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# PATHWISE DIFFERENTIABILITY WITH RESPECT TO A PARAMETER OF SOLUTIONS OF STOCHASTIC DIFFERENTIAL EQUATIONS

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*Abstract*

*We consider a stochastic differential equation*

$$X^u(t) = V^u(t) + \int_0^t \sigma(u,s,X^u_{s^-})dS_s + \int_0^t f(u,s,X^u_{s^-},x)\ q(ds,dx)$$

*where* S *is a semimartingale and* q *a random measure and where the "coefficients" depend on a parameter* u . *We prove under suitable differentiability-conditions that the solution* $X^u(t,\omega)$ *can be choosen for each* u *in such a way that the mapping* $u \rightsquigarrow X^u(t,\omega)$ *is continuously differentiable for every* $(t,\omega)$ .

## I - INTRODUCTION

The goal of this paper is to prove that under sufficient differentiability conditions on the coefficients, stochastic differential equations of the type

$$(1.1)\quad X^u(t) = V^u(t) + \int_0^t \sigma(u,s,X^u_{s^-})dS_s + \int_0^t f(u,s,X^u_{s^-},x)\ q(ds,dx)$$

where S is a semimartingale, q a random measure with zero dual predictable projection and u a parameter taking its values in a bounded open subset G of $\mathbb{R}^d$ , admit for each u a solution which can be determined in such a way that P.a.s. the functions $u \rightsquigarrow X^u(t,\omega)$ are for every t continuously differentiable.

This is a concept of differentiability different from the one considered by Sikhmann (see [ 3 ] and [ 4 ]), who studied the differentiability of the mapping $u \rightsquigarrow X^u_t(.)$ as a mapping from G into $L^p(\Omega)$ for some p and in the framework of Ito-equations. Recently Bichteler took the same point of view and considered equations of the type (1.1) with $q = u$ and $S$ and $X^u$ possibly infinite dimensional. J. Jacod in [6] considered differentiability "in probability".

Pathwise differentiability was considered by P. Malliavin and M. Bismut for the solutions of Ito-Stratonovitch equation as functions of the initial conditions (see [ 2 ] and [ 8 ]). In [ 7 ] H. Kunita proved pathwise differentiability with respect to the initial conditions for the solutions of an equation driven by a continuous martingale. In [11] P.A. Meyer proved the same result for equations driven by a semimartingale (equations of Doleans-Dade-Protter type).

We consider here equations of type (1.1) and of a more general type with coefficients depending on a parameter $u$ .

In section II we recall a few facts on the type of equations which are studied here. In section III we give sufficient conditions for the continuity of solutions with respect to $u$ and in section IV we deal with differentiability.

## II - THE EQUATION UNDER CONSIDERATION

### 2.1. - Inequalities for stochastic integrals

We assume that the random measure $q$ in (1.1) is of the form $\mu(\omega;ds;du) - \nu(\omega;ds;du)$ where $\mu(\omega;]0,t],du)$ is for each $\omega$ and $t$ a borelian measure in an open subset $\mathbb{E}$ of $\mathbb{R}^m - \{0\}$ such that for some $\alpha > 0$ $\int \frac{|x|^\alpha}{1+|x|^\alpha} |\mu|(\omega;]0,t],du) < \infty$ ($|\mu|$ denotes the variation of $\mu$ and $\alpha$ does not depend on $\omega$ and $t$ ) and where $\nu$ is the dual predictable projection of $\mu$ ).

$\mathbb{H}$ denotes a separable Hilbert space. We have shown in [ 9 ] (see also J. Jacod [ 5 ]) the existence of an increasing positive adapted process $b$ and of a process $\{\overset{o}{q}(\omega,s,.) : (\omega,s) \in \Omega \times \mathbb{R}^+\}$ the values of which are measures on $\mathbb{E} \times \mathbb{E}$ such that :

i) For each $\mathbb{H}$-valued function $h$ on $\mathbb{E}$ such that $\langle h(x),h(y)\rangle_{\mathbb{H}}$ is $\overset{o}{q}(\omega,s,dx \otimes dy)$ integrable, the integral $\int \langle h(x),h(y)\rangle_{\mathbb{H}} \overset{o}{q}(\omega,s,dx \otimes dy)$ defines a positive optional process ;

ii) If Y is an $\mathbb{H}$-valued $\mathscr{P}\otimes\mathscr{B}_{\mathbb{E}}$ measurable(*) function on $\mathbb{R}^+\times\Omega\times\mathbb{E}$ and if we denote by $\lambda_s(Y)$ the $\mathbb{H}$-valued positive random variable

$$\lambda_s(Y) := \int \langle Y(s,\ ,x),\ Y(s,\ ,y)\rangle_{\mathbb{H}}\ \overset{o}{q}(.,s,dx\otimes dy)$$

(set to be equal to $+\infty$ when the integral does not exist) and

iii) the following inequality holds for every stopping time $\tau$

$$(2.1)\qquad E\left(\sup_{t<\tau}\left\|\int_{]0,t]\times\mathbb{E}} Y(s,.x)q(.,ds,dx)\right\|^2\right) \leq 4\ E\left(\int_{[0,\tau[} \lambda_s(Y)db_s\right)$$

where $\left(\int_{]0,t]\times\mathbb{E}} Y(s,.,x)q(.,ds,dx)\right)_{t\geq 0}$ is the stochastic integral process of Y with respect to q which is defined as soon as the process $\left(\int_{]0,t]} \lambda_s(Y)db(s)\right)_{t\geq 0}$ is finite.

If S is a $\mathbb{K}$-valued ( $\mathbb{K}$ : separable Hilbert space) right continuous semimartingale we know that there exist two positive increasing adapted processes a and $\tilde{a}$ such that for every $\mathcal{L}(\mathbb{K};\mathbb{H})$-valued locally bounded predictable process $\{f(s,\omega);(s,\omega)\in\mathbb{R}^+\times\Omega\}$ and every stopping time $\tau$ :

$$(2.2)\qquad E\left(\sup_{t<\tau}\left\|\int f(s,.)dS_s\right\|^2\right) \leq E\left(\tilde{a}_{\tau^-}\cdot\int_{[0,\tau[}\|f(s)\|^2da(s)\right)$$

To simplify the writing we shall call $Z_t$ the process $Z_t := (S_t,q(.,]0,t],dx))$ which takes its values in $(\mathcal{L}(\mathbb{K};\mathbb{H})\times\mathcal{M}^\alpha)$ where $\mathcal{M}^\alpha$ is the space of borelian measures $\nu$ on $\mathbb{E}$ such that $\int_{\mathbb{E}}\frac{|x|^\alpha}{1+|x|^\alpha}\,|\nu|\,(du) < \infty$ .

Setting $A_t := b(t) + a(t)$ $\tilde{A}_t := 8 + 2\tilde{a}_t$ $\Phi := (f,Y)$

$$(2.3)\qquad \int_{]0,t]}\Phi(s)dZ_s := \int_{]0,t]} f(s,.)dS_s + \int_{]0,t]\times\mathbb{E}} Y(s,.,x)q(.,ds,dx)$$

and

$$(2.4)\qquad \lambda_s(\Phi) := \|f(s,.)\|^2 + \lambda_s(Y)$$

the following inequality holds for every stopping time

$$(2.5)\qquad E\left(\sup_{t<\tau}\left\|\int_{]0,t]}\Phi(s)dZ_s\right\|^2\right) \leq E\left(\tilde{A}_{\tau^-}\cdot\int_{]0,\tau]}\lambda_s(\Phi)dA_s\right)$$

(*) $\mathscr{P}$ is the $\sigma$-algebra of predictable subsets of $\mathbb{R}^+\times\Omega$ and $\mathscr{B}_{\mathbb{E}}$ of Borel subsets of $\mathbb{E}$.

Extending a classical argument on martingales (see [13]) it is also easy to see that for every $p \geq 2$ exists an increasing positive adapted process $(\tilde{A}^p_t)_{t\geq 0}$ such that for every stopping $\tau$

$$(2.6) \qquad E\left(\sup_{t<\tau} \left\| \int_{]0,t]} \Phi(s)dZ_s \right\|^p\right) \leq E\left(\tilde{A}^p_{\tau^-} \cdot \int_{[0,\tau[} \left(\lambda_s(\Phi)\right)^{p/2} dA_s\right)$$

2.2. - Hypothesis on equation (1.1)

The space of parameters $u$ is an open bounded subset $G$ of $\mathbb{R}^d$ .

In equation (1.1) $\sigma$ is a mapping from $(G \times \mathbb{R}^+ \times \Omega \times \mathbb{H})$ into $\mathcal{L}(\mathbb{K};\mathbb{H})$ which is continuous on $\mathbb{H}$ and such that for every $h \in \mathbb{H}$ and $u \in G$ the process $\{\sigma(u,s,\omega,h) : (s,\omega) \in \mathbb{R}^+ \times \Omega\}$ is predictable. $f$ is a mapping of $(G\times\mathbb{R}^+\times\Omega\times\mathbb{H}, \mathbb{E})$ into $\mathbb{H}$ which if continuous on $\mathbb{H}$ and such that for every $u \in G$ , $h \in \mathbb{H}$ the mapping $(s,\omega,x) \rightsquigarrow f(u,s,\omega,h,x)$ is $\mathcal{P}\otimes\mathcal{B}_{\mathbb{E}}$ measurable

In the sequel we shall call $g$ the couple $(\sigma,f)$ and according to the notations of (2.1) the equation (1.1) will be written in the *abreviated form* :

$$(2.7) \qquad X^u(t) = V^u(t) + \int_0^t g(u,s,X^u_{s^-})dZ_s$$

Here $V^u$ is for each $u \in G$ a given $\mathbb{H}$-valued adapted cad-lag process.

III - CONTINUITY OF THE SOLUTIONS WITH RESPECT TO u .

3.1. - Hypothesis

$L$ is an increasing positive adapted process and $p$ is a positive real number with $p \geq d + \varepsilon$ for some $\varepsilon > 0$ .

If $\xi$ is a cad-lag $\mathbb{H}$-valued adapted process we write $g(u,\xi)$ for the process $(t,\omega) \rightsquigarrow g(u,s,\omega,\xi_{s^-}(\omega))$ and $\lambda_s \circ g(u,\xi)$ for the positive functional of this process defined by formula (2.4).

With these notations we formulate the following hypotheses :

$(H_1)$ $\sup_{s\leq t} \|V_s^u - V_s^v\| \leq L_t \|u-v\|$ for all $t$ , $u$ and $v \in G$

and

$\sup_{u\in G}\ \sup_{s<t} \|V_t^u\| < \infty$

$(H_2)$ (Lipschitz hypotheses) :

$$\forall t\in\mathbb{R}^+ \quad \int_{]0,t]} [\lambda_s o (g(u,\xi)-g(u,\xi'))]^{p/2} dA_s \leq \int_{]0,t]} \sup_{r\leq s} \|\xi_r - \xi'_r\|^p dL_s$$

for every couple $(\xi,\xi')$ of $\mathbb{H}$-valued adapted cad-lag processes, $\mathbb{P}$.a.s.

$(H_3)$ $$\int_{]0,t]} [\lambda_s \ o\ g(u,\xi)]^{p/2} dA_s \leq \int_{]0,t]} (1 + \sup_{r\leq s} \|\xi_s\|^p) dL_s$$

for every $u \in G$ every $\mathbb{H}$-valued adapted cal-lag $\xi$ , P.a.s.
(Note that $(H_3)$ is implied by $(H_2)$ in most classical cases).

$(H_4)$ $\Psi$ being a given positive increasing (possibly constant) function on $\mathbb{R}^+$ , for every stopping time $\tau$ the following inequality holds for every $\mathbb{H}$-valued cad-lag adapted $\xi$ every $u$ and $v$ in $G$ :

$$E\left(\sup_{t<\tau^-}\left[\lambda_t o\ [g(u,\xi)-g(v,\xi)]\right]^{p/2}\right) \leq \|u-v\|^{d+\varepsilon}\Psi\left(E(\sup_{t<\tau}\|\xi_t\|^p)\right)$$

3.2. - Theorem

*1°) Under the above hypotheses* $(H_1)$ *to* $(H_4)$ *, the equation* (2.7) *has for each* $u$ *a unique strong solution* $X^u$ *on* $\mathbb{R}^+$ *and the random function* $(t,\omega,u) \rightsquigarrow X_t^u(\omega)$ *can be determined in such a way that* $u \rightsquigarrow X_t^u(\omega)$ *is continuous on* $G$ *for every* $t$ *and* $\omega$ *while the mapping* $t \rightsquigarrow X_t^{(\cdot)}(\omega)$ *is for each* $\omega$ *cad-lag from* $\mathbb{R}^+$ *into the set* $C_b^{\mathbb{H}}(G)$ *of bounded continuous* $\mathbb{H}$*-valued functions on* $G$ *endowed with the uniform topology.*

*2°) There exists an increasing sequence* $(\sigma_n)$ *of stopping times and constants* $K(\Psi,n,p,Z)$ *such that*

*a)* $\lim_n P\{\sigma_n < T\} = 0$ *for every* $T > 0$

*b)* $E\left(\sup_{t<\sigma_n} \|X^u(t) - X^v(t)\|^p\right) \leq K(Y,n,p,Z)\ \|u-v\|^p$

Proof.

The stopping times $\sigma_n$ are defined as follows :

$$\sigma_n := \inf \{t : \widetilde{A}^p_t \vee L_t \vee \sup_{\substack{u\in G \\ s\leq t}} \|V^u_t\|^p \vee A_t > n\}$$

Next we have the following lemmas

3.3. - Lemma 1

$$E\left(\sup_{t<\sigma_n} \|X^u_t\|^p\right) \leq 2^p(n+n^2) \sum_{j=0}^{2^P n^2} (2^P n^2)^j$$

Proof of Lemma 1

We remark that $A^p_{\sigma_n^-} \leq n$, $L_{\sigma_n^-} \leq n$, $\sup_{t<\sigma_n} \sup_u \|V^u_t\|^p \leq n$

We then apply inequality (2.6) to the second member of (2.7) and get

$$E\left(\sup_{t<\sigma_n} \|\mathbf{X}^u_t\|^p\right) \leq 2^{(P-1)}\ n+2^{(P-1)}\ E\left(\widetilde{A}^p_{\sigma_n^-} \int_{]0,\sigma_n[} [\lambda_s \circ g(u,\mathbf{X}^u)]^{P/2}\ dA_s\right)$$

and property $(H_3)$ gives for every stopping time $\tau \leq \sigma_n$

$$E\left(\sup_{t<\sigma_n} \|\mathbf{X}^u_t\|^p\right) \leq 2^{(P-1)}(n+n^2) + 2^{(P-1)}\ n\ E\left(\int_{]0,\tau[} (\sup_{s<t} \|\mathbf{X}^u_s\|^P)dL_s\right)$$

Applying the "Gronwall stochastic lemma" as in [10] section 7.1 we get the inequality of the lemma.

3.4. - Lemma 2

There exist constants $K(\Psi,n,p,A,\widetilde{A}^p)$ such that

$$\forall\ u,v \quad E\left(\sup_{t<\sigma_n} \|X^u_t - X^v_t\|^p\right) \leq K(\Psi,n,p,A,\widetilde{A}^p)\ \|u-v\|^p$$

Proof of Lemma 2

Applying again inequality (2.6) to the stochastic integrals
$\int_{]0,t]} (g_s(u,X^u_{s^-}) - g_s(v,X^u_{s^-}))dZ_s$ and

$\int_{]0,t]} [g_s(v,X^u_{s^-}) - g_s(v,X^v_{s^-})]dZ_s$

and using properties $(H_1)$, $(H_2)$ and $(H_4)$ we can write for every stopping time $\tau \leqslant \sigma_n$ :

$$E\left(\sup_{s<\tau} \|X^u(s)-X^v(s)\|^p\right) \leqslant 3^{P-1}n^P\|u-v\|^p + 3^{(P-1)}n\ \Psi\left(E\left(\sup_{s<\tau}\|X^u_s\|^p\right)\right)$$
$$+ 3^{(P-1)}n\ E\left(\int_{]0,\tau[} (\sup_{t<s}\|X^u(s)-X^v(s)\|^P)\,dL_s\right)$$

Applying as above the same "Gronwall-inequality" we obtain the lemma.

Theorem 3.2 is now a direct consequence of the following lemma which is a straight-forward extension of a lemma as stated by Neveu in [12] (see also P. Priouret [13] chap. 3. lemme 13 :

3.5 - Lemma 3

*Let* $\{Y(t,\omega,u) : t \in \mathbb{R}^+,\ \omega \in \Omega,\ u \in G\}$ *an* $\mathbb{H}$*-valued random function such that for every* $u : t \rightsquigarrow Y(t,\omega,u)$ *is a.s. cad-lag and such that for every* $t$ *:*

$$E\left(\sup_{s\leqslant t}\|Y_{s,u} - Y_{s,v}\|^p\right) \leqslant a_{t,p}\ \|u-v\|^{d+\varepsilon}$$

*Then there exists a mapping* $Y^* : (t,\omega,u) \rightsquigarrow Y^*(t,\omega,u) \in \mathbb{H}$ *such that*

*a)* $u \rightsquigarrow Y^*(t,\omega,u)$ *is continuous*

*b)* $\forall\, u \in G$ , $Y(t,u,.) = Y^*(t,u,.)$ *for all* $t$ *a.s.*

*b)* $t \rightsquigarrow Y^*(t,.,\omega)$ *is for* P*-almost all* $\omega$ *a cad-lag mapping from* $\mathbb{R}^+$ *into* $C_b^{\mathbb{H}}(G)$ *endowed with the uniform topology.*

Proof.

We omit the proof which is pretty similar to the one given in [13].

This finishes the proof of theorem 3.2. ■

## IV - PATHWISE DIFFERENTIABILITY

### 4.1. - Hypothesis

We consider the same equation (1.1) or in abreviated notation : (2.7).

For a couple $g := (\sigma,f)$ of "coefficients" as in (1.1) we write to simplify :

$$\|g(u,s,\omega,h,.)\|_{\Lambda} := \Big[\|\sigma(u,s,\omega,h)\|^2_{\mathcal{L}(\mathbb{K};\mathbb{H})} + \int_{\mathbb{E}\times\mathbb{E}} \langle f(u,s,\omega,h,x),f(u,s,\omega,h,y)\rangle_{\mathbb{H}}\ \overset{\circ}{q}(\omega,s,dx\otimes dy)\Big]^{\frac{1}{2}}$$

We set $v^*_t := \sup_{u\in G}\sup_{s<t} \|D_uV^u_s\| + \|V^u_s\| + \|D^2_{u^2}V^u_s\|$

were $D_u\Phi$ denotes the first order derivative and $D^2_{u^2}\Phi$ the second order derivative with respect to $u$ of a function $\Phi$ on $u$.

In the hypothesis below $C$ is a constant and $(K_t)_{t\geq 0}$ is an increasing positive process.

[$D_1$] For all $t$ and $\omega$ the derivatives $D_uV^u(t,\omega)$ and $D^2_{u^2}V^u(t,\omega)$ exist and $v^*_t < \infty$

[$D_2$] The derivatives $D_ug(s,u,x)$ $D_u\,g(s,u,x)$ $D_{u,x}g(s,u,x)$ and $D_xg(s,u,x)$ exist and

$$\sup_{u,s,x} (\|D_ug(s,u,x)\|_\Lambda + \|D^2_{u^2}g(s,u,x)\|_\Lambda + \|D^2_{ux}g(s,u,x)\|_\Lambda + \|D_xg(s,u,x)\|_\Lambda) \leq C$$

[$D_3$] For all $x,y$ $u$ and $v$ :

$$\|D_xg(s,u,x) - D_xg(s,v,y)\|_\Lambda \leq C(\|y-x\| + \|u-v\|)$$

### 4.2. - Theorem

*Under the above hypothesis* [$D_1$] *to* [$D_3$] *equation* (2.7) *has a unique (up to indistinguability) solution* $X^u$ *on* $\mathbb{R}^+$ *and there exists a version* $(\omega,t,u) \rightsquigarrow X^u_t(\omega)$ *of this random function such that for* P*-almost all* $\omega$ *:*

*a)* $u \rightsquigarrow X^u_t(\omega)$ *is continuously differentiable for every* $t$

*b)* $t \rightsquigarrow X^{(.)}_t(\omega)$ *and* $t \rightsquigarrow D_uX^{(.)}_t(\omega)$ *are cad-lag for the uniform norm on* $C_b(G\,;\mathbb{H})$ *and* $C_b(G\,;\mathcal{L}(G\,;\mathbb{H}))$ *respectively.*

*c) For every* $u$ *the stochastic process* $(D_uX^u_t)_{t\geq 0}$ *is a strong solution of the following stochastic equation (where* $X^u$ *is the process solution of* 2.7 *as in theorem* 3.2*) :*

$$(4.1)\qquad Y^u(t) = D_uV^u_t + \int_{]0,t]}\Big(D_ug(s,u,X^u_{s^-}) + D_xg(s,u,X^u_{s^-}) \circ Y^u_s\Big)dZ_s$$

Proof.

The proof is in several steps corresponding to lemmas 4 and 5 and section 4.5 bellow :

4.3. - Lemma 4

*Under hypothesis* $[D_1]$, $[D_2]$, $[D_3]$, *equations* (2.7) *and* (4.1) *satisfy the conditions* $[H_1]$ *to* $[H_4]$ *of section* 3.1 *for every* $p \geq 2$ *on any interval* $]0,\sigma_n]$ *as defined in theorem* 1.

Proof.

Let us first consider equation (2.7). $(H_1)$ is trivially implied by $[D_1]$. $[D_2]$ implies also the Lipschitz property $(H_2)$ and conditions $(H_3)$ and $(H_4)$ which is here expressed in the much stronger form $\| g(s,u,x)-g(s,v,x)\|_\Lambda \leq C \| u-v\|$.

We turn now to equation (4.1). The only condition $(H_i)$ which is not immediately implied by the hypothesis of the lemma is condition $(H_4)$. We write

$$\| D_u g(s,v,X^v_{t-}) - D_u g(s,u,X^u_{t-}) + D_x g(s,v,X^v_{t-}) \circ \xi_t - D_x g(s,u,X^u_{t-}) \circ \xi_{t-}\|^p_\Lambda$$

$$\leq 4^{p-1} \{\| D_u g(s,v,X^v_{t-}) - D_u g(s,u,X^v_{t-})\|^p_\Lambda\} +$$

$$+ 4^{p-1} \{\| D_u g(s,u,X^v_{t-}) - D_u g(s,u,X^u_{t-})\|^p_\Lambda\}$$

$$+ 4^{p-1} \{\| [D_x g(s,v,X^v_{t-}) - D_x g(s,u,X^v_{t-})] \circ \xi_{t-}\|^p_\Lambda\}$$

$$+ 4^{p-1} \{\| [D_x g(s,u,X^v_{t-}) - D_x g(s,u,X^u_{t-})] \circ \xi_{t-}\|^p_\Lambda\}$$

$$\leq 4^{p-1} c^p (\| u-v\|^p + \| X^v_{t-} - X^u_{t-}\|^p +$$

$$+ 4^{p-1} c^p \| u-v\|^p \|\xi_{t-}\|^p + 4^{p-1} c^p \| (X^v_{t-} - X^u_{t-}) \circ \xi_{t-}\|^p$$

One knows from proposition 2 that there exists an increasing sequence $(\sigma_n)$ of stopping times and constants $C_n$ such that

$$E \sup_{s<\sigma_n} \| Y^u(s) - Y^v(s)\|^{2p} \leq C_n \| u-v\|^{2p}$$

If we write for every stopping time $\tau$

$$E\Big(\sup_{t<\tau\wedge\sigma_n} \| (X^v_t - X^u_t) \circ \xi_{t-}\|^p\Big) \leq$$

$$\left[E\left(\sup_{t<\tau\wedge\sigma_n} \| X_t^v - X_t^u \|^{2p}\right)\right]^{\frac{1}{2}} \left[E\left(\sup_{t<\tau\wedge\sigma_n} \| \xi_t \|^{\frac{2p}{2p-1}}\right)\right]^{\frac{2p-1}{2}}$$

$$\leqslant C_n^{\frac{1}{2}} \| u-v\|^p \; E\left(\sup_{t<\tau\wedge\sigma_n} \| \xi_t \|^{\alpha}\right)^{p/\alpha}$$

with $\alpha = \frac{2p}{2p-1}$

Therefore

$$E\left(\sup_{s<\tau\wedge\sigma_n} \| g(s,u,\xi_{s^-}) - g(s,v,\xi_{s^-})\|_{\Lambda}^{p}\right) \leqslant 4^{p,1} \; C^p \| u-v\|^p [1+C_n + E(\sup_{s<\tau\wedge\sigma_n} \| \xi_s \|^p)]$$

$$+ C_n^{\frac{1}{2}} [E(\sup_{t<\tau\wedge\sigma_n} \| \xi_t \|^{\alpha})]^{p/\alpha}$$

If we remark that $E(\sup_{t<\tau\wedge\sigma_n} \| \xi_{t^-} \|^p) \geqslant [E(\sup_{t<\tau\wedge\sigma_n} \| \xi_{t^-} \|^{\alpha})]^{p/\alpha}$

we see that property $(H_4)$ holds with

$$\Psi(\rho) = 1 + C_n + (1 + C_n^{\frac{1}{2}})\rho$$

4.4. - Lemma 2

*If we define*

$$\Phi_t(e,u,\lambda) = \frac{1}{\lambda} [X_t^{u+\lambda e} - X_t^u - \lambda Y_t^u \circ e]$$

*there exists an increasing sequence* $(\tau_n)$ *of stopping times such that* $\lim_n P\{\tau_u < T\} = 0$ *and a sequence* $C_n$ *of constants such that*

$$E\left\{\sup_{t<\tau_n} \| \Phi_t(e,.,\lambda)\|_{L^2(G)}^2\right\} \leqslant C_n \lambda^2$$

Proof.

For each $u$ the process $(\Phi_t(e,u,\lambda))_{t\leqslant T}$ is solution of

$$(4.2) \quad \Phi_t(e,u,\lambda) = \frac{1}{\lambda} (V_t^{u+\lambda e} - V_t^u - \lambda D_e V_t^u) +$$

$$+ \int_{]0,t]} \frac{1}{\lambda} \Big[ g(s,u+\lambda e, X_{s^-}^{u+\lambda e}) - g(s,u,X_{s^-}^u) -$$

$$\lambda D_e g(s,u,X_{s^-}^u) - \lambda D_x g(s,u,X_{s^-}^u) \circ Y_{s^-}^u \circ e\Big] dS_s$$

We may write for $x,y \in \mathbb{H}$ and $\eta \in \mathcal{L}(\mathbb{H};\mathbb{H})$

$$(4.3)\quad g(s,u+\lambda e,y) - g(s,u,x) - \lambda D_e g(s,u,x) - \lambda D_x g(s,u,x) \circ \eta \circ e =$$

$$\lambda D_e g(s,u,y) + D_x g(s,u,x) \circ (y-x) - \lambda D_e g(s,u,x) - \lambda D_x g(s,u,x) \circ \eta \circ e +$$

$$+ h(s,u,x,y,\eta,\lambda,e)$$

$$= D_x g(s,u,x) \circ (y-x-\lambda\eta \circ e) + \widetilde{h}(s,u,x,y,\eta,\lambda)$$

with

$$(4.4)\quad \|\widetilde{h}(s,u,x,y,\eta,\lambda)_\Lambda \leqslant |\lambda|\ K\ (\|y-x\| + |\lambda|)$$

for some constant $K$

The equation (4.2) can therefore be written

$$(4.5)\quad \Phi_t(e,u\lambda) = H_t(u,\lambda,e) + \int_{]0,t]} D_x g(s,u,X^u_{s^-}) \circ \Phi_{s^-}(e,u,\lambda)\, dZ_s$$

where the process $H(u,\lambda,e)$ satisfies

$$(4.6)\quad \|H_t(u,\lambda,e)\|_{\mathbb{H}} \leqslant |\lambda|\ \ v^k_t + \left\| \int_{]0,t]} \frac{1}{\lambda}\, h(s,u,X^{u+\lambda e}_{s^-}, X^u_{s^-}, Y^u_{s^-} \circ e)\, dZ_s \right\|$$

Using (4.5) we obtain from (4.6) for every stopping time $\sigma$ :

$$E\left(\sup_{t<\sigma} \|H_t(u,\lambda,e)\|^2\right) \leqslant 2\lambda^2 v^*_{\sigma^-} + E\left(\widetilde{A}_{\tau^-} \cdot \int_{]0,\tau[} [\lambda^2 + C^2\|X^{u+\lambda e}_{s^-} - X^u_s\|^2]\, dA_s\right)$$

Using then theorem we see that there exists a sequence $(\sigma_n)$ of stopping times and a sequence of constants $(K_n)$ such that

$$(4.7)\quad \sup_{s<\sigma_n} (\widetilde{A}_s \vee A_s) \leqslant n \quad \text{and}$$

$$(4.8)\quad E(\sup_{t<\sigma_n} \|H_t(u,\lambda,e)\|^2) \leqslant K_n \lambda^2$$ (use a standard stopping procedure for processes $v^*, \widetilde{A}$ and $A$ ).

This implies

$$(4.9)\qquad E\Big(\sup_{t<\sigma_n}\int_G \|H_t(u,\lambda,e)\|^2\, du\Big) \leq \int_G K_n\, \lambda^2\, du \leq \tilde{K}_n\, \lambda^2$$

We next consider the $L^2(G)$-valued process $(\Phi_t(e,.,\lambda))_{t\leq T}$

As $D_x g$ is bounded by some constant $C$, inequality (4.6) shows that the $L^2(G)$-valued process $\Phi_t$ satisfies an inequality of the following type for every stopping time $\tau \leq \sigma_n$

$$E\Big\{\sup_{t<\tau}\|\Phi_t(e,.,\lambda)\|_{L^2(G)}\Big\} \leq 2\,\tilde{K}_n\,\lambda^2 + 2\,E\Big(\tilde{A}_\tau - \int_{[0,\tau[} C^2 \sup_{s<t}\|\Phi_s(e,.,\lambda)\|^2_{L^2(G)}\, dA_s\Big)$$

$$\leq 2\,\tilde{K}_n\,\lambda^2 + 2n\,C^2\int_{[0,\tau[}\sup_{s<t}\|\Phi_s(e,.,\lambda)\|_{L^2(G)}\, dA_s$$

The already used "Gronwall inequality" of [10] shows immediately the existence of a constant $C_n$ as in the lemma.

4.5. - End of the proof of the theorem

We make use of the following easily proved property : let $f \in L^2_{\mathbb{H}}(\bar{G})$ let $f \in L^2(\mathbb{G}\,;\mathbb{H}) \cap C(G\,;\mathbb{H})$ and $\bar{f} \in L^2(G\,;\mathcal{L}(\mathbb{H}\,;\mathbb{H}) \cap C(G\,;\mathcal{L}(\mathbb{H}\,;\mathbb{H}))$ such that for all $e \in \mathbb{R}^d$, all $u \in \mathbb{R}^d$ and some decreasing sequence $\lambda_k \downarrow 0$ :

$$\lim_{k\to\infty}\|f(u+\lambda_k e) - f(u) - \lambda_k\,\bar{f}(u)\circ e\|_{L^2(G\,;\mathbb{H})} = 0$$

Then $\bar{f}$ is the derivative of $f$ in the sense of distributions and therefore in the ordinary sense in every point $u \in G$. Let us consider for each $\omega$ and $n$ a P-negligeable set $\Omega_n$ and a sequence $\lambda_k$ such that $\lambda_k \downarrow 0$ and

$$\lim_{k\to\infty}\sup_{t<\tau_n(\omega)}\|\Phi_t(e,.,\omega,\lambda_k)\|_{L^2(G)} = 0 \quad \text{for every } \omega \notin \Omega_n$$

The above property shows that for every $\omega \notin \Omega_n$ and $t < \tau_n(\omega)$ $Y^u_t(\omega)$ is the derivative of $u \rightsquigarrow X^u_t(\omega)$ at point $u$. Therefore $Y^u_t(\omega)$ is the derivative of $u \rightsquigarrow X^u_t(\omega)$ for all $t<\tau_n(\omega)$ and $\omega \notin (\bigcup_n \Omega_n)$.

This proves the theorem. ■

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