SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

MING LIAO

Path continuity and last exit distributions

Séminaire de probabilités (Strasbourg), tome 18 (1984), p. 119-126

<http://www.numdam.org/item?id=SPS_1984__18__119_0>

Path Continuity And Last Exit Distributions

Ming Liao

It is well known that a Hunt process is determined by its hitting distributions up to a random time change, see [1, V5]. It was proved in [4] that the similar conclusion holds for its last exit distributions provided the process is transient. It is not difficult to show, see [3], that a Hunt process is continuous if and only if the hitting distributions are concentrated on the boundaries, i.e.

$\forall$ relatively compact open set A and $x \in A$,

$$P^x\left[X(T_{A^c}) \notin \partial A,\ T_{A^c} < \infty \right] = 0 \tag{1}$$

Naturally a question arises: Do we have the similar conclusion for the last exit distributions? To be precise, given a transient Hunt process X_t, is it true that the path continuity is equivalent to the following condition:

$\forall$ relatively compact open set A and $x \in A$,

$$P^x\left[X(\zeta_A -) \notin \partial A,\ \zeta_A < \zeta \right] = 0 \tag{2}$$

where $\zeta_A = \sup\{t : X_t \in A\}$ with $\sup \phi = 0$ and ζ is the lifetime of X_t. It is clear that the continuity implies (2). The purpose of this paper is to show that in general (2) does not imply continuity: An example is presented in Sec 1. (If killing is allowed, we can obtain a much simpler example as is given at the end of Sec 3.) In Sec 2, we show that under an additional condition, (2) does guarantee continuity and consequently the process is continuous if and only if the equilibrium measures are concentrated on the boundaries. In Sec 3, we establish two other results under the assumption (2).

Sec 1 . Let $E = \{ (x,y) \in R^2, \ -\infty < x < \infty \ , \ 0 \leqslant y \leqslant 1 \}$. We construct a process X_t with E as its state space, and, roughly speaking, having the following properties : If X_t starts from (x,y) with $y > 0$, then it moves at unit speed along a vertical line down to the x-axis; if X_t starts from a point on the x-axis, then it moves at unit speed to the right except that it may have several jumps along its paths and each jump brings X_t to a point one unit above its current position.

Let us first write down its transition functions.

For $0 \leqslant t \leqslant 1$, $z \in E$ and $f \geqslant 0$ measurable on E, define $P_t f(z)$ by

$$\begin{cases} P_t f(x,0) = \int_0^t e^{-u} du f(x+u, 1-t+u) + e^{-t} f(x+t,0) \\ P_t f(x,y) = f(x,y-t) \qquad \text{if } y \geqslant t \\ \qquad\qquad = P_{t-y} f(x,0) \qquad \text{if } y < t \, . \end{cases} \tag{3}$$

Lemma 1 : For $t,s \geqslant 0$ and $t + s \leqq 1$, $P_t P_s f(z) = P_{t+s} f(z)$.

proof : We only show this for $z = (x,0)$.

$$P_t P_s f(x,0) = \int_E P_t((x,0), dw) P_s f(w)$$

$$= \int_0^t e^{-u} du P_s f(x+u, 1-t+u) + e^{-t} P_s f(x+t,0) \qquad (\text{Since } s \leqslant 1-t)$$

$$= \int_0^t e^{-u} du f(x+u, 1-t-s+u) + e^{-t} \int_0^s e^{-v} dv f(x+t+v, 1-s+v) +$$

$$e^{-(t+s)} f(x+t+s,0) = \int_0^{t+s} e^{-u} du f(x+u, 1-(t+s)+u) +$$

$$e^{-(t+s)} f(x+t+s,0) = P_{t+s} f(x,0) \, . \qquad \text{QED}$$

For any $t > 0$, write $t = \sum_{k=1}^{n} t_k$ with $0 \leqslant t_k \leqslant 1$, let $P_t f = P_{t_1} P_{t_2} \cdots P_{t_n} f$. By Lemma 1, $P_t f$ is well defined and $\{ P_t \}$ form a semi-group of probabilities. By (3), we see easily that $\{ P_t \}$

is a Feller semi-group; hence there is a Hunt process X_t with $\{P_t\}$ as its transition semi-group.

For $h > 0$, let $r(h) = P^{(x,0)}\{X_t \text{ hits } (x+h,0)\}$. (4)

It is easy to see that $r(h)$ is independent of x.

Lemma 2 : For any $h > 0$, $r(h) = 1$.

Proof : By the strong Markov property, $r(h+k) = r(h)r(k)$, so it is enough to show $r(t) = 1$ for $0 \leqslant t \leqslant 1$. By (3),

$$r(t) \geqslant e^{-t} \quad \text{and} \quad r(t) = e^{-t} + \int_0^t e^{-u}\, du\, r(t-u) \tag{5}$$

So $r(t) \geqslant e^{-t} + \int_0^t e^{-u}\, du\, e^{-(t-u)} = e^{-t}(1+t)$. Substituting this in (5), we obtain

$$r(t) \geqslant e^{-t} + \int_0^t e^{-u}\, du\, e^{-(t-u)}(1 + (t-u)) = e^{-t} + e^{-t}t + \frac{1}{2!}e^{-t}t^2$$

By induction we can prove

$$r(t) \geqslant e^{-t}\left(1 + t + \frac{1}{2!}t^2 + \frac{1}{3!}t^3 + \cdots\right) = 1 .$$ QED

Lemma 2 shows that X_t is a transient Hunt process. We can check that X_t satisfies the properties prescribed in the first paragraph of this section, and from this it is easy to see that (2) holds but X_t is not continuous.

Sec 2 . From now on , we assume X_t is a Hunt process with state space E and that it is transient in the following sense :

$$\forall x \in E \text{ and compact } K \subset E, \quad P^x\left[\zeta_K = \infty\right] = 0 \tag{6}$$

Lemma 3 : Suppose for any compact set K and $y \notin K$, there exists a neighborhood U of y such that $\forall z \in K$, $P_U 1(z) < 1$. Then (2) implies the continuity of X_t .

Proof : It suffices to prove (1). Let A be a relatively compact open set, $x \in A$. Let $T = T_{A^c}$ and suppose

$$P^x\left[X(T) \notin \partial A,\ T < \infty\right] > 0 .$$

There exists a compact set $K \subset (\bar{A})^c$ such that

$$P^x\left[X(T) \in K\right] > 0 \tag{7}$$

Let u be the measure on E defined by

$$u(dz) = P^x\left[X(T-) \in dz,\ X(T) \in K\right] \tag{8}$$

u is carried by $\bar{A}$ and is non-trivial. We may assume $\operatorname{supp}(u) \cap A \neq \phi$, otherwise we can replace A by A' with $\bar{A} \subset A' \subset K^c$.

Let $y \in \operatorname{supp}(u) \cap A$. By the assumption, there exists an open set W with $y \in W \subset A$ and $P_W 1 < 1$ on K. Let U,V be open sets with $y \in V \subset \bar{V} \subset U \subset \bar{U} \subset W$. Since $y \in \operatorname{supp}(u)$,

$$P^x\left[X(T-) \in V,\ X(T) \in K\right] > 0. \tag{9}$$

Let $T_0 = 0$, $T_1 = T_V$, $T_2 = T_1 + T_{U^c} \circ \theta_{T_1}$ and inductively let $T_{2K+1} = T_{2K} + T_V \circ \theta_{T_{2K}}$, $T_{2K+2} = T_{2K+1} + T_{U^c} \circ \theta_{T_{2K+1}}$. Since a.s. $t \to X_t$ has left limits for $t < \infty$, P^x-a.s.

$$\left[X(T-) \in V,\ X(T) \in K\right] \subset \bigcup_{k=1}^{\infty} \left[T_{2K-1} < T,\ T_{2K} = T,\ X(T) \in K\right]. \tag{10}$$

By (9), for some k,

$$P^x\left[T_{2K} = T,\ X(T) \in K\right] > 0 \tag{11}$$

Since $P_W 1 < 1$ on K, $P^z\left[T_W = \infty\right] > 0$ for $z \in K$, hence

$$P^x\left[X(T_{2K}) \in K,\ T_W \circ \theta_{T_{2K}} = \infty\right] = E^x\left[X(T_{2K}) \in K,\ P^{X(T_{2K})}\left[T_W = \infty\right]\right] > 0.$$

$$E^x\left[P^{X(T_{2K-1})}\left[X(T_{U^c}) \in K,\ T_W \circ \theta_{T_{U^c}} = \infty\right]\right] > 0 \tag{12}$$

Since $X(T_{2K-1}) \in \bar{V}$ on $\left[T_{2K-1} < \infty\right]$, for some $z \in \bar{V}$,

$$P^z\left[X(T_{U^c}) \in K,\ T_W \circ \theta_{T_{U^c}} = \infty\right] > 0 \tag{13}$$

On $[X(0) = z,\ X(T_{U^c}) \in K,\ T_W \circ \theta_{T_{U^c}} = \infty]$,

$T_{U^c} = T_{W^c} = \zeta_W < \infty$ and $X(\zeta_W -) = X(T_{U^c}-) \in \overline{U}$, therefore

$$P^z\left[X(\zeta_W -) \in \overline{U}\right] > 0 \tag{14}$$

This contradicts (2), hence (1) is proved. QED

Corollary : Suppose for any compact F, $P_F 1$ is continuous on F^c and $\forall x \in E$, $P_{\{x\}} 1(y) < 1$ for $y \neq x$. Then (2) implies continuity.

Proof : Let K be compact and $x \notin K$. Choose D_n relatively compact open, $D_n \ni x$ and $\overline{D}_n \downarrow \{x\}$. We may assume $\overline{D}_1 \cap K = \emptyset$. $\{P_{\overline{D}_n} 1\}$ is a sequence of continuous functions on K and it decreases to the continuous function $P_{\{x\}} 1$ pointwise on K. By Dini's Theorem, $P_{\overline{D}_n} 1 \to P_{\{x\}} 1$ uniformly on K. Since $P_{\{x\}} 1 < 1$ on K, for some n, $P_{\overline{D}_n} 1 < 1$ on K. Hence the condition of Lemma 3 is satisfied. QED

Remark : By going through the proof of Lemma 3, we see that this Lemma and its corollary still hold with (2) replaced by

$\forall$ relatively compact open A and $x \in A$,

$$P^x\left[X(\zeta_A) \notin \partial A,\ \zeta_A < \zeta\right] = 0 \tag{2'}$$

Now we suppose our process X_t has a potential density u(x,y) with respect to an excessive Radon measure m on E, i.e.

$$\forall f \geq 0 \text{ measurable}, \quad \int_0^\infty P_t f(x)\,dt = \int_E u(x,y) f(y)\, m(dy) \tag{15}$$

Assume : $\forall x \in E$, $u(x,\cdot)$ and $u(\cdot,x)$ are strictly positive and extended continuous, and $u(x,y) = \infty$ if and only if $x = y$.

Our hypothesis is slightly stronger than that in [2] and to which we refer the readers for a complete account of the related theory. We know that any compact set K has an equilibrium measure μ_K which is the unique measure characterized by

$$\forall x \in E, \quad P_K 1(x) = \int u(x,y)\mu_K(dy) \tag{16}$$

Furthermore μ_K satisfies : $\forall x \in E$ and $f \geqslant 0$ measurable,

$$E^x\left[f(X(\zeta_K -)); \ \zeta_K > 0\right] = \int_E u(x,y)f(y)\mu_K(dy) \tag{17}$$

It is easy to check the condition of the above corollary in the present situation hence by (17) we have

Proposition 1 : Under the above hypothesis, X_t is continuous if and only if for any compact K, μ_K is concentrated on ∂K.

Sec 3 . We say X has no killing inside E if $X(\zeta -) \notin E$ a.s. on $[\zeta < \infty]$.
Under the assumption of transience, this is equivalent to the following :

$$\forall \text{ relatively compact open } A \text{ and } x \in A, \quad P^x\left[T_{A^c} < \infty\right] = 1 \tag{18}$$

Proposition 2 : Assume (2) and X_t has no killing inside E. Then

$$\forall \text{ relatively compact open } A \text{ and } x \in A, \quad P^x\left[T_{\partial A} < \infty\right] = 1. \tag{19}$$

Proof : Let $T_0 = 0$, $T_1 = T_{A^c}$, $T_2 = T_1 + T_A \circ \theta_{T_1}$ and inductively let $T_{2k+1} = T_{2k} + T_{A^c} \circ \theta_{T_{2k}}$, $T_{2k} = T_{2k-1} + T_A \circ \theta_{T_{2k-1}}$. Then P^x-a.s. we have three possible cases :

Case 1 : $\exists k$, $T_k = T_{k+1} < \infty$.

Case 2 : $T_1 < T_2 < \cdots < T_k < T_{k+1} < \cdots$

Case 3 : $\exists k$ such that $T_1 < T_2 < \cdots < T_{2k+1} < T_{2k+2} = \infty$.

Observe that it is not possible to have $T_{2k} < T_{2k+1} = \infty$ because of (18).

In Case 1 , $X(T_k) = X(T_{k+1}) \in \overline{A} \cap \overline{(A^c)} = \partial A$.

In Case 2 , let $T = \lim_k T_k$, then $T \leqslant \zeta_A < \infty$, by the quasi-left continuity, $X(T) \in \partial A$.

In Case 3, $T_{2k+1} = \zeta_A$, $T_{2k} < T_{2k+1}$ and by (2),

$$X(T_{2k+1}-) = X(\zeta_A -) \in \partial A . \tag{20}$$

Let B_n be open sets with $\overline{B}_n \subset A$ and $B_n \uparrow A$, $S_n = T_{2k} + T_{B_n^c} \circ \theta_{T_{2k}}$, then $S_n \uparrow T_{2k+1}$. By (20), $X(S_n) \in A - B_n$. By the quasi-left continuity,

$$X(T_{2k+1}) = \lim_n X(S_n) \in \partial A .$$ QED

Proposition 3 : Assume (2) and X_t has no killing inside E. Then X_t has no holding points.

Proof : Fix $x \in E$, let D_n be a sequence of relatively compact open sets such that $\overline{D}_{n+1} \subset D_n$, $D_n \ni x$ and $\overline{D}_n \downarrow \{x\}$. For each $n \geq 1$, define

$$S_n = T_{\partial D_n},\ S_{n-1} = S_n + T_{\partial D_{n-1}} \circ \theta_{S_n}, \cdots, S_1 = S_2 + T_{\partial D_1} \circ \theta_{S_2}$$ and let

$$T_k^{(n)} = S_k \quad \text{for } k = 1, 2, \cdots, n.$$

For each k, $T_k^{(n)}$ is defined for $n \geq k$ and $T_k^{(n)} \uparrow$ as $n \uparrow$. By (19), $T_k^{(n)} < \infty$ a.s. so $T_k^{(n)} \leq \zeta_{\overline{D}_1} < \infty$ a.s. Let $T_k = \lim_n T_k^{(n)}$ then a.s. $T_k < \infty$. We see easily that $T_k \downarrow$ as $k \uparrow$. Let

$$T = \lim_k T_k . \tag{21}$$

We have $T < T_k$ and $T_k = T + T_k \circ \theta_T$ (22)

By (21) and (22), $\lim_k T_k \circ \theta_T = 0$ so

$$1 = P^x\left[\lim_k T_k \circ \theta_T = 0\right] = E^x\left[P^{X(T)}\left[\lim_k T_k = 0\right]\right]$$

Since $X(T_k) \in \partial D_k$, $X(T) = x$ by the right continuity, hence $P^x\left[\lim_k T_k = 0\right] = 1$. This implies x is not a holding point. QED

Remark : The assumption that no killing occurs inside E cannot be dropped. To see this, construct a transient Hunt process according to the following description : Let $[0, 1]$ be the state space. If the process starts from $x < 1$, it moves to the right with unit speed until it reaches 1. 1 is a holding point with the exponentially distributed holding time and when it leaves 1, it jumps to 0 or kills itself with the equal probability $\frac{1}{2}$.

Acknowledgement : The author wishes to express his gratitude to Professor K.L. Chung who suggested this subject.

References :

1. R.M. Blumenthal and R.K. Getoor, "Markov Processes and Potential Theory", Academic Press, New York (1968).

2. K.L. Chung and M. Rao, "A New Setting for Potential Theory", Ann. Inst. Fourier 30 (1980), pp. 167-198.

3. Courrège et Priouret, "Axiomatique du Problème de Dirichlet et Processus de Markov", Seminaire de Théorie du Potentiel (Brelot-Choquet-Deny) V.8, 1963/64.

4. Joseph Glover, "Markov Processes with Identical Last Exit Distributions", Z. Wahrscheinlichkeitstheorie Verw. Geb. 59 (1982), pp. 67-75.

Ming Liao

Department of Mathematics

Stanford University

Stanford, California 94305.

U. S. A.