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THE GAUGE AND CONDITIONAL GAUGE THEOREM

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Let $\{X_t, t \geq 0\}$ be the Brownian motion in R^d, $d \geq 1$. Let D be a bounded domain in R^d, $\bar{D}$ its closure, ∂D its boundary; and let q be a Borel function defined in R^d and satisfying the following condition:

$$\lim_{t \downarrow 0} \sup_{x \in D} E^x\{\int_0^t 1_D|q|(X_s)ds\} = 0 \tag{1}$$

where 1_D is the indicator of D. Such a function is said to belong to the Kato class K_d. The equivalent condition (1) is given by Aizenman and Simon [1].

The *gauge* for (D,q) is defined to be the function u on $\bar{D}$ below:

$$u(x) = E^x\{\exp(\int_0^{\tau_D} q(X_s)ds)\} \ . \tag{2}$$

From here on we write for abbreviation:

$$e_q(t) = \exp(\int_0^t q(X_s)ds) \ . \tag{3}$$

For a domain D with $m(D) < \infty$ (where m denotes the Lebesgue measure), without any regularity hypothesis on ∂D, and a bounded q, we proved the following theorem in [3].

<u>The Gauge Theorem</u>. *If* $u(\cdot) \not\equiv \infty$ *in* D, *then* u *is bounded in* $\bar{D}$.

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Actually, if ∂D is regular in the sense of the Dirichlet problem, then u is continuous and strictly positive in $\bar{D}$. However, in this note we shall concentrate on the main thing, as stated above. Zhao [6] extended the theorem to $q \in K_d$ for a bounded domain in R^d, $d \geq 3$; he also did the case $d = 2$ in yet unpublished notes. For $d = 1$ and D a half-line, see [2]. Prior to Zhao's work, Falkner extended the theorem in another direction by considering the *conditional gauge* for (D,q) as follows:

$$(4) \qquad u(x,z) = E_z^x\{e_q(\tau_D)\}, \qquad (x,z) \in D \times \partial D ;$$

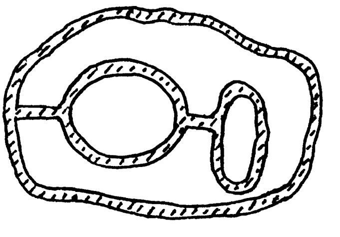
where E_z^x is the expectation associated with the Brownian motion killed outside D, starting from x and conditioned to converge to z (at its life-time τ_D). For a class of bounded domains including those with C^2 boundary, and bounded q, he proved the following theorem in [5].

<u>Conditional Gauge Theorem</u>. *If* $u(\cdot,\cdot) \not\equiv \infty$ *in* $D \times \partial D$, *then it is bounded there. This is the case if and only if* $u(\cdot) \not\equiv \infty$ *in* D, *as in the gauge theorem.*

I gave a simpler proof of Falkner's theorem in [4]. Subsequently Zhao [7] proved that if $u(\cdot) \not\equiv \infty$, then $u(\cdot,\cdot) \not\equiv \infty$, for bounded C^2 domains. He has since proved the conditional gauge theorem as stated above for bounded $C^{1,1}$ domains. In this note I shall show that the conditional gauge theorem actually follows in a general way and rather quickly from the gauge theorem.

The basic probabilistic argument turns out to be an old one in [2] (see the proof of Theorem 1 there), easily adapted to the multi-dimensional case. The sole difficulty encountered in extending the class of bounded q to the class K_d is contained in Lemma 1 below.

We begin by setting up the framework of the probabilistic argument involving a sequence of hitting times. Let D_1 and D_2 be subdomains of D such that $\overline{D}_1 \subset D_2$, $\overline{D}_2 \subset D$, and $C \triangleq D-\overline{D}_1$ is connected and $m(C) < \varepsilon$ for an arbitrary $\varepsilon > 0$. This is possible if ∂D is Lipschitzian for instance. For then each connected component of $R^d - \overline{D}$ must contain a ball of fixed size, hence there are at most a finite number of "holes" inside the outer boundary of D. Since D is connected, it is easy to see how to construct D_1 and D_2 as desired. A picture illustrates the result. I am indebted to Falkner for alerting me to the necessity of making C connected.



(The shaded portion represents C.)

Lemma 1. *If* ∂D *is sufficiently smooth, then for any given* $\varepsilon > 0$*, there exists* $\delta(\varepsilon)$ *such that if the* C *described above has* $m(C) \leqslant \delta(\varepsilon)$*, then*

(5) $$\sup_{\substack{x \in C \\ z \in \partial D}} E_z^x\{\int_0^{\tau_C} |q|(X_t)dt\} \leqslant \varepsilon \ ;$$

(6) $$\sup_{\substack{x \in C \\ z \in \partial D}} E_z^x\{e_q(\tau_C)\} \leqslant \frac{1}{1-\varepsilon} \ .$$

In [7], Zhao proved that C^2 boundary is sufficient for the lemma to hold; more recently he has improved this result to require only $C^{1,1}$ boundary. In this connection it should be mentioned that the gauge theorem for an arbitrary bounded domain D, and $q \in K_d$, follows quickly from an easier analogue of (5) for a small ball B, as follows:

(7) $$\sup_{\substack{x \in B \\ z \in \partial B}} E_z^x\{\int_0^{\tau_B} |q|(X_t)dt\} \leqslant \varepsilon \ .$$

This was proved in Zhao [6]. The deduction of (6) from (5) is standard Markovian calculation.

Lemma 2 is a strengthened form of an argument I have indicated elsewhere (see [5], Remark 2.13). The constants $a_1, a_2, \cdots$ below are strictly positive, depending only on D_1, D_2 and D. We assume ∂D to be Lipschitzian below.

Lemma 2. *For all* $y \in \partial D_2$ *and* $z \in \partial D$, *we have*

(8) $$a_1 \leqslant E_z^y\{\tau_C = \tau_D;\ e_q(\tau_D)\} \leqslant a_2 \ .$$

Proof: Recall that

(9) $$P_z^y\{\tau_C < \tau_D\} = \frac{f(y,z)}{K(y,z)}$$

where K is the Poisson kernel for D, and

$$f(y,z) = E^y\{\tau_C < \tau_D;\ K(X(\tau_C),z)\} .$$

For each $y \in \partial D_2$, $f(y,\cdot)$ is continuous on ∂D, because on $\{\tau_C < \tau_D\}$ we have $X(\tau_C) \in \partial D_1$ almost surely, and K is bounded continuous in $\partial D_1 \times \partial D$. For each $z \in \partial D$, $f(\cdot,z)$ is harmonic in C. Hence f is continuous on $\partial D_2 \times \partial D$ since ∂D_2 and ∂D are disjoint closed sets. It follows that the function of (y,z) in (9) is continuous and positive on $\partial D_2 \times \partial D$. The function $K(\cdot,z) - f(\cdot,z)$ is harmonic in C and unbounded in the neighborhood of z, because K is unbounded while f is bounded. Hence it is strictly positive in C by harmonicity, because C is connected and $z \in \partial C$. Therefore we have by continuity

(10) $$b \triangleq \inf_{\substack{y \in \partial D_2 \\ z \in \partial D}} P_z^y\{\tau_C = \tau_D\} > 0 .$$

Now it follows by Jensen's inequality and (15) that for $(y,z) \in \partial D_2 \times \partial D$:

(11) $$\begin{aligned} E_z^y\{e_q(\tau_D) \mid \tau_C = \tau_D\} &\geq E_z^y\{e_{-|q|}(\tau_D) \mid \tau_C = \tau_D\} \\ &\geq \exp\{-E_z^y[\int_0^{\tau_D} |q|(X_t)dt \mid \tau_C = \tau_D]\} \\ &\geq \exp\{-\frac{1}{b}\int_0^{\tau_C} |q|(X_t)dt\} \geq e^{-\varepsilon/b} . \end{aligned}$$

Combining (10), (11) and (16), we have proved (8) with $a_1 = b\ e^{-\varepsilon/b}$, $a_2 = \frac{1}{1-\varepsilon}$.

We are ready to prove the conditional gauge theorem for a bounded Lipschitzian domain for which the conclusions of Lemma 1 hold true, thus at least when ∂D belongs to $C^{1,1}$. Put $T_o \equiv 0$, and for $n \geqslant 1$:

$$T_{2n-1} = T_{2n-2} + \tau_{D_2} \circ \theta_{T_{2n-2}},$$

$$T_{2n} = T_{2n-1} + \tau_C \circ \theta_{T_{2n-1}}.$$

For any $(x,z) \in D \times \partial D$, we have $P_z^x\{\tau_D < \infty\} = 1$. This nontrivial result has recently been proved by M. Cranston for a bounded Lipschitzian domain; for a bounded C^1-domain D it follows from the fact that $K(\cdot,z)$ is integrable over D, by a remark communicated to me by Kenig. It follows that for some $n \geqslant 1$, $X(T_{2n}) \in \partial D$. Therefore we have by the strong Markov property of the conditioned process:

$$\text{(12)}\quad E_z^x\{e_q(\tau_D)\} = \sum_{n=1}^{\infty} E_z^x\{T_{2n} = \tau_D;\ e_q(\tau_D)\}$$

$$= \sum_{n=1}^{\infty} E_z^x\{T_{2n-2} < \tau_D;\ e_q(T_{2n-1}) E_z^{X(T_{2n-1})}[\tau_C = \tau_D; e_q(\tau_D)]\}.$$

Observe that $\partial C = \partial D_1 \cup \partial D$. On the set $\{T_{2n-2} < \tau_D\}$, $X(T_{2n-1}) \in \partial D_2$. Hence by Lemma 2

$$\text{(13)}\quad a_1 \sum_{n=1}^{\infty} E_z^x\{T_{2n-2} < \tau_D; e_q(T_{2n-1})\} \leqslant E_z^x\{e_q(\tau_D)\}$$

$$\leqslant a_2 \sum_{n=1}^{\infty} E_z^x\{T_{2n-2} < \tau_D; e_q(T_{2n-1})\}.$$

The general term in the series above is explicitly:

$$(14)\qquad \frac{1}{K(x,z)} E^x\{T_{2n-2} < \tau_D;\ e_q(T_{2n-1})K(X(T_{2n-1}),z)\} .$$

Since K is continuous and strictly positive on $\overline{D}_2 \times \partial D$, we have for (x,z) and (x',z') in $\overline{D}_2 \times \partial D$, almost surely

$$(15)\qquad a_3 \frac{K(X(T_{2n-1}),z')}{K(x',z')} \leqslant \frac{K(X(T_{2n-1}),z)}{K(x,z)} \leqslant a_4 \frac{K(X(T_{2n-1}),z')}{K(x',z')}$$

where a_3 and a_4 depend only on $\overline{D}_2$ and D. It follows from (13), (14) and (15) that

$$(16)\qquad \sup_{x \in \overline{D}_2} \sup_{z \in \partial D} u(x,z) \leqslant \frac{a_2 a_4}{a_1 a_3} \inf_{x \in \overline{D}_2} \inf_{z \in \partial D} u(x,z) .$$

Since $u(x)$ is a probability average of $u(x,z)$ over $z \in \partial D$, we have

$$(17)\qquad \inf_{z \in \partial D} u(x,z) \leqslant u(x) \leqslant \sup_{z \in \partial D} u(x,z) .$$

Now by hypothesis of the theorem, there exists $(x_o,z_o) \in D \times \partial D$ such that $u(x_o,z_o) < \infty$. Without loss of generality we may suppose $x_o \in D_2$. Hence by (16)

$$(18)\qquad \sup_{z \in \partial D} u(x_o,z) \leqslant \frac{a_2 a_4}{a_1 a_3} u(x_o,z_o) < \infty .$$

Next by (17), $u(x_o) < \infty$. Hence by the gauge theorem, $\sup_{x \in \overline{D}} u(x) < \infty$. It follows then by (16) and (17) that

$$(19)\qquad \sup_{x \in \overline{D}_2} \sup_{z \in \partial D} u(x,z) < \infty .$$

For $x \in D-\overline{D}_2$, we use the old argument in [3] adapted to the conditioned process, as follows:

$$u(x,z) = E_z^x\{\tau_C = \tau_D;\ e_q(\tau_C)\} + E_z^x\{\tau_C < \tau_D;\ u(X(\tau_C),z)\}$$

$$\leqslant \frac{1}{1-\varepsilon} + \sup_{x \in \overline{D}_1} \sup_{z \in \partial D} u(x,z) < \infty .$$

This establishes the first assertion of the conditional gauge theorem. The second assertion has also been proved between the lines above.

Remark : Conditional gauge theorem is also true for a bounded C^1 domain, and bounded q, using a hard inequality of Kenig's to prove lemma 1.

References

[1] Aizenman, N., Simon, B.: Brownian motion and Harnack inequality for Schrödinger operators, Comm. Pure Appl. Math. 35 (1982), 209-273.

[2] Chung, K.L.: On stopped Feynman-Kac functionals, Séminaire de Probabilités XIV, 1978/79, Lecture Notes in Mathematics No. 784, Springer-Verlag.

[3] Chung, K.L., Rao, K.M.: Feynman-Kac functional and Schrödinger equation, Seminar on Stochastic Processes 1, 1-29, Birkhäuser 1981.

[4] Chung, K.L.: Conditional gauges, Seminar on Stochastic Processes 3, 1983.

[5] Falkner, N.: Feynman-Kac functionals and positive solutions of $\frac{1}{2}\Delta u+qu = 0$, Z. Wahrsch. Verw. Gebiete 65 (1983), 19-33.

[6] Zhao, Z.: Conditional gauge with unbounded potential, Z. Wahrsch. Verw. Gebiete 65 (1983), 13-18.

[7] Zhao, Z.: Uniform boundedness of conditional gauge and Schrödinger equations, Comm. Math. Phys 93 (1984), 19-31.