taken into account by letting $b \downarrow 0$ (or $\lambda \to +\infty$).

<u>Remark 3</u>. In the original theorem of Novikov (with $d = 1$, see [6]) one makes use of the fact that

$$(t + b)^{\nu}.\, H(\nu,\alpha,\frac{S_t - c}{(t + b)^{\frac{1}{\alpha}}}),\ t \geqq 0,\ b > 0,$$

is a complex-valued martingale (w.r.t. F_t^S, $t \geqq 0$) for every complex ν with $\mathrm{Re}\,\nu < 1$. This fact involves an analytical continuation in contrast to our Theorem 1.

As a second example we consider an optimal stopping problem originally treated in more general setting in [5], [7] and [9]. This problem admits a simple solution in terms of stopping times $T_{\mu}(x)$.

Under the hypothesis stated at the beginning of this section (H_α and $\bar{b} = 0$) the quantity

(6.5) $$v(x,b,\tau) = E\,\frac{x + S_\tau}{b + \tau},\ b > 0,\ -\infty < x < \infty,$$

is to be maximized on stopping times $\tau = \tau(\omega)$ w.r.t. F_t^S, $t \geqq 0$.

By Lemma 1 we have

$$v(x,b,\tau) = v(x,b,\tilde{\tau}) = E\,\frac{x + \tilde{S}_{\tilde{\tau}}}{b + \tilde{\tau}},$$

using $\tilde{S}_t = Y_{\rho(t)}$, $t \geqq 0$, and $\tilde{\tau}$ in the place of S_t, $t \geqq 0$, and τ. Now taking $\lambda = \frac{1}{\alpha b}$ and $t = \delta(s)$, $s \geqq 0$, we get

$$\frac{x + \tilde{S}_t}{b + t} = \frac{x + Y_{\rho(t)}}{b + t} = \frac{e^{\lambda\rho(t)}.X_{\rho(t)}}{\frac{1}{\alpha\lambda} + t} = \frac{e^{\lambda s}.X_s}{\frac{1}{\alpha\lambda}e^{\alpha\lambda s}} = \alpha\lambda e^{-(\alpha - 1)\lambda s}.X_s\ .$$

Consequently, it is equivalent to consider the problem of maximizing the quantity

(6.6) $$V(x,b,T) = \frac{1}{b}\,E\,e^{-\beta T}.X_T\ ,\ \beta = \frac{\alpha - 1}{\alpha b} > 0,$$

on stopping times $T = T(\omega)$ w.r.t. F_s^X, $s \geqq 0$, provided that $T = \rho(\tau)$, because $V(x,b,T) = v(x,b,\tau)$.

By [7] for $\alpha = 2$ and [5] for $1 < \alpha < 2$ one knows the solution of the original problem of maximizing (6.5) is one of the stopping times $\tau(a,b,-x)$ or the stopping time $\tau_0 = 0$.

Let us denote

$$\Psi(\mu) = \frac{\int_0^\infty y^{\alpha-2}\exp(\mu y - \bar{\sigma} b y^\alpha)\,dy}{\int_0^\infty y^{\alpha-1}\exp(\mu y - \bar{\sigma} b y^\alpha)\,dy},\ -\infty < \mu < \infty\ .$$

As far as $\Psi(\mu)$ is positive, decreasing and continuous and $\Psi(0) = \Gamma(\frac{\alpha-1}{\alpha}) > 0$, the equation $\mu = \Psi(\mu)$ has a unique solution $\tilde{\mu}$ (moreover, $0 < \tilde{\mu} < \Psi(0)$). The corresponding result in our case is given below without proof because it can be justified as in [5] and [7] (see also [9], Example 2, for the case $\alpha = 2$ and $\lambda = 1$).

Theorem 5. For every real x and $b > 0$, either the stopping time $T_{\tilde{\mu}}(x)$, or the stopping time $T_x(x) = 0$ maximizes the quantity (6.6). More precisely,

$$\sup_T V(x,b,T) = V(x,b,T_{\tilde{\mu}}) = \frac{\tilde{\mu}}{b} \gamma_{\tilde{\mu}}(\beta,x) \quad \text{if} \quad x \leqq \tilde{\mu},$$

and

$$\sup_T V(x,b,T) = V(x,b,0) = \frac{x}{b} \quad \text{if} \quad x > \tilde{\mu}.$$

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