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HYPOTHESIS (B) OF HUNT

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For a strong Markov process X_t with a locally compact econd Countable State Space, Hunt's Hypothesis (B) may be tated

$$P_G P_K = P_K$$

or all compact K and open G containing K.

There are equivalent statements of hypothesis (B):

1) The hitting time to any set of the process X_{t-} is the same as that of X_t;

2) The probability is zero that the process belongs to a given Semipolar set at any time of discontinuity;

3) If $\alpha > 0$, Hypothesis (B) is equivalent to [2]

0) $$P_G^\alpha P_K 1 = P_K^\alpha 1.$$

In this note we remove the restriction that $\alpha > 0$, assuming hat we have a transient Markov process satisfying Hypothesis L). There are instances where it is easiest to verify the above hen $\alpha = 0$ hence such a result is not without interest.

In the proof sets of the form $(P_K 1 = 1$ for thin sets K lay an important role. We show that non-existence of such sets mplies hypothesis (B) provided of course that (0) is valid hen $\alpha = 0$. It is also shown in the end that a set of the type $P_K 1 = 1)$ is finely open so that unless empty it is rather "large".

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Notation will be as in [1].

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Let $K = K_0$ be a Borel set contained in a given compact set. Define for each countable ordinal γ a set K_γ as follows

$$K_{\gamma+1} = (x \in K_\gamma : P_{K_\gamma} 1(x) = 1)$$

$$K_\gamma = \bigcap_{\beta<\gamma} K_\beta \quad \text{if } \gamma \text{ is a limit ordinal.}$$

Put

$$A = \bigcap_\gamma K_\gamma .$$

<u>Lemma 1</u>. The set A is Borel and

(1) $$A \subset (x : P_A 1(x) = 1).$$

<u>Proof</u>. Hypothesis (L) implies that K_γ is a Borel set for all countable ordinals γ. Let ξ denote a probability reference measure. As $\phi(\gamma) = E^\xi(\exp(-T_{K_\gamma}))$ is non-increasing there is a countable ordinal β such that $\phi(\gamma) = \phi(\beta)$ for all $\gamma \geq \beta$.

If $x \in K_{\beta+1}$, $P^x(T_{K_\beta} < \infty) = 1$ and hence $P^x(T_{K_{\beta+1}} < \infty) = 1$ i.e. $x \in K_{\beta+1}$ and hence $x \in K_{\beta+2}$ etc.

Therefore for all $\gamma \geq \beta$ K_γ is the same $K_{\beta+1}$. That is to say $A = K_{\beta+1}$ is Borel. The assertion is proved.

A Borel set B is called thin if $P_B^1 1(x) = E^x(\exp(-T_B)) < 1$ for all x. It is called totally thin if there exists $\eta < 1$ such that

$$P_B^1 1(x) \leq \eta < 1 \quad \text{for all } x \in B.$$

Using Theorem 11.4 p.62 of [1] it is seen that the successive hitting times to a totally thin set must increase to infinity almost surely.

Lemma 2. Let A be as in Lemma 1. Assume the process is transient. If A is totally thin then A is empty.

Proof. A being relatively compact the last exit time L from A is finite almost surely. But A being totally thin the successive hitting times tend to infinity almost surely. But by (1) for $x \in A$ all successive hitting times to A are finite almost surely. Since all these are less or equal to L, transience is violated. The Lemma follows.

Theorem 3. Assume transience and hypothesis (L). If for all compact K and all open G containing K

$$(2) \qquad P_G P_K 1 = P_K 1$$

then $P_G P_K = P_K$, namely hypothesis (B) holds.

Proof. The arguments of p.p. 70-71 of [2] show that for the validity of hypothesis (B) it is sufficient to prove that for each totally thin set K contained in an open set G we have for each x

$$(3) \qquad P^x(T_G = T_K,\ T_G < \infty) = 0.$$

Using the notation above we now show that on the set $(T_G = T_K < \infty)$

we have

(4) $$X_{T_G} \in K_\gamma \text{ for every } \gamma \text{ countable ordinal.}$$

This is trivial if $\gamma = 0$. Assuming (4) is valid for a particular γ. On the set $(T_G = T_K < \infty)$ we have $T_G = T_K$.

By (2) with $K = K_\gamma$

(5) $$P_G P_{K_\gamma} 1(x) = P_{K_\gamma} 1(x).$$

From (5) we deduce

$$E^x[P_{K_\gamma} 1(X_{T_G}), T_G = T_{K_\gamma} < \infty]$$

$$= P^x[T_G = T_{K_\gamma} < \infty]$$

which implies that $X_{T_G} \in K_{\gamma+1}$ on $T_G = T_K < \infty$.

Next if γ is a limit ordinal, $X_{T_G} \in K_\beta$ for $\beta < \gamma$, trivially implies $X_{T_G} \in K_\gamma$. Thus $X_{T_G} \in A$. But by Lemma 2 this set is empty.

The proof is complete.

Complements

The assumptions will be as above.

Theorem 4. Let K denote a thin Borel set. Then the set

(6) $$B = \{P_K 1 = 1\}$$

is a finely open and closed Borel set. In particular it has positive ξ-measure unless it is empty. ξ is an exessive reference measure.

Proof. B is Borel and finely closed by definition. Since does not have regular points, it is sufficient to show that or all $x \in B$,

(7) $$P^x[X_t \in B \text{ for all } 0 < t < T_K] = 1.$$

ut

$$s = P_K 1.$$

hen $x \notin B$ iff $s(x) < 1$. In other words

$$B^c = \bigcup_n A_n, \quad A_n = (s \le 1 - \tfrac{1}{n}).$$

(7) follows if we show

(8) $$P^x[T_n < T] = 0, \quad x \in B$$

where $T_n = T_{A_n}$ and $T = T_K$.

But by strong Markov property and the fact that $s(x) = 1$ for $x \in B$ we have

$$P^x[T_n < T] = E^x[s(X_{T_n}), T_n < T]$$

$$\le (1 - \tfrac{1}{n}) P_x[T_n < T < \infty]$$

because A_n being finely closed, X_{T_n} belongs to A_n. That completes the proof.

If B and K are as above and B is not empty, it is intuitively clear that the last exit from K is at least as large as the last exit time from B. Let us supply a proof. Since B is finely open it is clear that the last exit time L from B satisfies

$$L = \sup(t > 0,\ t \in Q,\ x_t \in B)$$

where Q denotes the set of rationals. Write

$$A = ((t,w): X_t \in B \text{ and } t \in Q).$$

A is optional with countable sections. There exists stopping times T_n with disjoint graphs $[T_n]$ such that

$$A = \bigcup_n [T_n].$$

For every x, M denoting the last exit from K

$$P^x(M \geq T_n,\ T_n < \infty) \geq P^x[T_n + T_K(\theta_{T_n}) < \infty]$$

$$= E^x[P_K 1(X_{T_n}),\ T_n < \infty] = P^x[T_n < \infty]$$

namely $M \geq T_n$ on the set $T_n < \infty$, P^x - a.s.

That completes the proof.

References

[1] R.M.Blumenthal and R.K.Getoor: Markov Processes and Potential theory. Academic Press (1968).

[2] P.A.Meyer: Processes du Markov et la Frontiere du Martin. Springer Lecture Notes Vol 77 (1968).