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On Local Times of a Diffusion

by P. Salminen

Abstract

In this note we consider local time of a regular, transient diffusion as a density of a occupation measure, on the one hand, and as a dual predictable projection, on the other hand. The essential tool in our discussion is the Doob-Meyer decomposition for submartingales.

1. Introduction

Let X be a regular, canonical, one-dimensional diffusion on an interval $I \subseteq (-\infty,+\infty)$. It is well-known that for every $y \in I$ there exist a local time process L_t^y . By the usual definition L^y is an integrable, increasing stochastic process defined over the same probability space as X such that, with probability one,

(1.1) $\quad (t,y) \rightsquigarrow L_t^y$ is continuous,

(1.2) $\quad$ for every $A \in \mathcal{B}(I)$ (= Borel subsets of I) and $t \geq 0$

$$\int_0^t 1_A(X_s)\,ds = \int_A L_t^y \, m(dy) ,$$

where 1_A is the indicator function of the set A and m is the speed measure of X

A classical and elegant proof for the existence of L^y for a Brownian motion is via Tanaka's formula (see, for example, [5]). This proof extends for an arbitrary regular diffusion by a standard random time change argument and a scale transformation.

Tanaka's approach to Brownian local times has been generalized for semi-martingales by Meyer (see [9]): Let X be a continuous semi-martingale on $\mathbb{R}$. For every $y \in \mathbb{R}$ there exists an increasing, continuous process $\hat{L}_t^y$, called local time, such that

$$(1.3) \qquad (X_t - y)^+ = (X_0 - y)^+ + \int_0^t 1_{\{X_s > a\}} dX_s + \tfrac{1}{2} \hat{L}_t^y .$$

Further the measure $d\hat{L}_t^y$ is almost surely supported by $\{s : X_s = y\}$ and satisfies

$$(1.4) \qquad \int_0^t 1_A(X_s) d\langle X,X\rangle_s = \int_A \hat{L}_t^y \, dy ,$$

where $A \in \mathcal{B}(\mathbb{R})$ and $\langle X,X\rangle$ is the quadratic variational process of the continuous semi-martingale X.

Meyer's approach is not directly applicable and always suitable for diffusions. Firstly a regular diffusion is not in general a semi-martingale (see [3]). Secondly for a semi-martingale diffusion the local time $\hat{L}_t^y$ given by (1.3) is not in general jointly continuous (see [14]). Consequently, (1.2) does not hold with $\hat{L}_t^y$, and, if the speed measure is not absolutely continuous with respect to the Lebesque measure, we cannot deduce (1.2) from (1.4).

We may consider the existence of $\hat{L}_t^y$ in (1.3) as a consequence of the Doob-Meyer decomposition. The aim of this note is to explore this connection for a regular diffusion.

2. Preliminaries

We assume that the diffusion X is transient: for all $x,y \in I$ $\mathbb{P}_x(\lambda_y < \infty) = 1$ where

$$\lambda_y = \begin{cases} \sup\{t : X_t = y\} & \text{if } \{\cdot\} \neq \emptyset , \\ 0 & \text{otherwise,} \end{cases}$$

and $\mathbb{P}_x$ is the probability measure associated with X, $X_0 = x$. The non-transient case can be treated by killing the diffusion in some fashion so that it becomes transient. We note that our definition of transience, in the case there are no absorbing points, is equivalent with the condition of Itô and McKean (see [6] p. 124 and 134): Let

$$\tau_y = \begin{cases} \inf\{t : X_t = y\} & \text{if } \{\cdot\} \neq \emptyset , \\ +\infty & \text{otherwise.} \end{cases}$$

Then for all $x,y \in I$ either

(2.1) $$\mathbb{P}_x(\tau_y < \infty)\, \mathbb{P}_y(\tau_x < \infty) = 1$$

or

(2.2) $$\mathbb{P}_x(\tau_y < \infty)\, \mathbb{P}_y(\tau_x < \infty) = 1 .$$

When (2.1) holds X is called recurrent and in the case (2.2) transient.

Let $\zeta = \inf\{t : X_t \notin I\}$. Introduce a fictious state Δ and extend X to $I \cup \{\Delta\}$ by setting $X_t = \Delta$ for $t \geq \zeta$. We use the usual convention that $f(\Delta) = 0$ for any function f defined on I . The left- and right-hand end-point of I are denoted with a and b , respectively. We assume that a killing boundary does not belong to I . Obviously, if I is a open, finite interval then a sequence of points in I converges to Δ if and only if it converges (in the usual topology) to a , or b , "or both".

We remark that X being a Feller process is quasi-left-continuous. Therefore

(2.3) ζ is not predictable on the set $\{ X_{\zeta-} \in I\}$.

Clearly $\{X_{\zeta-} \notin I\} = \{X_{\zeta-} = a$ or $b\}$, and hence ζ is predictable on $\{X_{\zeta-} \notin I\}$ and X is quasi-left-continuous in the topology of $I \cup \{\Delta\}$.

It can be proved (see [6] p 159) that X is transient if and only if

$$G(x,y) := \lim_{\alpha \downarrow 0} G^{\alpha}(x,y) = \int_0^{\infty} p(t;x,y)dt < \infty ,$$

where $p(t;x,y)$, $t \geq 0$, $x,y \in I$, is the jointly continuous transition density (with respect to the speed measure m) of X and

$$G^{\alpha}(x,y) = \int_0^{\infty} e^{-\alpha t} p(t;x,y)dt .$$

Further (see [6] p 160)

$$G(x,y) = \begin{cases} \phi^{\uparrow}(x)\phi^{\downarrow}(y) & x \leq y \\ \phi^{\uparrow}(y)\phi^{\downarrow}(x) & x \geq y , \end{cases}$$

where $\phi^{\uparrow}$ ($\phi^{\downarrow}$) is a continuous, positive, increasing (decreasing) solution of the equation

$$(2.4) \qquad u^{+}(y) - u^{+}(x) = \int_{(x,y]} u(z)k(dz)$$

with $a < x < y < b$, and on I

$$(2.5) \qquad \phi^{\uparrow +} \phi^{\downarrow} - \phi^{\uparrow}\phi^{\downarrow +} \equiv 1 .$$

Here k is the killing measure, and

$$(2.6) \qquad u^{+}(x) = \lim_{y \downarrow x} \frac{u(y) - u(x)}{S(y) - S(x)}$$

where S is the scale function of X. For left derivatives

$$(2.6)' \qquad u^{-}(x) = \lim_{y \uparrow x} \frac{u(x) - u(y)}{S(x) - S(y)}$$

(2.4) and (2.5) take the forms

$$(2.4)' \qquad u^-(y) - u^-(x) = \int_{[x,y)} u(z)k(dz) \; ,$$

$$(2.5)' \qquad \phi^{\uparrow -}\phi^{\downarrow} - \phi^{\uparrow}\phi^{\downarrow -} \equiv 1 \; , \text{ respectively.}$$

Finally we need the following result (see [13] (3.3) Proposition) : For $a < \alpha < x < \beta < b$

$$(2.7) \qquad \begin{aligned} \mathbb{P}_x(X_{\zeta-} > \beta) &= -\phi^{\uparrow}(x)\phi^{\downarrow +}(\beta) \; , \\ \mathbb{P}_x(X_{\zeta-} < \alpha) &= \phi^{\downarrow}(x)\phi^{\uparrow -}(\alpha) \; . \end{aligned}$$

In fact, in [13] this result is only proved in the case $k \equiv 0$; however it is easily seen that it is valid also in the general transient case.

<u>Example</u>: Let $I = \mathbb{R}$, $m(dx) = 2dx$, $S(x) = x$, and $k(dx) = \varepsilon_{\{0\}}(dx)$ (= Dirac's measure at 0). Then

$$\phi^{\uparrow}(x) = \begin{cases} 1 & x \le 0 \\ x + 1 & x \ge 0 \end{cases}$$

and

$$\phi^{\downarrow}(x) = \begin{cases} 1 - x & x \le 0 \\ 1 & x \ge 0 \; . \end{cases}$$

This process is a Brownian motion which is killed "elastically" at 0 . Note that for all x $G(x,0) = 1$. Further ζ is not predictable, and $\zeta < \infty$ $\mathbb{P}_x$-a.s.

<u>3. Existence of local time as a density of a occupation measure</u>

For $x \in I$ and $A \in \mathcal{B}(I)$ let $L_t^A = \int_0^t 1_A(X_s)ds$ and $G(x,A) = \int_A G(x,y)m(dy)$. Introduce for $t \ge 0$

$$M_t^A = G(x,A) - G(X_t,A) - L_t^A \; .$$

Recall the convention $G(\Delta,A) = 0$. We have the following easy

(3.1) Proposition. The process M^A is a $(\mathbb{P}_x, F_t)$-martingale, where $(F_t)_{t\geq 0}$ are the natural, completed filtrations of X .

Proof. Let ξ_s be a F_s-measurable, bounded, and positive random variable. We have for $t > s$

$$\mathbb{E}_x(\xi_s(G(X_s,A) - G(X_t,A)))$$

$$= \mathbb{E}_x(\xi_s(G(X_s,A) - \mathbb{E}(G(X_t,A)(F_s)))$$

$$= \mathbb{E}_x(\xi_s(G(X_s,A) - \mathbb{E}_{X_s}(G(X_{t-s},A)))$$

$$= \mathbb{E}_x(\xi_s(\int_0^\infty \mathbb{P}_{X_s}(X_u \in A)dn - \int_{t-s}^\infty \mathbb{P}_{X_s}(X_u \in A)dn))$$

$$= \mathbb{E}_x(\xi_s \int_0^{t-s} \mathbb{P}_{X_s}(X_u \in A)dn)$$

$$= \mathbb{E}_x(\xi_s \mathbb{E}_{X_s}(\int_0^{t-s} 1_A(X_u)dn))$$

$$= \mathbb{E}_x(\xi_s \int_s^t 1_A(X_u)dn) = \mathbb{E}_x(\xi_s(L_t^A - L_s^A)) ,$$

where we used the Markov property and the convention $\mathbb{P}_\Delta(X_t \in A) = 0$ for all t and $A \in \mathcal{B}(I)$.

Note that $t \rightsquigarrow L_t^A$ is continuous and, hence, we have

(3.2) Corollary. The process L^A is the unique increasing, and integrable process associated with the $(\mathbb{P}_x, F_t)$-sub-martingale $S_t^A = G(x,A) - G(X_t,A)$ by the Doob-Meyer decomposition.

For $y \in I$ and $\varepsilon > 0$ let $A^\varepsilon = (y - \varepsilon, y + \varepsilon)$, $\hat{L}_t^\varepsilon = \frac{1}{m\{A^\varepsilon\}} L_t^{A^\varepsilon}$, $\hat{G}(x,A^\varepsilon) = \frac{1}{m\{A^\varepsilon\}} G(x,A^\varepsilon)$, and $\hat{S}_t^\varepsilon = \frac{1}{m\{A^\varepsilon\}} S_t^{A^\varepsilon}$. The following lemma allows us (roughly speaking) to take the limit of $\hat{L}_t^\varepsilon$ as $\varepsilon \downarrow 0$.

<u>(3.3) Lemma</u>. The family $\{\hat{L}^{\varepsilon}_{\infty};\varepsilon > 0\}$ of random variables is uniformly integrable $(\hat{L}^{\varepsilon}_{\infty} = \lim_{t\to\infty} \hat{L}^{\varepsilon}_{t})$.

<u>Proof</u>. We shall argue as M. Rao in [11] p. 70. For $\lambda > 0$ let $T^{\varepsilon}_{\lambda} = \inf\{t : \hat{L}^{\varepsilon}_{t} > \lambda\}$. Then $T^{\varepsilon}_{\lambda}$ is a F_t-stopping time, and $\hat{L}^{\varepsilon}_{\infty} > \lambda$ if and only if $T^{\varepsilon}_{\lambda} < \infty$. By the Doob-Meyer decomposition we have $\mathbb{P}_x$-a.s. (we drop "ε" from our notation)

$$\hat{S}_{T_\lambda} = \mathbb{E}\,(\hat{S}_{\infty} - \hat{L}_{\infty} | F_{T_\lambda})$$

This implies

$$\hat{L}_{T_\lambda} = \mathbb{E}\,(\hat{L}_{\infty} | F_{T_\lambda}) - \hat{G}(X_{T_\lambda}, A) \quad ,$$

and, consequently,

$$(3.4) \qquad \mathbb{E}_x\,(\hat{L}_{\infty};\hat{L}_{\infty} > \lambda) = \mathbb{E}_x\,(\hat{L}_{T_\lambda};T_\lambda < \infty) + \mathbb{E}_x\,(Y_{T_\lambda};T_\lambda < \infty)$$

$$= \lambda\mathbb{P}_x\,(\hat{L}_{\infty} > \lambda) + \mathbb{E}_x(Y_{T_\lambda};T_\lambda < \infty) \quad ,$$

where $Y_t = \hat{G}(X_t,A)$. Further we obtain

$$\mathbb{E}_x\,(\hat{L}_{\infty} - \lambda;\hat{L}_{\infty} > \lambda) = \mathbb{E}_x\,(Y_{T_\lambda};T_\lambda < \infty)$$

and, therefore,

$$\mathbb{E}_x\,(\hat{L}_{\infty} - \lambda;\hat{L}_{\infty} > 2\lambda) \leq \mathbb{E}_x\,(\hat{L}_{\infty} - \lambda;\hat{L}_{\infty} > \lambda)$$

$$= \mathbb{E}_x\,(Y_{T_\lambda};T_\lambda < \infty) \quad .$$

This gives

$$\lambda\mathbb{P}_x\,(\hat{L}_{\infty} > 2\lambda) \leq \mathbb{E}_x(Y_{T_\lambda};\,T_\lambda < \infty)$$

and

$$2\lambda\mathbb{P}_x\,(\hat{L}_{\infty} > 2\lambda) \leq 2\mathbb{E}_x\,(Y_{T_\lambda};T_\lambda < \infty) \quad .$$

Replacing λ by 2λ in (3.4) we obtain

$$\mathbb{E}_x(\hat{L}_\infty;\hat{L}_\infty > 2\lambda) = 2\lambda \mathbb{P}_x(\hat{L}_\infty > 2\lambda) + \mathbb{E}_x(Y_{T_{2\lambda}};T_{2\lambda} < \infty)$$

$$\leq 2\mathbb{E}_x(Y_{T_\lambda};T_\lambda < \infty) + \mathbb{E}_x(Y_{T_{2\lambda}};T_{2\lambda} < \infty) .$$

Further

$$\lambda \mathbb{P}_x(T_\lambda < \infty) \leq \mathbb{E}_x(\hat{L}_\infty;T_\lambda < \infty)$$

$$\leq \mathbb{E}_x(\hat{L}_\infty) = \hat{G}(x,A) .$$

We take now "ε" back to the notation. Because $x \rightsquigarrow G(x,y)$ is bounded and jointly continuous we have $\sup_{\varepsilon>0} \hat{G}(x,A^\varepsilon) < \infty$. Consequently $\mathbb{P}_x(T^\varepsilon_\lambda < \infty)$ is uniformly small for large λ . Also there exists a constant K such that for all $\varepsilon > 0$ and every ω we have for all $t \geq 0$ $|Y_t| = |G(X_t,A^\varepsilon)| < K$. Hence we obtain

$$\mathbb{E}_x(\hat{L}^\varepsilon_\infty;\hat{L}^\varepsilon_\infty > 2\lambda) \leq 2K\,\mathbb{P}_x(T^\varepsilon_\lambda < \infty) + K\,\mathbb{P}_x(T^\varepsilon_{2\lambda} < \infty) . \tag{3.5}$$

This shows that for every $\delta > 0$ there exists a λ such that the right hand side of (3.5) is less than δ for all $\varepsilon > 0$, and the proof is complete.

<u>(3.6) Remark</u>. Note that for a fixed $t > 0$ $\hat{L}^\varepsilon_t \leq \hat{L}^\varepsilon_\infty$, and, therefore, also the family $\{\hat{L}^\varepsilon_t;\varepsilon > 0\}$ is uniformly integrable.

Next we show how the results above can be used to prove the well-known

<u>(3.7) Theorem</u>. For every $y \in I$ there exists a process $t \rightsquigarrow L^y_t$, which is continuous, increasing, integrable, and F_t-adapted. Further $(y,t) \rightsquigarrow L^y_t$ is $\mathcal{B}(I) \times \mathcal{B}([0,\infty))$-measurable and $\mathbb{P}_x$-a.s. for all $t \geq 0$ and $A \in \mathcal{B}(I)$

$$\int_0^t 1_A(X_s)\,ds = \int_A L^y_t\, m(dy) . \tag{3.8}$$

<u>Proof</u>. Note that $(y,t) \rightsquigarrow \hat{L}^\varepsilon_t$ is $\mathcal{B}(I)\times\mathcal{B}([0,\infty))$-measurable and $t \rightsquigarrow \hat{L}^\varepsilon_t$ is predictable. Therefore by the Dunford-Pettis criterion (see [15] p. 51)

there exists a family of random variables $\{L_t^y; y \in I, t \geq 0\}$ with the same measurability properties as $\{\hat{L}_t^\varepsilon; y \in I, t \geq 0\}$, and such that for every $y \in I$ and $t \geq 0$

(3.8) $$\mathbb{E}_x(\xi \hat{L}_t^\varepsilon) \to \mathbb{E}_x(\xi L_t^y)$$

as $\varepsilon \downarrow 0$ along a sub-sequence. Here ξ is an arbitrary bounded random variable. By the path-continuity and the fact that m is finite on compact subsets of I it is seen that (3.8) holds in general as $\varepsilon \downarrow 0$.

Next let M^y be a right-continuous modification of the martingale $G(x,y) - \mathbb{E}_x(L_\infty^y | F_t)$. We have

$$\begin{aligned} \mathbb{E}_x(\xi(M_t^y + L_t^y)) &= \mathbb{E}_x(\xi(G(x,y) - \mathbb{E}_x(L_\infty^y | F_t) + L_t^y)) \\ &= \lim_{\varepsilon \downarrow 0} \mathbb{E}_x(\xi(\hat{G}(x,A^\varepsilon) - \mathbb{E}_x(\hat{L}_\infty^\varepsilon | F_t) + \hat{L}_t^\varepsilon)) \\ &= \lim_{\varepsilon \downarrow 0} \mathbb{E}_x(\xi \hat{S}_t^\varepsilon) \\ &= \mathbb{E}_x(\xi(G(x,y) - G(X_t,y))) , \end{aligned}$$

where we used the weak continuity of conditional expectation (see [15] p. 55) and (3.3). Consequently for all $t \geq 0$ $\mathbb{P}_x$-a.s.

(3.9) $$S_t^y = M_t^y + L_t^y ,$$

where $S_t^y = G(x,y) - G(X_t,y)$. By the right-continuity of $t \rightsquigarrow S_t^y$ and $t \rightsquigarrow M_t^y$ (3.9) holds $\mathbb{P}_x$-a.s. for all $t \geq 0$. But $t \rightsquigarrow S_t^y$ is a $(\mathbb{P}_x, F_t)$-sub-martingale and $t \rightsquigarrow L_t^y$ is predictable. Therefore (3.9) is the unique Doob-Meyer decomposition of S^y. Further S^y is regular because the diffusion X is quasi-left-continuous. This implies (see [11]) that $t \rightsquigarrow L_t^y$ is in fact continuous.

It remains to prove (3.8). Because $(y,t) \rightsquigarrow L_t^y$ is $\mathcal{B}(I) \times \mathcal{B}([0,\infty))$-

measurable we can integrate in (3.9) over a set $A \in \mathcal{B}(I)$ to obtain $\mathbb{P}_x$-a.s. for all $t \geq 0$

$$(3.10) \qquad S_t^A = \int_A M_t^y \, m(dy) + \int_A L_t^y \, m(dy) .$$

But $t \rightsquigarrow \int_A L_t^y \, m(dy)$ is increasing, integrable, and continuous, and hence (3.10) is the unique Doob-Meyer decomposition of S^A. This together with (3.2) gives

$$\int_0^t 1_A(X_s) ds = \int_A L_t^y \, m(dy) ,$$

and the proof is complete.

Remark. From (3.8) it follows that for almost all (Lebesque) y we have $\mathbb{P}_x$-a.s. for all $t \geq 0$ $L_t^y = \lim_{\varepsilon \downarrow 0} \hat{L}_t^\varepsilon$. To extend this statement for all y requires at least right-continuity of $y \rightsquigarrow L_t^y$. However it seems to us that this kind of regularity properties (or Trotter's theorem) are not reachable in our framework.

4. Local time as a dual predictable projection

In [1] p. 8 Azema and Yor remark that the local time at the point 0 of a continuous uniformly integrable martingale can be interpreted, roughly speaking, as a dual predictable projection of the last exit time from 0. In this section we study the local times of a diffusion from this point of view, and show some applications.

Consider the process $Z_t^y = 1_{\{\lambda_y \leq t\}}$, where λ_y is the last exit time from a point $y \in I$. By our transience assumption $\lambda_y < \infty$ $\mathbb{P}_x$-a.s. the process $t \rightsquigarrow Z_t^y$ is increasing and non-adapted to $(\mathcal{F}_t)_{t \geq 0}$. We have

(4.1) Proposition. The dual predictable projection of Z_t is $\hat{L}_t^y = \frac{1}{G(y,y)} L_t^y$

where L^y is the local time of X constructed in (3.7).

Proof. Denote the optional projection of X^y with ${}^{\circ}Z^y$ (see [7] (1.23) p. 14). Then we have for all $t \geq 0$ $\mathbb{P}_x$-a.s.

$$(4.2)\qquad {}^{\circ}Z_t^y = E(Z_t^y \mid F_t)$$

$$= \mathbb{P}_{X_t}(\tau_y = +\infty) = 1 - \mathbb{P}_{X_t}(\tau_y < +\infty)$$

$$= \begin{cases} 1 - \dfrac{\phi^{\uparrow}(X_t)}{\phi^{\uparrow}(y)} & , X_t \leq y \\ 1 - \dfrac{\phi^{\downarrow}(X_t)}{\phi^{\downarrow}(y)} & , X_t \geq y , \end{cases}$$

where we used the Markov property. The right-continuity of Z^y implies the right-continuity of ${}^{\circ}Z^y$ (see [7] 1.27 p. 14), and, hence, (4.2) is valid $\mathbb{P}_x$-a.s. for all $t \geq 0$. Further the right-hand side of (4.2) is predictable. It follows that

$${}^{p}Z_t^y - {}^{p}Z_0^y = {}^{\circ}Z_t^y - {}^{\circ}Z_0^y = \frac{1}{G(y,y)}\, S_t^y ,$$

where ${}^{p}Z^y$ is the predictable projection of Z^y. By (3.9) $S^y - L^y$ is a martingale and L^y is the only predictable increasing process with this property. By Dellacherie's formula (see [4] T30 p. 107) the dual predictable projection $\hat{L}^y$ is such that ${}^{p}Z^y - {}^{p}Z_0^y - L^y$ is a martingale. Therefore $\hat{L}^y = \frac{1}{G(y,y)} L^y$.

The following result (see also [10] p 326 and [12]) is now an easy consequence of (4.1).

(4.3) Corollary. For $x,y \in I$ and $t > 0$

$$\mathbb{P}_x(0 < \lambda_y \leq t) = \int_0^t \frac{p(s;x,y)}{G(y,y)}\, ds .$$

Proof. By Dellacherie's formula we have

$$\mathbb{P}_x(0 < \lambda_y \le t) = \mathbb{E}_x\left(\int_0^t dZ_s^y\right)$$
$$= \mathbb{E}_x\left(\int_0^t d\hat{L}_s^y\right)$$
$$= \frac{1}{G(y,y)} \mathbb{E}_x(L_t^y) = \int_0^t \frac{p(s;x,y)}{G(y,y)}\, ds .$$

Remark. Note that $\mathbb{E}_x(\hat{L}_\infty^y) = \frac{G(x,y)}{G(y,y)}$ and $\mathbb{E}_x(L_\infty^y) = G(x,y)$. Therefore $\hat{L}^y$ and L^y may be considered as the local times with the Blumenthal-Getoor and Itô-McKean normalizations, respectively (see [2] and [6]).

Next we consider the process $Z_t = 1_{\{\zeta \le t\}}$, where ζ is the life time of X . This process is increasing and adapted to $(F_t)_{t \ge 0}$. We have

(4.4) Proposition. Let k be the killing measure of X. Then the process

$$A_t = \int_I L_t^y\, k(dy) + 1_{\{X_{\zeta-} \notin I\}}\, 1_{\{\zeta \le t\}}$$

is the dual predictable projection of the process Z .

Proof. Let $\tilde{A}_t = \int_I L_t^y\, k(dy)$, and $\tilde{Z}_t = 1_{\{X_{\zeta-} \in I\}} 1_{\{\zeta \le t\}}$. Then $\tilde{A}_t$ is the unique, increasing, and predictable process associated with the sub-martingale

$$S_t^k = \int_I G(x,y)k(dy) - \int_I G(X_t,y)k(dy)$$

by the Doob-Meyer decomposition. Consequently, if for every bounded, positive, and F_s-measurable variable ξ_s we have $(t > s)$

$$(4.5) \qquad \mathbb{E}_x(\xi_s(\tilde{Z}_t - \tilde{Z}_s)) = \mathbb{E}_x(\xi_s(S_s^k - S_t^k))$$

then $\tilde{A}$ is the dual predictable projection of $\tilde{Z}$ by the uniqueness of the Doob-Meyer decomposition. To prove (4.5) we note that for $[\alpha,\beta] \subset I$ and $x \in [a,\beta]$

$$\int_{[\alpha,\beta]} G(x,y)k(dy) = \int_{[\alpha,x)} \phi^{\uparrow}(y)\phi^{\downarrow}(x)k(dy) + \phi^{\uparrow}(x)\phi^{\downarrow}(x)k\{x\}$$
$$+ \int_{(x,\beta]} \phi^{\downarrow}(y)\phi^{\uparrow}(x)k(dy)$$
$$= \phi^{\downarrow}(x)(\phi^{\uparrow-}(x) - \phi^{\uparrow-}(\alpha)) + \phi^{\uparrow}(x)\phi^{\downarrow}(x)k\{x\}$$
$$+ \phi^{\uparrow}(x)(\phi^{\downarrow+}(\beta) - \phi^{\downarrow+}(x))$$
$$= \mathbb{P}_x(\alpha \le X_{\zeta-} < x) + \phi^{\uparrow}(x)\phi^{\downarrow}(x)k\{x\} +$$
$$+ \mathbb{P}_x(x < X_{\zeta-} \le \beta) \ ,$$

where we have used (2.4), (2.4)', and (2.7). This implies

$$\int_I G(x,y)k(dy) = \mathbb{P}_x(X_{\zeta-} \in I) \ , \tag{4.6}$$

and we obtain (4.5):

$$\mathbb{E}_x(\xi_s(\tilde{Z}_t - \tilde{Z}_s)) = \mathbb{E}_x(\xi_s 1_{\{X_{\zeta-}\in I\}} 1_{\{s<\zeta\le t\}})$$
$$= \mathbb{E}_x(\xi_s 1_{\{X_{\zeta-}\in I\}}(1_{\{s<\zeta\}} - 1_{\{t<\zeta\}}))$$
$$= \mathbb{E}_x(\xi_s(\mathbb{P}_{X_s}(X_{\zeta-} \in I) - \mathbb{P}_{X_t}(X_{\zeta-} \in I)))$$
$$= \mathbb{E}_x(\xi_s(S^k_s - S^k_t)) \ .$$

Next consider the process $\hat{Z}_t = 1_{\{X_{\zeta-}\notin I\}} 1_{\{\zeta\le t\}}$; but on the set $\{X_{\zeta-} \notin I\}$ the life time ζ equals to the first hitting time of the set $\{a,b\}$. Therefore $\hat{Z}$ is predictable, and because it is increasing its dual predictable projection coincides with it (by the uniqueness of the Doob-Meyer decomposition). Because $Z = \tilde{Z} + \hat{Z}$ the proof is complete.

Now we apply (4.4) and Dellacherie's formula to prove

(4.7) Corollary (see [6] p. 184). Let f be a measurable, positive, and bounded function on I . Then

$$\mathbb{E}_x(f(X_{\zeta-})1_{\{X_{\zeta-}\in I\}};\ \zeta \le t) = \int_0^t \int_I f(y)p(s;x,y)k(dy)ds\ .$$

Proof. By Dellacherie's formula we have

$$\begin{aligned}\mathbb{E}_x(f(X_{\zeta-})1_{\{X_{\zeta-}\in I\}};\ \zeta \le t) &= \mathbb{E}_x\Big(\int_0^t f(X_s)d\tilde{Z}_s\Big)\\ &= \mathbb{E}_x\Big(\int_0^t f(X_s)d\tilde{A}_s\Big)\\ &= \mathbb{E}_x\Big(\int_I\int_0^t f(X_s)dL_s^y\,k(dy)\Big)\ ,\end{aligned}$$

where we used Fubini's theorem. But dL_s^y is supported by $\{s : X_s = y\}$. Therefore

$$\begin{aligned}\mathbb{E}_x\Big(\int_I\int_0^t f(X_s)dL_s^y\,k(dy)\Big) &= \mathbb{E}_x\Big(\int_I f(y)\int_0^t dL_s^y\,k(dy)\Big)\\ &= \int_I f(y)\,\mathbb{E}_x(L_t^y)k(dy)\\ &= \int_I f(y)\int_0^t p(s;x,y)ds\,k(dy)\ ,\end{aligned}$$

which is the desired result.

As our final application we prove the following well-known representation theorem for continuous additive functionals. For simplicity we assume that $k(I) = 0$.

(4.8) Theorem. Let A_t be a continuous additive funtional of a transient diffusion X with $k(I) = 0$. Then there exists a measure μ finite on compact subsets of I such that $\mathbb{P}_x$-a.s. for all $t \ge 0$

$$A_t = \int_I L_t^y\,\mu(dy)\ ,$$

where L^y is the local time for X .

Proof. Let T be an exponentially distributed random variable, which is independent of X and has a parameter $\beta > 0$. Introduce

$$\hat{\zeta} = \begin{cases} \inf\{t : A_t > T\} & \text{if } \{\cdot\} \neq \emptyset , \\ \zeta & \text{if } \{\cdot\} = \emptyset , \end{cases}$$

and consider the process

$$Z_t^\beta = \begin{cases} X_t & t < \hat{\zeta} , \\ \Delta & t \geq \hat{\zeta} . \end{cases}$$

The process Z^β is a diffusion. Denote its killing measure with $\hat{k}_\beta$, and let $\hat{k} \equiv \hat{k}_1$. The claim is that $\hat{k}$ has the required properties. To prove this note first that $\hat{k}_\beta \equiv \beta \hat{k}$, and, therefore, (4.7) gives us

$$(4.9) \qquad \mathbf{E}_x(Z^\beta_{\hat{\zeta}-} \in I \ ; \ \hat{\zeta} \leq t) = \mathbb{E}_x\left(\int_I L^y_{t\wedge\hat{\zeta}} \, \beta \, \hat{k}(dy)\right) ,$$

because $L^y_{t\wedge\hat{\zeta}}$ is the local time for Z^β . But

$$\begin{aligned} \mathbf{E}_x(Z^\beta_{\hat{\zeta}-} \in I \ ; \ \hat{\zeta} \leq t) &= \mathbf{E}_x(A_t > T) \\ &= \mathbf{E}_x(1 - e^{-\beta A_t}) , \end{aligned}$$

and so (4.9) takes the form

$$\mathbf{E}_x(1 - e^{-\beta A_t}) = \mathbb{E}_x\left(\int_I L^y_{t\wedge\hat{\zeta}} \, \beta \hat{k}(dy)\right) .$$

i.e.

$$\mathbf{E}_x\left(\frac{1-e^{-\beta A_t}}{\beta}\right) = \mathbb{E}_x\left(\int_I L^y_{t\wedge\hat{\zeta}} \, \hat{k}(dy)\right) .$$

As $\beta \downarrow 0$ $\hat{\zeta} \uparrow \zeta$ $\mathbb{P}_x$-a.s. , and by monotone convergence we obtain

$$\mathbf{E}_x(A_t) = \mathbb{E}_x\left(\int_I L^y_t \, \hat{k}(dy)\right) .$$

This implies that the process

$$\int_I G(x,y)\hat{k}(dy) - \int_I G(Z^\beta_t,y)\hat{k}(dy) - A_{t\wedge\hat{\zeta}}$$

is a $\mathbf{P}_x$-martingale, where G is the Green function for the process Z^β .

Because $t \rightsquigarrow A_t$ is continuous we obtain by the uniqueness of the Doob-Meyer decomposition that $\mathbb{P}_x$-a.s. for $t \geq 0$

$$A_{t\wedge\hat{\zeta}} = \int_I L^y_{t\wedge\hat{\zeta}}\hat{k}(dy) .$$

Letting $\beta \downarrow 0$ gives

$$A_t = \int_I L^y_t \hat{k}(dy) ,$$

and the proof is complete.

Remark. The results in (4.1) and (4.4) (for a killed Brownian motion) may also be found in a recent paper by Jeulin (see [8]). Techniques in [8] are however quite different.

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