RICHARD F. BASS

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MARKOV PROCESSES AND CONVEX MINORANTS

Richard F. Bass
Department of Mathematics
University of Washington
Seattle, WA 98195 USA

1. INTRODUCTION:

Let W_t be Brownian motion and let C_t be the convex minorant of Brownian motion. That is, for each ω, $t \mapsto C_t(\omega)$ is the function that is the convex minorant of the function $s \mapsto W_s(\omega)$, $0 \le s < \infty$. Call T a vertex time for C_t if (T, C_T) is an extreme point for the graph of C_t. Recently Groeneboom [5] studied the properties of the convex minorant (and majorant) of Brownian motion. He found the distribution of the extreme points of the convex minorant, showed that there are only finitely many vertex times in any closed subinterval of $(0, \infty)$, and then proved the following remarkable fact:

Theorem 1. *Let $S < T$ be the first two consecutive vertex times after a fixed time $t_0 > 0$, $c = T-S$. Then $c^{-\frac{1}{2}}(W_{ct+S} - C_{ct+S})$, $0 < t \le 1$ is a scaled Brownian excursion, scaled to be 0 at $t = 0$ and $t = 1$.*

For the transition densities of scaled Brownian excursion, see [6, p.76].

Groeneboom's proof consists of a direct calculation of the joint densities. Pitman [9] has given a proof of this theorem and has also derived many properties of C_t itself using arguments that rely on Williams' path decompositions and on time inversion. Our aim here is to give a proof of Theorem 1 that uses the decomposition of general Markov processes at splitting times. The method can be easily extended to the study of other diffusions, although the calculations quickly become unmanageable.

Our method is to define a strong Markov process X_t such that the vertex times of C_t turn out to be last exit times from sets for X_t, and the well-known theory of last exit time decompositions applies. This argument is carried out in section 2. In section 3 we show how this method can be used to find the distribution of the slopes and vertex times of C_t. In section 4 we briefly discuss generalizations to other diffusions.

I would like to thank R. M. Blumenthal for many very helpful discussions.

2. BROWNIAN EXCURSION:

We first prove that space-time Brownian motion with drift δ, conditioned to be 0 at time 1 and positive up to time 1, is scaled Brownian excursion, regardless of the size of δ. Our conditioning is done using h-path transforms. For the definitions and properties of these, see Doob [2]. Recall that to condition a Markov process to exit a set A at a given point x_0, one h-path transforms the process by a function h, where h is harmonic (invariant) on A and has boundary values 0 everywhere except x_0. In the cases of interest in this paper, such h are unique up to a multiplicative constant.

Let (W_t, Y_t) be space-time Brownian motion with drift δ, with probability measures N^x, $x \in \mathbb{R} \times [0, \infty)$. Let N_t be the transition probabilities, with densities

$$(1) \qquad n_t((w_1, y_1), (w_2, y_2)) = (2\pi t)^{-\frac{1}{2}} \exp(-(w_2 - (w_1 + \delta t))^2/2t)$$

$$\text{if } y_2 - y_1 = t, \ 0 \text{ otherwise.}$$

Let $\tau_0 = \inf\{t: W_t = 0\}$. It is known that τ_0 has a C^1 density under $N^{(w,y)}$ (see, e.g., section 3). Let

$$H(w, y) = \frac{\partial}{\partial u} N^{(w,y)}(\tau_0 \le u - y)\big|_{u=1} .$$

Let $N_t^H(x, dz) = H(z)N_t(x, dz)/H(x)$, $x, z \in \mathbb{R} \times [0, \infty)$. H is harmonic since it is the limit of hitting propabilities, the boundary values of H are 0 on the lines $w = 0$ and $y = 1$, except at the point $w = 0, y = 1$, and the N_t^H are thus the transition probabilities for (W_y, Y_t) conditioned to hit 0 for the first time at time 1.

<u>Proposition 2</u>. <u>The</u> N_t^H <u>are the transition probabilities for scaled Brownian excursion</u>.

<u>Proof</u>. Let $K(w, y) = n_{1-y}((w, y),(0, 1))$ for $y < 1$. Let $N_t^K(x, dz) = K(z)N_t(x, dz)/K(x)$, $x, z \in \mathbb{R} \times [0, \infty)$. K is harmonic with boundary values 0 on the line $y = 1$, except at the point $w = 0, y = 1$. So the N_t^K are the transition probabilities for (W_t, Y_t) conditioned to be 0 at time 1. A direct calculation shows that

$$N_t^K((w_x, y_x), (dw_z, y_z)) = (2\pi t)^{-\frac{1}{2}}\left(\frac{1 - y_x}{1 - y_z}\right)^{\frac{1}{2}} \times$$

$$\exp(-(w_z - w_x)^2/2t - w_z{}^2/2(1 - y_z) + w_x{}^2/2(1 - y_x))dw_z$$

if $y_z - y_x = t$, 0 otherwise.

Note that N_t^K is independent of δ, and it is therefore no surprise to recognize that the N_t^K are the transition probabilities for Brownian bridge. Let N_K^x be the corresponding probability measures.

Now let $N_t{}^{KL}(x, dz) = L(z)N_t^K(x, dz)/L(x)$ for $x, z \in \mathbb{R} \times [0, \infty)$, where $L(w, y) = (\pi/2)^{\frac{1}{2}} \lim_{r \to 0} N_K^{(w,y)}(1 - y \geq \tau_0 \geq 1 - r - y)/r^{\frac{1}{2}}$.

We will show in a moment that $r^{\frac{1}{2}}$ is the appropriate normalization so that the limit is finite. Given that, L is a harmonic function for N^K since it is the limit of hitting probabilities. The boundary values of L are 0 on the lines $w = 0$ and $y = 1$, except at the point $w = 0, y = 1$, and so

(W_t, Y_t) under N_t^{KL} is Brownian bridge conditioned to hit 0 for the first time at time 1. Fix (w, y), and let $f(s) = \frac{\partial}{\partial u} N^{(w,y)}(\tau_0 \le u)|_{u=s}$. Recall that f is continuous.

By properties of h-path transforms,

$$\begin{aligned}
L(w, y) &= \lim_{r\to 0} (\pi/2)^{\frac{1}{2}} \int_{1-r}^{1} \frac{\partial}{\partial u} N_K^{(w,y)} (\tau_0 \le u - y)|_{u=s} \, ds/r^{\frac{1}{2}} \\
&= \lim_{r\to 0} (\pi/2)^{\frac{1}{2}} \int_{1-r}^{1} K(0,s)\, f(s-y)\, ds/r^{\frac{1}{2}} K(w,y) \\
&= \lim_{r\to 0} \int_{1-r}^{1} (1-s)^{-\frac{1}{2}} \exp(-\, \delta^2(1-s)/2)\, f(s-y)\, ds/2r^{\frac{1}{2}} K(w,y) \\
&= f(1-y)/K(w,y) \ .
\end{aligned}$$

So then
$$\begin{aligned}
N_t^{KL}(x,dz) &= L(z)\, K(z)\, N_t(x,dz)/K(x)\, L(x) \\
&= H(z)\, N_t(x,dz)/H(x) \\
&= N_t^H(x,dz) \ .
\end{aligned}$$

To complete the proof, it suffices to show that the N_t^{KL} are transition probabilities for scaled Brownian excursion, i.e., that Brownian bridge conditioned to be positive on $(0,1)$ is Brownian excursion. This has been shown by Durrett, Iglehart, and Miller [4] and by Blumenthal [1]. The idea of Blumenthal's proof is to perform a one-to-one mapping of the state space so that the assertion is equivalent to showing that Brownian motion h-path transformed to be positive is the three-dimensional Bessel process. This last is well-known and is easily verified by a simple calculation with infinitesimal generators. □

The proof of the following lemma is immediate from the definitions.

Lemma 3. *Suppose* (X_t, Y_t) *is a strong Markov process with state space* $\underline{\underline{X}} \times \underline{\underline{Y}}$, *where* Y_t *is measurable with respect to the right continuous completion of* $\sigma(X_s, s \le t)$. *Suppose the transition probabilities* N_t *satisfy:*

$N_t((x,y),\ A \times \underline{\underline{Y}})$ <u>is independent of</u> y .

<u>Then</u> X_t <u>is strong Markov with transition possibilities</u>

$$P_t(x,A) = N_t((x,y),\ A \times \underline{\underline{Y}}) \ .$$

<u>Lemma 4</u>. <u>Suppose</u> N_t <u>is the transition probability for a strong Markov process</u>. <u>Suppose</u> $\lambda(d\eta)$ <u>is a measure</u>, $h(\eta,x)$ <u>is a jointly measurable nonnegative function and</u> $\int h(\eta,x)\lambda(d\eta) = 1$ <u>for all</u> x . <u>Suppose</u> $N_t^\eta(x,dz) = h(\eta,z)\ N_t(x,dz)/h(\eta,x)$ <u>does not depend on</u> η . <u>Then</u> $N_t^\eta(x,dz) = N_t(x,dz)$.

<u>Proof</u>. $h(\eta,x)\, N_t^\eta(x,A) = \int_A h(\eta,z)\, N_t(x,dz)$.

Integrate with respect to $\lambda(d\eta)$, use Fubini, and use the fact that N^η does not depend on η to get

$$N_t^\eta(x,A) = \int N_t^\eta(x,A)h(\eta,x)\lambda(d\eta) = \int\int_A h(\eta,z)N_t(x,dz)\lambda(d\eta) = N_t(x,A) \ . \qquad \square$$

<u>Proof of Theorem 1</u>. We begin by first defining a state space and then defining the process X_t . Let $\underline{\underline{S}}$ be the set of nonincreasing, continuous, convex, bounded functions g on $[0,\infty]$ with

$$\lim_{\delta\downarrow 0}\ \sup_{0<|t-s|<\delta} |g(t)-g(s)|/|t-s|^{1/4} < \infty \qquad \text{and} \qquad \limsup_{t\uparrow\infty} |g(\infty)-g(t)|\,t^{1/4} < \infty \ ;$$

furnish $\underline{\underline{S}}$ with the sup norm. $\underline{\underline{S}}$ is σ-compact since $\underline{\underline{S}} = \cup_{k=1}^\infty A_k$, where

$$A_k = \{g: \sup_t |g(t)| \le k, \quad \sup_{0<|t-s|<1/k} |g(t)-g(s)|/|t-s|^{1/4} \le k\ , \quad \sup_{t\ge k} |g(\infty)-g(t)|\,t^{1/4} \le k\} \tag{2}$$

is compact by Ascoli-Arzela. Since $\underline{\underline{S}}$ is a metric space with the relative topology inherited from $C[0,\infty]$, $\underline{\underline{S}}$ has a countable basis.

Let $\underline{\underline{E}} = \{(w,y,g) \in \mathbb{R} \times [0,\infty) \times \underline{\underline{S}}: w \geq g(y)\}$. Clearly $\underline{\underline{E}}$ is also σ-compact with a countable basis when given the product topology.

Let V be the functional defined on real-valued bounded Borel measurable functions on $[0, \infty]$ that maps V to $V(g)$, the convex minorant of g . Two obvious properties of V are:

(3) (i) $V(g_1 \wedge g_2) = V(V(g_1) \wedge g_2) = V(g_1 \wedge V(g_2))$;

(ii) If g_1 and g_2 are two functions that agree up to some time t_0 , that are each constant from t_0 on, and $g_i(\infty) \geq \inf\{g_i(r): r < t_0\}$, $i = 1,2$, then $V(g_1) = V(g_2)$.

Let (W_t, Y_t) be space-time Brownian motion with probabilities $P^{(w,y)}$. The transition probabilities P_t are given by (1) with $\delta = 0$. Fix t . Let $s \longmapsto W_t^0(s)$ be the function on $[0, \infty]$ given by $W_t^0(s) = W_s$ if $s \leq t$, $W_t^0(s) = W_0$ if $t < s \leq \infty$. Let $V_t = V(W_t^0(\cdot))$, and let $X_t = (W_t, Y_t, V_t)$. V_t may be thought of as the best guess for the convex minorant of Brownian motion based on the path up to time t .

For all t and ω , the function V_t is constant from some point on, and so it is not hard to see that $X_t \in \underline{\underline{E}}$ for all t, ω . X_t is a process with continuous paths. As usual, let Δ be a "cemetery" and set $X_\infty = \Delta$.

Before proceeding further, we must verify that X_t is strong Markov. Let ω be fixed. If Θ_t is the shift operator for (W_t, Y_t) , $V_s \circ \Theta_t$ will be the image under V of the function f where $f(r) = W_{r+t}$ if $r \leq s$, $f(r) = W_t$ if $r > s$. Let g be the function defined by $g(r) = V_t(r)$ if $r \leq t$, $g(r) = V_s \circ \Theta_t(r-t)$ if $r > t$. Then by (3), one sees that $V(g) = V_{s+t}$, and thus V_{s+t} is a measurable function of V_t and $V_s \circ \Theta_t$.

By the proof of Theorem (6) of Millar [8] (see also the remarks immediately following that proof), X_t is strong Markov. Let $Q_t(x,dz)$ be the transition probabilities for X_t, Q^x the associated probability measures.

Although $\underline{\underline{E}}$ is not locally compact, the state space for $X_{t\wedge T_k}$ is, where $T_k = \inf\{t: V_t \notin A_k\}$ and A_k is given by (2). By properties of Brownian motion paths, $T_k \uparrow \infty$, a.s., and from this one can conclude that results like the measurability of hitting times to Borel sets hold for X_t.

We now show how the vertex times for C_t can be constructed as last exit times for X_t. For any g in $\underline{\underline{S}}$, let $\sigma_g = \inf\{t: g(t) = g(\infty)\}$. Let t_0 be fixed, and let

$$A = \{(w,y,g) : w = g(\infty),\ D^-g(\sigma_g) = D^+g(s) \text{ for some } s \le t_0\},$$

where D^-, D^+ denote left and right hand derivative, respectively.
Let $L = \sup\{t > t_0 : X_t \in A\}$. L is a last exit time, and by Meyer, Smythe, and Walsh [7], X_{t+L}, $t > 0$ is a strong Markov process with transition probabilities

$$R_t(x,dz) = H(z)Q_t(x,dz)/H(x),$$

where $H(x) = Q^x(X_t \notin A \text{ for all } t < \infty)$. Let R^x denote the corresponding probabilities. Note that L is the first vertex time of C_t after time t_0

Now let $\rho_g = \sup\{t \ge t_0 : D^-g(t) = D^+g(s) \text{ for some } s \le t_0\}$, let

$$B = \{(w,y,g) : \sigma_g > \rho_g,\ w = g(\infty), \text{ and } D^+g(\rho_g) = D^-g(\sigma_g)\},$$

and let $M = \sup\{t > L : X_t \in B\}$. Again by Meyer, Smythe, and Walsh, X_{t+L}, $0 < t < M - L$ is a strong Markov process with transition probabilities

$$S_t(x,dz) = K(z)R_t(x,dz)/K(x),$$

where $K(x) = R^x(X_t \in B$ for some $t < \infty)$. Let the associated probabilities be denoted S^x . M will be the next vertex time of C_t after L . (The distribution of $M - L$ will be computed in section 3.)

Since the law of X_{t+L} potentially depends on L, C_L, and $D^- C_L$, we decompose the range of X_{t+L} . Let

$$F(a,y_0,w_0) = \{(w,y,g) \in \underline{\underline{E}} - A : D^- g(\rho_g) = -a,\ \sigma_g > \rho_g,\ \rho_g = y_0,\ g(\rho_g) = w_0\},\ a > 0.$$

Suppose we start the process X_t at a point x in $F(a,y_0,w_0)$. With Q^x probability 1, X_t will remain in $F(a,y_0,w_0)$ up until the time X again hits A . And X_t started at x in $F(a,y_0,w_0)$ will again hit A when and only when W_t hits the line through (y_0,w_0) of slope $-a$. There is a positive probability of this never happening, and so $H(x) > 0$ if $x \in F(a,y_0, w_0)$. But there is also positive probability of W_t hitting this line; since X_t started in $F(a,y_0,w_0)$ cannot again hit A without hitting B , $K(x)$ is also > 0 . Finally, note that if $t > L$, X_t will be in $F(a,y_0,w_0)$ with $a = -D^- V_L$, $y_0 = Y_L$, and $w_0 = W_L$.

We would now like to condition $\{X_t, S^x\}$ so that $D^+ V_L = -b$, $M - L = c$. To do the conditioning, we use h-path transforms. Let

$$B_{ru} = ((Y_t, W_t) \text{ hits the line segment connecting } (y_0,w_0) \text{ and } (y_0 + u, w_0 - ru)).$$

It is known that $P^{(w,y)}(B_{ru})$ is a C^2 function of r and u (see, e.g. section 3). So $Q^{(w,y,g)}(B_{ru})$, which does not depend on g , will also be a C^2 function of r,u . Since $P^{(w,y)}(B_{ru})$ is the probability that (W_t, Y_t) hits a set, this function is harmonic in (w,y) for P and hence for Q .

Now, for $x,z \in F(a,y_0,w_0)$, let

$$J_{bc}(x) = \frac{\partial^2}{\partial r \partial u} Q^x(B_{ru})\Big|_{r=b,u=c} ,$$

$$G_{bc}(x) = J_{bc}(x)/H(x)K(x), \qquad \text{and} \qquad N_t(x,dz) = G_{bc}(z)S_t(x,dz)/G_{bc}(x) .$$

$J_{bc}(x)$, being the uniform limit on compacts of harmonic functions, is also harmonic for P and Q . We then have

$$(4) \qquad N_t(x,dz) = G_{bc}(z)\ K(z)\ R_t(x,dz)/G_{bc}(x)\ K(x)$$
$$= G_{bc}(z)\ K(z)\ H(z)\ Q_t(x,dz)/G_{bc}(x)\ H(x)\ K(x)$$
$$= J_{bc}(z)\ Q_t(x,dz)/J_{bc}(x)\ .$$

Therefore the N_t are the transition probabilities for $\{X_t, Q^x\}$ conditioned so that (Y_t, W_t) stays above the line segment from (y_0,w_0) to $(y_0 + c, w_0 - bc)$ but hits the point $(y_0 + c, w_0 - bc)$, i.e., W_t conditioned to stay above the line through (y_0, w_0) of slope $-b$ until time $c + L$. Starting at $x \in F(a,y_0,w_0)$, if $L < t \leq c + L$, W_t will also be strictly above the line through (y_0,w_0) of slope $-a$; hence X_t will not hit A before time $c + L$, and X_t will still be in $F(a,y_0,w_0)$.

Let N^x be the probability measures associated with N_t . Let $x = (w,y,g) \in F(a,y_0,w_0)$. We have already mentioned that $J_{bc}(x)$ does not depend on g . And the law under Q^x of (W_t, Y_t) does not depend on g . Using (4) and applying lemma 3, we see that for $L < t \leq L + c$, (W_t, Y_t) under N^x is a strong Markov process; and we have already observed that this process is space-time Brownian motion conditioned to stay above the line through (y_0,w_0) of slope $-b$ until time $c + L$. Using the translation invariance of Brownian motion, under N^x , $(W_t - w_0 + bt, Y_t - y_0)$, $0 < t \leq c$, is space-time Brownian motion with drift b conditioned to be above the line $w = 0$ and to hit it for the first time at time c , and by scaling, $((W_{ct} - w_0 + bct)/c^{\frac{1}{2}}, (Y_{ct} - y_0)/c)$, $0 < t \leq 1$, is space-time Brownian motion with drift $\delta = bc^{\frac{1}{2}}$ conditioned to be positive and to hit the line $w = 0$ for the first time at time 1. By proposition 2, $c^{-\frac{1}{2}}(W_{ct} - w_0 + bct)$, and so $c^{-\frac{1}{2}}(W_{ct+L} - C_{ct+L})$, $0 < t \leq 1$ as well, is scaled Brownian excursion regardless of b and c .

Now for $x = (w,y,g) \in F(a,y_0,w_0)$,

$$\int_0^a \int_0^\infty J_{bc}(x)\, dc\, db = P^{(w,y)}(W_t \text{ hits some line through } (y_0,w_0) \text{ with slope between } 0 \text{ and } -a) = 1 ,$$

and so by lemma 4, $c^{-\frac{1}{2}}(W_{ct+L} - C_{ct+L})$, $0 < t \le 1$, is scaled Brownian excursion under S^x, $x \in F(a,y_0,w_0)$. Since the law of this process under S^x is scaled Brownian excursion for x in any $F(a,y_0,w_0)$, the proof is complete. □

3. SLOPES AND VERTEX TIMES OF C_t .

We would like to calculate the distributions of $M - L$ and $D^+C_L = -(W_M - W_L)/(M - L)$. First we find that of the latter. If $x = (w,y,g) \in F(a,y_0,w_0)$, we know the law of X_{t+L} is given by R^x , where R is Q h-path transformed by the function

$$H(w,y,g) = P^{(w,y)}(W_t \text{ never again hits the line through } (y_0,w_0) \text{ with slope } -a) .$$

Using translation invariance, we may as well assume $(y_0,w_0) = (0,0)$, and then [3]

$$h(w,y) = H(w,y,g) = 1 - e^{-2a(w+ay)} .$$

Still for $x \in F(a,0,0)$, let

$$R_c^x = \frac{\partial}{\partial u} R^x(W_t \text{ hits the line } -bt \text{ before time } u)\big|_{u=c} ,$$

and define Q_c^x analogously. Properties of h-path transforms, the continuity of h near the point $w = -bc$, $y = c$, and an easy limit argument show that $R_c^x = h(-bc,\ c)Q_c^x/h(w,y)$. Then

$$\int_0^\infty e^{-\lambda c}\, R_c^x\, dc = \int_0^\infty e^{-\lambda c}\,(1 - e^{-2a(a-b)c})\, Q_c^x\, dc/h(w,y) .$$

$f(x) = \int_0^\infty e^{-\alpha c} Q_c^x \, dc$ is the Laplace transform for the distribution of the time when W_t first hits the line $-bt$, which is the same as the time when $W_t + bt$ first hits 0. $W_t + bt$ has infinitesimal generator $\frac{1}{2}f'' + bf'$, and so [6] $f(x)$ is the solution to the differential equation with constant coefficients:

$$\frac{1}{2} f'' + bf' - \alpha f = 0 \quad ; \quad f(\infty) = 0 \ , \ f(0) = 1 \ ;$$

therefore $f(x) = \exp((-b - (b^2 + 2\alpha)^{\frac{1}{2}})x)$. Applying this with $\alpha = \lambda$ and then $\alpha = \lambda + 2a(a - b)$, we get, letting $y = 0$,

$$\int_0^\infty e^{-\lambda c} R_c^{(w,0)} \, dc = \frac{\exp((-b - (b^2 + 2\lambda)^{\frac{1}{2}})w) - \exp((-b - (b^2 + 2(\lambda + 2a(a-b)))^{\frac{1}{2}})w)}{1 - \exp(-2aw)} \ .$$

Taking the limit as $w \to 0$, and then setting $\lambda = 0$, we get

$$R^{(0,0)}(W_t \text{ ever hits the line } -bt) = 1 - (b/a) \ .$$

Thus, the most negative value β for which W_t, under $R^{(0,0)}$, hits the line βt, has a uniform distribution on $(-a,0)$, or given that $D^-C_L = -a$, D^+C_L is uniform on $(-a,0)$.

Now let us find the distribution of $M - L$, given that $D^-C_L = -a$ and $D^+C_L = -b$. We thus need to consider Brownian motion conditioned to hit $-bt$ but to stay above $-(b + \varepsilon)t$ for every $\varepsilon > 0$ and to find the distribution of the time when this process hits the line $-bt$.

Again, $P^{(w,y)}(W_t \text{ ever hits } -rt) = e^{-2r(w+ry)}$. Taking the derivative with respect to r at $r = b$, let

$$k(w,y) = (2w + 4by)e^{-2b(w+by)} \ .$$

k is the value of the density of the random variable $\psi = \sup\{r: W_t \text{ ever hits } -rt\}$ at b, and P_t^k would be P_t conditioned so that $\psi = b$. Letting $U_t = P_t^k$,

$$U_c^{(w,y)} = \frac{\partial}{\partial u} U^{(w,y)}(W_t \text{ hits } -bt \text{ before } u)\big|_{u=c} ,$$

and similarly for $P_c^{(w,y)}$, we get as above $U_c^{(w,y)} = k(-bc,\ c)P_c^{(w,y)}/k(w,y)$.

Then

$$\begin{aligned}\int_0^\infty e^{-\lambda c}\, U_c^{(w,0)} dc &= \int_0^\infty e^{-2\lambda c}\ 2bc\ P_c^{(w,0)} dc/k(w,0)\\ &= -2b \frac{\partial}{\partial\lambda}\left(\int_0^\infty e^{-\lambda c}\ P_c^{(w,0)} dc\right)/k(w,0)\\ &= -2b \frac{\partial}{\partial\lambda}\left(\exp((-b - (b^2 + 2\lambda)^{\frac{1}{2}})w\right)/k(w,0)\\ &= \frac{2b\ \exp((-b - (b^2 + 2\lambda)^{\frac{1}{2}})w)w/(b^2 + 2\lambda)^{\frac{1}{2}}}{2w\ \exp(-2bw)} .\end{aligned}$$

Letting $w \to 0$,

$$\int_0^\infty e^{-\lambda c}\ U_c^{(0,0)} dc = \left(\frac{b^2/2}{b^2/2 + \lambda}\right)^{\frac{1}{2}} ,$$

and therefore $U_c^{(0,0)}$ is a gamma density with parameters $\frac{1}{2}$, $b^2/2$.

4. GENERALIZATIONS.

Although properties of Brownian Motion were used throughout, the only places these properties were essential were in determining the explicit form of the distributions of $W_t - C_t$, of the slopes of C_t, and of the vertex times of C_t. The rest of the argument could be easily modified to be applicable to a large class of recurrent diffusions on the real line. In general, the distribution of W_{t+L}, $t \le M - L$, given $W_L = w$, $D^+C_L = -b$, and $M - L = c$ will be that of the diffusion started at w and conditioned to stay above the line through $(0,w)$ of slope $-b$ before time c and to hit this line at time c ; unfortunately, one would not expect to be able to arrive at any simpler description in most cases.

In section 2 we used the fact that $P^{(w,y)}(B_{ru})$ is a C^2 function of r and u ; with a bit more work, the requirement could be weakened considerably. What is needed is the existence of a harmonic function $J_{bc}(x)$ such that Q_t^J is the diffusion conditioned to stay above the appropriate line of slope $-b$ until time c . Similar comments apply to the other places where we used C^1 or C^2 smoothness.

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