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Skorokhod Imbedding via Stochastic Integrals

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Given a Brownian motion L_t and a probability measure μ on $\mathbb{R}$ with mean 0, a Skorokhod imbedding of μ is a stopping time T adapted to the sigma fields of L_t such that L_T has distribution μ. We give here a new method of constructing such an imbedding using results from the representation of martingales as stochastic integrals.

We first construct a Brownian motion N_t and a stopping time W such that N_W has law μ. We then show how, given an arbitrary Brownian motion L_t, one can construct a stopping time T such that L_T has law μ.

Define $p_t(y) = (2\pi t)^{-1/2} e^{-y^2/2t}$, $q_t(y) = \partial p_t(y)/\partial y = -(2\pi t)^{-1/2}(y/t)e^{-y^2/2t}$. Let X_t be a Brownian motion, $\underline{\underline{F}}_t$ its filtration, and g a real-valued function.

<u>Lemma 1</u>. <u>Suppose</u> $E|g(X_1)|<\infty$. <u>Then</u>

a) $\sup_{|y|\le y_0} \int g(z)|z-y|^k e^{-(z-y)^2/2t}dz < \infty$ <u>for all positive</u> k, <u>all</u> y_0, all $t<1$.

b) $g(X_1) = Eg(X_1) + \int_0^1 a(s,X_s)dX_s$, <u>where</u> $a(s,y) = \int q_{1-s}(z-y)g(z)dz$ <u>for</u> $s<1$; <u>furthermore</u> $\int_0^1 a^2(s,X_s)ds < \infty$, a.s.

c) $E(g(X_1)|\underline{\underline{F}}_s) = b(s,X_s)$ <u>for</u> $s<1$, <u>where</u> $b(s,y) = \int p_{1-s}(z-y)g(z)dz$.

<u>Proof</u>. a) follows from the formula for the normal density and the fact that $|z-y|^k e^{-(z-y)^2/2t} \le e^{-z^2/2}$ for z large.

b) Suppose first that g is bounded, has compact support, and is in C^2. By Clark's formula [1] applied to the functional $g(X_1)$,

$$g(X_1) = Eg(X_1) + \int_0^1 E[g'(X_1)|\underline{\underline{F}}_s]dX_s .$$

(Another derivation of this representation is to use Ito's lemma to take care of the case $g(x) = e^{iux}$ and then use linearity and a limiting process.)

By the Markov property, if $s < 1$,

$$E[g'(X_1)|\underline{\underline{F}}_s] = \int g'(z)p_{1-s}(X_s - z)dz \ .$$

An integration by parts gives the result for such g ; the result for general g follows by a limit argument.

c) By the Markov property, if $s < 1$,

$$E[g(X_1)|\underline{\underline{F}}_s] = \int g(z)p_{1-s}(X_s - z)dz \ . \qquad \square$$

<u>Lemma</u> 2. <u>Suppose</u> g <u>is nondecreasing and not identically constant</u>. <u>Then</u>

a) <u>On compact subsets of</u> $[0,1) \times \mathbb{R}$, $a(s,y)$ <u>is bounded above</u>, <u>bounded below away from</u> 0 , <u>and uniformly Lipschitz in</u> s <u>and</u> y .

b) <u>For each</u> $s < 1$, $b(s,y)$ <u>is continuous and strictly increasing as a function of</u> y .

c) <u>For each</u> $s < 1$, <u>let</u> $B(s,\cdot)$ <u>be the inverse of</u> $b(s,\cdot)$; <u>then on compact subsets of its domain</u>, $B(s,y)$ <u>is uniformly Lipschitz in</u> s <u>and jointly continuous in</u> s <u>and</u> y .

<u>Proof</u>. a) Suppose $|y| \le y_0, s \le s_0 < 1$. $a(s,y)$ is bounded above by lemma 1a. An integration by parts argument shows that $a(s,y) = \int p_{1-s}(y-z)dg(z)$, hence a is bounded below. Using the definition of $a(s,y)$, appropriate bounds on $\partial q_{1-s}/\partial s$ and $\partial q_{1-s}/\partial y$, and lemma 1a gives the uniformly Lipschitz result.

b) The definition of b shows that $b(s,\cdot)$ is continuous. Since we also have $b(s,y) = \int g(y+z)p_{1-s}(z)dz$, it follows that $b(s,\cdot)$ is nondecreasing, and in fact, strictly increasing since g is not constant. Note that this implies that the range of $b(s,\cdot)$ must be an open (possibly infinite) interval.

c) Since $b(s,\cdot)$ is continuous and strictly increasing, we can define its inverse $B(s,\cdot)$ on the range of $b(s,\cdot)$. $B(s,y)$ will be continuous in y.

Integrating by parts,

$$\partial b/\partial y = \int p_{1-s}(y-z)dg(z) \ ,$$

which is uniformly > 0 for s,y in a compact subset of $[0,1) \times \mathbb{R}$. $\partial b/\partial s$ is bounded above on compact sets since $\partial p_{1-s}/\partial s$ is, using lemma 1a again.

We now show that B is uniformly Lipschitz in s , s,y in a compact subset of the domain of B . Let $w = B(s+h,y)$, $x = B(s,y)$, and suppose $w \le x$, the other case being similar. Then

$$0 = b(s+h,w) - b(s,x) = b(s+h,w) - b(s,w) + b(s,w) - b(s,x) \le C|h| - c(x-w),$$

or $|x-w| \le C|h| / c$,

where C and c are upper and lower bounds for $\partial b/\partial s$ and $\partial b/\partial y$, respectively. This proves that B is uniformly Lipschitz in s , and it follows immediately that B is jointly continuous. □

Now let μ be a probability measure on $\mathbb{R}$ and suppose $\int |x| \, d\mu(x) < \infty$ and $\int x \, d\mu(x) = 0$. Let $F(x) = \mu(-\infty,x]$, let $F^{-1}(y) = \inf\{x: F(x) \ge y\}$, let $\Phi(x) = \int_{-\infty}^{x} p_1(y)dy$, and let $g(x) = F^{-1}(\Phi(x))$. Then $g(X_1)$ has distribution μ and $Eg(X_1) = 0$.

Define $M_t = \int_0^t a(s,X_s)dX_s$, where $a(s,y)$ is given by lemma 1 for $s < 1$, $a(s,y) = 1$ for $s \ge 1$. Note $M_1 = g(X_1)$ has law μ , and if $s < 1$, $M_s = b(s,X_s)$. Let $R(t) = \int_0^t a^2(s,X_s)$, define $S(t) = \inf\{r:R(r) \ge t\}$, and let $N_t = M_{S(t)}$. Since the quadratic variation of the continuous martingale N is t , N is a Brownian motion.

$N_{R(1)} = M_1$, which has law μ . Letting $W = R(1)$, it suffices to show that $R(1)$ is a stopping time of the N_t process.

Proposition 3. (cf. Yershov, [2]). $(W \ge u)$ *is in the right continuous completion of* $\sigma(N_s;\ s \le u)$.

Proof. Since $W = R(1) = \lim_{s \to 1} R(s)$ by monotone convergence, it suffices to consider $R(s), s < 1$. $(R(s) \ge u) = (s \ge S(u))$.

It is not hard to see that $S(t)$ satisfies the equation

$$\frac{dS(t)}{dt} = a^{-2}(S(t),\ X_{S(t)})$$

if $S(t) < 1$. But $X_{S(t)} = B(S(t),\ M_{S(t)}) = B(S(t),\ N_t)$. Thus, for each ω , $S(t)$ satisfies the ordinary differential equation

(1) $$\frac{dS(t)}{dt} = a^{-2}(S(t),\ B(S(t),\ N_t)) .$$

For each ω , $\{(S(t),\ N_t): S(t) \le s\}$ is contained in a compact subset of the domain of B . This, lemma 2, and a theorem on uniqueness of solutions of differential equations [3, pp.1-6] show that there is a unique solution $S(t)$ to (1) up to the first t for which $S(t) = s$. Moreover, this solution may be constructed via Picard iteration. But then $(s \geq S(u))$ is in the right continuous completion of $\sigma(N_s;\ s \leq u)$ as required. $\square$

Suppose now that L is any Brownian motion. We construct an L-measurable stopping time T such that L has law μ. Let $V(t)$ be the unique solution to

$$\frac{dV(t)}{dt} = a^{-2}(V(t),\ B(V(t),\ L_t))$$

for each ω. (Since L_t has the same law as N_t, $\{(V(t), L_t): V(t) \leq s\}$ will be in a compact subset of the domain of B, a.s.) Let $U(t) = V^{-1}(t)$, $t < 1$ and let $T = U(1) = \sup_{s<1} U(s)$. Clearly the law of (L,T) is the same as the law of (N,W), and so L_T has distribution μ.

$T^{1/2}$ will satisfy certain moment conditions if μ does. For example, suppose $\Psi\colon [0,\infty) \to [0,\infty)$ is continuous, $\int \Psi(|x|)d\mu(x) < \infty$, and for some $\varepsilon > 0$, $\Psi^{1/(1+\varepsilon)}$ is convex and increasing. By Doob's inequality applied to the submartingale $\Psi^{1/(1+\varepsilon)}(|M_s|)$, $E \sup_{s\leq 1} \Psi(|M_s|) < \infty$ since $E\,\Psi(|M_1|) < \infty$. Then by Burkholder's inequality, $E\,\Psi(W^{1/2}) < \infty$.

If Y_t is a d-dimensional Brownian motion, $d \geq 2$, it is known that there are measures μ for which one cannot find a stopping time T with μ the law of Y_T: just take μ atomic and recall that d-dimensional Brownian motion does not hit points. However, one can always find $f : \mathbb{R} \to \mathbb{R}^d$ such that the law of $f(X_1)$ is μ, X_t a 1-dimensional Brownian motion. (The coordinate functions of f are not assumed to be nondecreasing.) One can then use lemma 1 to find a vector-valued function $A\colon[0,1) \times \mathbb{R} \to \mathbb{R}^d$ such that $f(X_1) = Ef(X_1) + \int_0^1 A(s,\ X_s)dX_s$.

REFERENCES

1. J.M.C. Clark, The Representation of Functionals of Brownian Motion by Stochastic Integrals, Ann. Math Stat 41 (1970) 1282-1295.

2. M.P. Yershov, On Stochastic Equations, Proceedings 2nd Japan-USSR Symposium on Probability Theory. Lec. Notes in Math. 330, 527-530, Springer-Verlag, Berlin, 1973.

3. K. Yosida, Lectures on Differential and Integral Equations, Interscience, New York, 1960.