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STOCHASTIC INTEGRALS AND PROGRESSIVE MEASURABILITY -- AN EXAMPLE

by

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In this note we construct a measurable set $D \subset [0,\infty) \times \Omega$, a 3-dimensional Bessel process, X, and a filtration, $\{F_t^B\}$, containing the canonical filtration, $\{F_t^X\}$, of X satisfying the following properties:

(i) X is an $\{F_t^B\}$ - semimartingale.

(ii) D is an $\{F_t^X\}$ - progressively measurable set, i.e., $D \cap [[0,t]] \in \text{Borel}([0,t]) \times F_t^X$ for all $t \geq 0$.

(iii) $\int_0^t I_D \, dX = X(t)$, where the left side is interpreted with respect to $\{F_t^X\}$, and I_D denotes the indicator function of D.

(iv) $\int_0^t I_D \, dX$ is an $\{F_t^B\}$ - Brownian motion when the stochastic integral is taken with respect to $\{F_t^B\}$.

As the local martingale part of X with respect to either filtration will be a Brownian motion (since $[X](t) = t$), $\int_0^t I_D \, dX$ may be defined in the obvious way even though D will not be predictable.

Let B be a 1-dimensional Brownian motion on a complete (Ω, F, P). If $M(t) = \sup_{s \leq t} B(s)$, $Y = M - B$ and $X = 2M - B$, then Y is a reflecting Brownian motion, and X is a 3-dimensional Bessel process by a theorem of Pitman [4]. $\{F_t^X\}$, respectively $\{F_t^B\}$, will denote the smallest filtration, satisfying the usual conditions, that makes X, respectively B, adapted. $F_\cdot^X \subseteq F_\cdot^B$ is clear, and since $M(t) = \inf_{s \geq t} X(s)$, the inf being assumed at the next zero of Y, we must have $F_t^X \subsetneq F_t^B$ for $t > 0$, as $M(t)$ cannot be F_t^X - measurable. Finally, define

$$D = \{(t,\omega) \mid \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} I(X(t+2^{-k}) - X(t+2^{-k-1}) > 0) = 1/2\} .$$

Property (i) is immediate and for (ii), fix $t \geq 0$ and note that

$$D \cap [[0,t]] = (\{t\} \times D(t)) \bigcup_{N=1}^{\infty} \{(s,\omega) | s \leq t - 2^{-N} ,$$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{k=N}^{\infty} I(X(s+2^{-k}) - X(s+2^{-k-1}) > 0) = 1/2\} \in \mathrm{Borel}([0,t]) \times F_t^X \ .$$

Here $D(t)$ is the t-section of D. To show (iii) choose $t > 0$ and note that

$$X(t+2^{-k}) - X(t+2^{-k-1}) = B(t+2^{-k-1}) - B(t+2^{-k}) \quad \text{for large } k \text{ a.s.}$$

Therefore the law of large numbers implies that

(1) $$P((t,\omega) \in D) = 1 \quad \text{for all } t > 0 \ .$$

The canonical decomposition of X with respect to $\{F_t^X\}$ is (see McKean [3])

(2) $$X(t) = W(t) + \int_0^t X(s)^{-1} ds \ ,$$

where W is an $\{F_t^X\}$ - Brownian motion. Therefore with respect to $\{F_t^X\}$ we have

$$\int_0^t I_D \, dX = \int_0^t I_D \, dW + \int_0^t I_D \, X_s^{-1} ds = X(t) \text{ a.s. (by (1))} \ .$$

It remains only to prove (iv). If

$$T(t) = \inf\{s | M(s) > t\} \ ,$$

we claim that

(3) $$P((T(t),\omega) \in D) = 0 \quad \text{for all } t \geq 0 \ .$$

Choose $t \geq 0$ and assume $P((T(t),\omega) \in D) > 0$. Since $X(T(t)+\cdot) - X(T(t))$ is equal in law to $X(\cdot)$, the 0-1 law implies that $P((T(t),\omega) \in D) = 1$. The dominated convergence theorem and Brownian scaling imply

$$1/2 = n^{-1} \sum_{k=1}^{n} P(X(2^{-k}) - X(2^{-k-1}) > 0)$$

$$= P(X(2) - X(1) > 0)$$

$$= P(B(2) - B(1) < 2(M(2) - M(1)))$$

$$> 1/2 \ .$$

Therefore (3) holds and, with respect to $\{F_t^B\}$, we have w.p.1

$$\int_0^t I_D \, dX = 2 \int_0^t I_D \, dM - \int_0^t I_D \, dB$$

$$= 2 \int_0^t I_D(T(s),\omega) ds - B(t) \quad \text{(by (1))}$$

$$= -B(t) \quad \text{(by (3))}$$

This completes the proof.

It is not hard to see that the above result implies that the optional projections of I_D with respect to $\{F_t^X\}$ and $\{F_t^B\}$ are distinct. In particular D cannot be $\{F_t^X\}$-optional. In fact, D is not $\{F_t^B\}$-optional and both optional projections may be computed explicitly.

Proposition (a) The optional projection of I_D with respect to $\{F_t^X\}$ is $I_{(0,\infty)\times\Omega}$.

(b) The optional projection of I_D with respect to $\{F_t^B\}$ is I_{Z^c} where Z is the zero-set of Y .

(c) D is not $\{F_t^B\}$ - optional.

Proof (a) Let $\infty \ge T \ge \varepsilon > 0$ be an $\{F_t^X\}$ stopping time. The law of large numbers implies that

$$(4) \qquad \lim_{n\to\infty} \frac{1}{n} \sum_{k=1}^{n} I(W(T+2^{-k}) - W(T+2^{-k-1}) > 0) = 1/2 \quad \text{a.s. on } \{T < \infty\} ,$$

where W is as in (2). Recall that $M(t) = \inf_{s \ge t} X(s)$. Therefore

$$E(|I(W(T+2^{-k}) - W(T+2^{-k-1}) > 0) - I(X(T+2^{-k}) - X(T+2^{-k-1} > 0)|I(T < \infty))$$

$$\le P(0 \ge W(T+2^{-k}) - W(T+2^{-k-1}) \ge \int_{T+2^{-k-1}}^{T+2^{-k}} X(s)^{-1} ds \ , \ T < \infty)$$

$$\le P(0 \ge (W(T+2^{-k}) - W(T+2^{-k-1}))2^{(-k-1)/2} \ge -2^{(-k-1)/2} M(\varepsilon)^{-1} \ , \ T < \infty)$$

$$\le CE(\min(1,\ 2^{-(k-1)/2}M(\varepsilon)^{-1}))$$

$$\le C(2^{-(k-1)/4} + P(M(\varepsilon) < 2^{-(k-1)/4}))$$

$$\le C(\varepsilon)2^{-(k-1)/4} .$$

The Borel-Cantelli lemma implies that

(5) $W(T+2^{-k}) - W(T+2^{-k-1}) > 0 \Longleftrightarrow X(T+2^{-k}) - X(T+2^{-k-1}) > 0$

for large k a.s. on $\{T < \infty\}$.

(4) and (5) imply that $(T,\omega) \in D$ a.s. Moreover by (3) with $t = 0$, $(0,\omega) \notin D$ a.s. Therefore if T is any $\{F_t^X\}$ - stopping time and

$$T' = \begin{cases} T & \text{if } T > 0 \\ \infty & \text{if } T = 0 \end{cases} ,$$

then

$$E(I_D(T,\omega)\ I(T < \infty)) = \lim_{\varepsilon\to 0^+} E(I_D(T'\vee\varepsilon,\omega)I(T' < \infty))$$

$$= P(T' < \infty) \quad \text{(since by the above } (T'\vee\varepsilon,\omega) \in D \text{ a.s. on } \{T'<\infty\})$$

$$= P(0 < T < \infty) .$$

This proves (a) .

(b) Let $T \le \infty$ be any $\{F_t^B\}$ - stopping time. Then just as in the derivation of (1) one has

(6) $(T,\omega) \in D$ a.s. on $\{Y(T) \neq 0 , T < \infty\}$.

Moreover just as in the derivation of (3) one has

(7) $(T,\omega) \notin D$ a.s. on $\{Y(T) = 0 , T < \infty\}$.

Therefore

$$E(I_D(T,\omega)I(T < \infty)) = P(Y(T) \neq 0 , T < \infty) ,$$

and (b) is proved.

(c) If D is $\{F_t^B\}$ - optional then $D = Z^c$ (up to indistinguishability) by the above. Therefore Z is on $\{F_t^X\}$ - progressively measurable set. $M(t)$ is the local time of Z and hence can be constructed from Z as Lévy's mesure du voisinage [2, p.225] . It follows easily from this construction that $M(t)$ is $\{F_t^X\}$ - adapted. As $M(t)$ is the future minimum of X , this is absurd. □

The above example was suggested by joint work with Michel Emery [1], in which the predictable set

$$\{(t,\omega) \mid \lim_{n\to\infty} n^{-1} \sum_{k=1}^{n} I((cM - B)(t-2^{-k}) - (cM - B)(t-2^{-k-1}) > 0) = 1/2\}$$

was used to show $F_\cdot^{cM-B} = F_\cdot^B \iff c \neq 2$.

List of References

1. Emery, M. and Perkins, E. La filtration de B+L. Z.f. Wahrscheinlichkeitstheorie 59, 383-390 (1982).

2. Lévy, P. Processus Stochastiques et Mouvement Brownien. Gauthier-Villars, Paris, 1948.

3. McKean, M.P. The Bessel motion and a singular integral equation. Mem. Coll. Sci. Univ. Kyoto. Ser. A Math. 33, 317-322 (1960).

4. Pitman, J. One-dimensional Brownian motion and the three-dimensional Bessel process. Adv. Appl. Prob. 7, 511-526 (1975).

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