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A MARTINGALE APPROACH TO SOME WIENER-HOPF PROBLEMS, I,

by

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This is one of two companion papers. This paper, I, studies how certain of "Feller's Brownian motions on $[0,\infty)$" may be obtained from Brownian motion via time-substitutions based on fluctuating clocks. Paper II starts afresh with a look at time substitutions for symmetrizable Markov chains; and in that context it is possible to see rather more clearly what is going on. Much of the fascination of Wiener-Hopf theory lies in the difficulty of obtaining explicit answers in concrete cases. The second half of Paper II is a detailed analysis, partially motivated by our study of the chain case, of a concrete example of the problem discussed here in Paper I; and whether or not it makes good reading, it was fun to do.

1. Introduction and summary

1.1. Let $\{B_t : t \geq 0\}$ be a Brownian motion on $\mathbb{R}$ with $B_0 = 0$. Let $\{L_t(x) : t \geq 0,\ x \in \mathbb{R}\}$ denote the jointly continuous local-time process of B, normalised so that for each x,

$$|B_t - x| - L_t(x)$$

is a martingale. Hence L is twice the standard Brownian local time of Itô-McKean [4].

Let m be a measure on $(-\infty,0]$. [Note. 'Measure' always implies: 'taking values in $[0,\infty]$'.]

Define the additive functionals:

$$\text{(1a)} \qquad \phi_t^+ \equiv \int_0^t I_{(0,\infty)}(B_s)\,ds, \qquad \phi_t^- \equiv \int_{(-\infty,0]} L_t(x)\,m(dx),$$

$$\phi_t \equiv \phi_t^+ - \phi_t^- ,$$

the random time changes:

(1b) $$\sigma_t^+ \equiv \inf\{u:\phi_u^+ > t\}, \qquad \sigma_t^- = \inf\{u:\phi_u^- > t\},$$

$$\tau_t^+ \equiv \inf\{u:\phi_u > t\}, \qquad \tau_t^- = \inf\{u:\phi_u < -t\},$$

and the time-changed processes:

(1c) $$X_t^+ \equiv B(\sigma_t^+), \qquad X_t^- \equiv B(\sigma_t^-),$$

$$Y_t^+ \equiv B(\tau_t^+), \qquad Y_t^- \equiv B(\tau_t^-).$$

In these definitions, we make the usual conventions:

$$t \geq 0, \qquad \inf\emptyset \equiv \infty, \qquad B_\infty \equiv \partial \quad \text{(coffin state)}$$

and allow the usual notational switches $B(t) \equiv B_t$, etc..

Note that ϕ^- can be the most general continuous increasing additive functional which grows only when $B \in (-\infty,0]$. See §5.9 of Itô-McKean [4]. Notice however that we do not require m to be σ-finite; for example, we allow m to assign infinite mass to a singleton set $\{\xi\}$ with $\xi \leq 0$. Define:

$$a \equiv \inf\{u:m[u,0] < \infty\} \geq -\infty.$$

Then as far as the process Y^+ is concerned, the values of m in $(-\infty,a)$ are irrelevant; they come into play only when B has entered $(-\infty,a)$, but by that time, ϕ^- is infinite and Y^+ is dead. We therefore make the convention:

(2) <u>if</u> $a \equiv \inf\{u:m[u,0] < \infty\} > \infty$, <u>then</u>

$m[a,0] = \infty$,

$m(-\infty,a) = 0$ <u>if</u> $m(a,0) < \infty$,

$m(-\infty,a] = 0$ <u>if</u> $m(a,0) = \infty$.

We emphasize that throughout the whole paper, m is understood to satisfy convention (2).

The possibility that ϕ^- can jump to infinity requires us to specify the lifetimes $\zeta(X^-)$ of X^- and $\zeta(Y^-)$ of Y^- more precisely.
Let $\rho^- \equiv \inf\{t:\phi_t^- = \infty\}$. Then

$$(3) \qquad \zeta(X^-) \equiv \lim_{s\uparrow\rho^-} \phi_s^- , \qquad \zeta(Y^-) \equiv -\inf_{s<\rho^-} \phi_s .$$

1.2. We wish to study the law of the process Y^+. It is easy to show that

4(i) Y^+ is a strong Markov process with state-space $[0,\infty)$;

and it is clear that

4(ii) Y^+ behaves as a Brownian motion while inside the open interval $(0,\infty)$,

so that if G^+ is the infinitesimal generator of Y^+, then $G^+f = \frac{1}{2}f''$ within $(0,\infty)$.

The results 4(i) and 4(ii) exactly comprise the statement that Y^+ is a Feller Brownian motion in the sense of §5.7 of Itô-McKean [4]. Now the domain of the infinitesimal generator of an arbitrary Feller Brownian motion Z is specified by a side condition of the following type:

$$(5) \qquad p_1 f(0) - p_2 f'(0) + \tfrac{1}{2}p_3 f''(0) = \int_{(0,\infty)} [f(x) - f(0)]p_4(dx)$$

where p_1, p_2 and p_3 are nonnegative constants, p_4 is a measure on $(0,\infty)$ such that

$$(6) \qquad \int (1 - e^{-x})p_4(dx) < \infty,$$

and $\quad f'(0) = f'(0+), \quad f''(0) = f''(0+).$

Condition (5) must be non-trivial in that $(p_1,p_2,p_3,p_4) \neq (0,0,0,0)$. The law of Z is completely determined by the quadruple (p_1,p_2,p_3,p_4) of 'characteristics'. Moreover, the law of Z determines the quadruple of 'characteristics' provided we consider quadruples projectively (identifying two quadruples which are (strictly positive) scalar multiples of a fixed quadruple). The number p_1 corresponds to the killing rate at 0, p_2 to the continuous-exit rate at 0, p_3 to the degree of stickiness at 0, and p_4 is the Lévy kernel describing jumps from 0 back into $(0,\infty)$. Different normalisations of the quadruple of characteristics correspond to different normalisations of the 'local time' of Z at 0. (We have put 'local time' in quotes because Z may visit 0 only at a discrete set of times.)

1.3. One of the principal results of this paper is the following theorem.

(7) THEOREM. *Let Z be a Feller Brownian motion with characteristics (p_1,p_2,p_3,p_4). Then there exists a measure m on $(-\infty,0]$ such that Z is identical in law to the time-changed process Y^+ if and only if*

$$(8) \qquad p_3 = 0 \quad \text{and} \quad p_4(dx) = dx\int_{(0,\infty)} e^{-rx}J(dr)$$

for some measure J on $(0,\infty)$ satisfying the following obvious equivalent to (6):

$$(9) \qquad \int [r(r+1)]^{-1}J(dr) < \infty .$$

Moreover, the measure m is then uniquely determined by the law of Y^+ (equivalently by the triple (p_1,p_2,J)).

[Note. The probabilistic reason why $p_3 = 0$ for Y^+ is 'obvious', for we need only show that

$$\text{measure } \{t : Y_t^+ = 0\} = 0,$$

and this 'must' hold since ϕ^+ has 'no $L_t(0)$ component'. We shall give a proper (analytic) proof later.]

Of course, the 'abstract' statement of Theorem 7 needs to be complemented by the more interesting solution to the 'practical' problem: How does one make explicit the one-one correspondence between measures m and triples (p_1, p_2, J) (considered projectively)? The solution is described in §1.7 after we have introduced the necessary terminology.

1.4. Our basic method is the 'martingale-problem' approach to this type of problem employed in Barlow-Rogers-Williams [1] and Rogers-Williams [6].

For each $\theta > 0$, we find a bounded function f_θ on $\mathbb{R}$ such that

(10) $\quad M_t^\theta \equiv \exp(\tfrac{1}{2}\theta^2\phi_t) f_\theta(B_t)$ defines a martingale M^θ.

Since M^θ is bounded on each interval of the form $[0, \tau_t^+]$, we may apply the optional-sampling theorem to deduce that

$$\exp(\tfrac{1}{2}\theta^2 t) f_\theta(Y_t^+) \text{ is a martingale,}$$

whence, with G^+ again denoting the infinitesimal generator of Y^+, we have

(11) $\quad f_\theta \in \mathcal{D}(G^+) \quad (\text{and } G^+ f_\theta = -\tfrac{1}{2}\theta^2 f_\theta).$

Our hope is that on feeding the information (11) into formula (5), we can determine the characteristics (p_1, p_2, p_3, p_4) of Y^+; and this proves to be justified.

Note. We need to be rather careful in checking the validity of the above application of the optional-sampling theorem because of the possibility that $\tau_t^+ = \infty$. Now, of course, $f_\theta(\partial) = 0$, by the usual convention. So the essential thing to prove is that (except on a null set of ω)

$$[-\infty < \phi_u < t\ (\forall u)] \implies [\lim_{u\to\infty} M_u^\theta = 0].$$

But it is easy to show that if $-\infty < \phi_u < t\ (\forall u)$, then $\phi_u \to -\infty$. (For example, consider the Lévy process $\phi \circ \Lambda^{-1}$ introduced later.)

1.5. Consider the problem of finding a bounded function f_θ on $\mathbb{R}$ such that M^θ, as defined at (10), is a martingale. Since we must have $\frac{1}{2}f_\theta'' = -\frac{1}{2}\theta^2 f_\theta$ on $(0,\infty)$, we can take:

$$f_\theta(x) = f_\theta(0)\cos\theta x + \theta^{-1}\sin\theta x \quad \text{on} \quad (0,\infty). \tag{12}$$

We have chosen the normalisation: $f_\theta'(0) = 1$ for reasons which will emerge later.

Recall the definitions of σ^+ and $X^+ \equiv B(\sigma^+)$ at (1b) and (1c). Of course X^+ is a reflecting Brownian motion with standard local time at 0 before t equal to $\Lambda(\sigma_t^+)$, where Λ is standard local time at 0 for B:

$$\Lambda(t) \equiv \tfrac{1}{2}L_t(0).$$

With apologies for the conflicting use of σ's (!), we define:

$$\mathcal{F}(t) \equiv \sigma\{B(s):s \le t\}, \qquad \mathcal{G}(t) \equiv \sigma\{X_s^+:s \le t\} \subset \mathcal{F}(\sigma_t^+).$$

Since the martingale M^θ is bounded on each interval of the form $[0,\sigma_t^+]$, the optional-sampling theorem implies that

$$M^\theta(\sigma_t^+) = \exp[\tfrac{1}{2}\theta^2 t - \tfrac{1}{2}\theta^2\phi^-(\sigma_t^+)]f_\theta(X_t^+)$$

is a martingale relative to the filtration $\{\mathcal{F}(\sigma_t^+)\}$. Hence, the 'optional projection'

$$E[M^\theta(\sigma_t^+)\,|\,\mathcal{G}_t]$$

defines a martingale relative to the filtration $\{\mathcal{G}(t)\}$. Utilising the independence of the 'up' and 'down' excursion processes from 0, and some standard independent-increment properties, we have:

$$(13) \qquad E[M^\theta(\sigma_t^+)|\mathcal{G}_t] = \exp[\tfrac{1}{2}\theta^2 t - \tfrac{1}{2}\theta^2 c_\theta \Lambda(\sigma_t^+)]f_\theta(X_t^+),$$

where c_θ is determined via the equation:

$$(14) \qquad \exp(-c_\theta t) = E\exp[-\tfrac{1}{2}\theta^2\phi^-(\Lambda^{-1}(t)],$$

where, of course, $\Lambda^{-1}(t) \equiv \inf\{u:\Lambda(u) > t\}$. The fact that the expression at (13) defines a martingale implies that

$$f_\theta \in \mathcal{D}(\mathcal{A}_\theta) \quad \text{and} \quad \mathcal{A}_\theta f = -\tfrac{1}{2}\theta^2 f ,$$

where $\mathcal{A}_\theta$ is the infinitesimal generator of elastic Brownian motion with killing constant c_θ. See §2.3 of Itô-McKean [4]. Hence, f_θ must satisfy the boundary condition:

$$f_\theta'(0) = c_\theta f_\theta(0),$$

and we have (using (14)):

$$\begin{aligned} f_\theta(0) = c_\theta^{-1} &= \int_{[0,\infty)} \exp(-c_\theta t)dt \\ &= E\int_{[0,\infty)} \exp[-\tfrac{1}{2}\theta^2\phi^-(\Lambda^{-1}(t))]dt \\ &= E\int_{[0,\zeta(X^-))} \exp(-\tfrac{1}{2}\theta^2 t)d\Lambda(\sigma_t^-) . \end{aligned}$$

By a further elementary application of the optional-sampling theorem to (10), the reader can easily show that

$$(15)\qquad f_\theta(x) = E^x \int_{[0,\zeta(X^-))} \exp(-\tfrac{1}{2}\theta^2 t)\, d\Lambda(\sigma_t^-) \qquad (a < x \le 0)$$

where a is as in (2), and E^x is the usual expectation associated with the law P^x of B started at x.

1.6. Suppose for a moment that m is finite and strictly positive on every compact subinterval of $(-\infty,0]$. Then X^- is a diffusion process on $(-\infty,0]$ and $\Lambda(\sigma_t^-)$ is the standard local time of X^- at 0. See §5.4 of Itô-McKean [4]. Hence,

$$(16)\qquad f_\theta(x) = r_\lambda(x,0) \qquad (\lambda \equiv \tfrac{1}{2}\theta^2)$$

where $r_\lambda(.,.)$ is the resolvent density function for X^- relative to the measure $2m$. In particular, the function f_θ on $(-\infty,0]$ may be calculated as the unique bounded non-negative solution of the equations:

$$(17)\qquad \frac{d}{dm}\frac{d}{dx} f_\theta = \theta^2 f_\theta \quad \text{on } (-\infty,0), \quad f_\theta'(0) = 1.$$

See §5.4 of Dym-McKean [2]. [Remark. Since the first equation at (17) implies that f_θ is absolutely continuous relative to Lebesgue measure with a density f_θ' satisfying,

$$f_\theta'(c) - f_\theta'(b) = \theta^2 \int_{(b,c]} f_\theta(x) m(dx) \qquad (b < c).$$

it follows that $f_\theta'(0)$ is well-defined.] It is a standard piece of spectral theory (see §5.5 of Dym-McKean [2]) that

$$f_\theta(0) = r_\lambda(0,0) = \int_{[0,\infty)} \frac{G(dr)}{r^2 + \theta^2}$$

for some measure G on $[0,\infty)$.

If we relax the assumption that m is finite and strictly positive on every compact subinterval of $(-\infty,0]$, then (17) still holds, but now we have

$$f_\theta(0) = \gamma + \int_{[0,\infty)} \frac{G(dr)}{r^2+\theta^2}, \tag{18}$$

where $-\gamma = \inf\{u \le 0 : m[u,0] = 0\}$ and G is again a measure on $[0,\infty)$.

[<u>Note</u>. A certain amount of poetic licence may be needed in the interpretation of (17) when $m[a,0] = \infty$ for some a. Then $f_\theta(a) = 0$, and we may need licence to interpret $0 \times \infty$].

We thought it instructive to derive the analytic form of f_θ from the assumption that M^θ is a martingale. We leave the reader to check the converse result, the one we really need: viz., that if f_θ has the analytic form we have described, then M^θ is indeed a martingale.

<u>1.7</u>. The deep and very remarkable <u>inverse spectral theorem of Krein</u> (see Dym-McKean [2]) tells us that <u>(17) and (18) put measures m satisfying (2) into one-one correspondence with pairs</u> (γ,G), <u>where</u>

$$\infty \ge \gamma \ge 0, \qquad \int (r^2+1)^{-1}G(dr) < \infty, \quad \underline{\text{and}} \quad G = 0 \quad \underline{\text{if}} \quad \gamma = \infty. \tag{19}$$

We shall prove that <u>if the pair</u> (γ,G) <u>satisfies (19)</u>, <u>and if</u> $f_\theta(0)$ <u>is defined by (18) and</u> f_θ <u>on</u> $(0,\infty)$ <u>via (12)</u>, <u>then the quadruple</u> (p_1,p_2,p_3,p_4) <u>is determined uniquely (modulo multiplication by scalars) by the fact that</u> f_θ <u>satisfies (5) for all</u> $\theta > 0$. If we temporarily <u>assume</u> (8), we are led via (18), (12), and (5) to the relation:

$$(20)\qquad \gamma + \int_{[0,\infty)} \frac{G(dr)}{r^2+\theta^2} = \frac{p_2 + \int_{(0,\infty)} \frac{J(dr)}{r^2+\theta^2}}{p_1 + \theta^2 \int_{(0,\infty)} \frac{J(dr)}{r(r^2+\theta^2)}}$$

But we shall prove analytically that <u>equation (20) sets up a one-one correspondence between pairs</u> (γ, G) <u>satisfying (19) and triples</u> (p_1, p_2, J) <u>(considered projectively) where</u> $p_1 \geq 0$, $p_2 \geq 0$, <u>and</u> J <u>satisfies</u> (9). Hence, of course (8) must hold, because of the fact that $(p_1, p_2, 0, p_4)$ is determined by the values $f_\theta(0)$.

We sketch a proof that

$$(21)\qquad p_2 + \int \frac{J(dr)}{r^2+\theta^2} = E \int_{[0,\zeta(Y^-))} \exp(-\tfrac{1}{2}\theta^2 t)\, d\Lambda(\tau_t^-),$$

so that the numerator on the right-hand side of (20) may be regarded as the 'resolvent density' $\tilde{r}_\lambda^-(0,0)$ for the process Y^-. This means that <u>the J measure arises from the spectral decomposition of the transition semigroup of the</u> Y^- <u>process</u>. At first sight, equation (21) is therefore rather surprising because, except in trivial cases, the process Y^- is <u>not</u> symmetrizable. (In general, Y^- will make jumps from 0, but not to 0.)

<u>1.8.</u> We are of course aware that (20) corresponds to a Wiener-Hopf factorization of the Lévy (independent-increments) process $\phi \circ \Lambda^{-1}$, and that, especially, the fine Greenwood-Pitman paper [3] provides much

probabilistic insight.

However, it would be totally wrong to imagine that everything of interest in the present paper can be attributed in some way to the 'dominant' rôle of the process $\phi \circ \Lambda^{-1}$. Indeed, Paper II makes it clear that the way in which the spectral decomposition of the transition semigroup of Y^- governs the law of Y^+ reflects a general principle for Markov processes. Paper II also gives some explanation, rather than only verification, of why the p_4 measure for Y^+ is completely monotone.

1.9. In §3, we show that the martingales M^θ at (10) form a 'full' family in a stronger sense than is implicit in various uniqueness assertions made above. In particular, we show that for $x < 0$, the P^x law of Y_0^+ is UNIQUELY determined by the Wald identity (optional-sampling result):

$$E^x f_\theta(Y_0^+) = f_\theta(x) \qquad \forall \theta > 0.$$

This key uniqueness theorem is obtained as a consequence of the Wiener-Hopf factorization (20) of $f_\theta(0)$.

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2. Proofs.

2.1. Let m be given. For $\theta > 0$, the value $f_\theta(0)$ corresponding to m has the form:

$$(22)\qquad f_\theta(0) = \gamma + \int_{[0,\infty)} \frac{G(dr)}{r^2 + \theta^2}$$

where γ and G are as at (19). Also,

$$f_\theta(x) = f_\theta(0)\cos\theta x + \theta^{-1}\sin\theta x \qquad (x \in [0,\infty)),$$

and f_θ satisfies Feller's side condition (5). Hence, we have the key relation:

$$(23)\qquad f_\theta(0)[p_1 - \tfrac{1}{2}\theta^2 p_3 + \int (1 - \cos\theta x)p_4(dx)]$$

$$= p_2 + \int (\theta^{-1}\sin\theta x)p_4(dx).$$

[Note. The 'extreme' cases:

$m(-\infty,0] = 0$ corresponding to $f_\theta(0) = \infty$ and $(p_1,p_2,p_3,p_4) = (0,1,0,0,)$,

and

$m\{0\} = \infty$ corresponding to $f_\theta(0) = 0$ and $(p_1,p_2,p_3,p_4) = (1,0,0,0)$,

will henceforth be ignored.]

(24) LEMMA. $p_3 = 0$.

Proof. We examine the orders of magnitude of the various expressions occurring in (23). First, note that

$$\left|\int (\theta^{-1}\sin\theta x)p_4(dx)\right| \le \int_0^\delta x p_4(dx) + \theta^{-1}\int_\delta^\infty p_4(dx).$$

Given $\varepsilon > 0$, we can first choose δ so that the first term on the right-hand side is less than $\frac{1}{2}\varepsilon$, and then choose θ_0 so large that the second term is less than $\frac{1}{2}\varepsilon$ when $\theta > \theta_0$. Hence,

$$\int (\theta^{-1}\sin\theta x)p_4(dx) = o(1) \qquad \text{as } \theta \uparrow \infty.$$

Next,

$$\left|\int (1 - \cos\theta x)p_4(dx)\right| \le \int_0^1 |1 - \cos\theta x|p_4(dx) + 2\int_1^\infty p_4(dx)$$

$$\le \theta\int_0^1 xp_4(dx) + 2\int_1^\infty p_4(dx) = O(\theta).$$

Since we are ignoring the case when $f_\theta(0) = 0, \forall\theta$, we see from (22) that $\theta^2 f_\theta(0) \uparrow K \in (0,\infty]$ as $\theta \uparrow \infty$. On dividing (23) by $\theta^2 f_\theta(0)$, we see that

$$-\tfrac{1}{2}p_3 + O(\theta^{-1}) = \frac{p_2}{\theta^2 f_\theta(0)} + o(1).$$

If $K = \infty$, we see that $p_3 = 0$; and if $K < \infty$, we obtain $-\frac{1}{2}p_3 = K^{-1}p_2$, so that (since $p_2 \ge 0$ and $p_3 \ge 0$) we must have $p_3 = p_2 = 0$. □

[Note. The reader should perform the exercise of spelling out the more informative probabilistic proof described after the statement of Theorem 7.]

(25) THEOREM. The quadruple $(p_1,p_2,0,p_4)$ is uniquely determined (modulo scalar multiples) by the fact that equation (23) holds for every $\theta > 0$.

Proof. This proof is a modification of the proof due to Kingman which was given in §5 of Rogers-Williams [6].

Let

$$\mathbb{H} \equiv \{z \in \mathbb{C} : \mathrm{Im}(z) \ge 0\}, \quad \mathbb{H}^+ \equiv \{z \in \mathbb{C} : \mathrm{Im}(z) > 0\}.$$

Define a mapping $h : \mathbb{H} \to \mathbb{C}$ as follows:

$$(26) \qquad h(z) \equiv p_1 - izp_2 + \int_{(0,\infty)} (1 - e^{izx})p_4(dx).$$

(Thus, $h(z)$ measures the extent to which $x \mapsto e^{izx}$ fails to satisfy Feller's condition (5).) Then h is continuous on $\mathbb{H}$ and analytic in $\mathbb{H}^+$. It is clear that

$$\mathrm{Re}(h) \geq 0 \text{ on } \mathbb{H}, \quad \mathrm{Re}(h) > 0 \text{ on } \mathbb{H}^+.$$

(Recall that $p_1 = 0$ and p_4 is the zero measure exactly when $m(-\infty,0] = 0$. This case is one which we have agreed to ignore.) It follows that the function:

$$\log h = \log |h| + i \arg (h)$$

may be defined as an analytic function on $\mathbb{H}^+$ with $\arg(h)$ taking values in $(-\frac{\pi}{2}, \frac{\pi}{2})$. Now, for $\theta \in \mathbb{R}\backslash\{0\}$, equation (23) states that

$$(27) \qquad f_\theta(0)\mathrm{Re}(h(\theta)) = -\theta^{-1}\mathrm{Im}(h(\theta)).$$

If $\mathrm{Re}(h(\alpha)) = 0$ for some $\alpha \in \mathbb{R}\backslash\{0\}$, then $p_1 = 0$ and p_4 is concentrated on a series of points in arithmetical progression, and $\{\theta \in \mathbb{R} : \mathrm{Re}(h(\theta)) = 0\}$ is countable. Hence, in every case, for almost all $\theta \in \mathbb{R}$,

$$(28) \qquad \lim_{z\to\theta} \arg(h(z)) = \arg(h(\theta)) = -\tan^{-1}[\theta f_\theta(0)].$$

Since the boundary values of the bounded harmonic function $\arg h(z)$ on $\mathbb{H}^+$ are known almost everywhere on $\mathbb{R}$, the function $\arg h(z)$ is determined in $\mathbb{H}^+$. Hence the function $\log h(z)$ is determined in $\mathbb{H}^+$ up to an additive constant, so that the function $h(z)$ is determined in $\mathbb{H}^+$ up to a multiplicative constant. In particular, the values

$$(29) \qquad h(i\theta) = p_1 + p_2\theta + \int (1 - e^{-\theta x})p_4(dx) \qquad (\theta > 0)$$

are determined up to a constant multiplier; and, by standard results, so too is the quadruple $(p_1, p_2, 0, p_4)$. □

Note. In §3 below, we present a deeper uniqueness result which is more useful in practice.

2.2. We continue on the course mapped out in §1.7. If we assume (8) and substitute (8) and (22) into (23), then we obtain the following equation, previously labelled as (20):

$$(30) \qquad \gamma + \int \frac{G(dr)}{r^2 + \theta^2} = \frac{p_2 + \int \frac{J(dr)}{r^2 + \theta^2}}{p_1 + \theta^2 \int \frac{J(dr)}{r(r^2 + \theta^2)}}$$

We shall prove the following theorem.

(31) THEOREM. Equation (30) sets up a one-one correspondence between pairs (γ, G) satisfying (19) and triples (p_1, p_2, J) (considered modulo scalar multiples) where $p_1 \geq 0$, $p_2 \geq 0$, and J satisfies (9).

Let us briefly recall the logic of the situation. A measure m determines a pair (γ, G). Part of Theorem 31 guarantees the existence of a triple (p_1, p_2, J) such that (30) holds. Theorem 25 guarantees that

$$(32) \qquad p_4(dx) = dx \int e^{-rx} J(dr)$$

and also that (p_1, p_2, J) is unique. Conversely, if a triple (p_1, p_2, J) is given, then Theorem 31 guarantees the existence of a unique pair (γ, G) such that (30) holds, and Krein's inverse spectral theorem guarantees existence and uniqueness of the corresponding m.

Remarks on Theorem 31.

(a) On substituting equation (32) into the definition of $h(iz)$, where h is as at (26), we obtain

$$h(iz) = p_1 + p_2 z + \int (1 - e^{-zx}) p_4(dx)$$

$$= p_1 + p_2 z + \int \frac{zJ(dr)}{r(r+z)},$$

so that if $z = \alpha + i\beta$, then

$$\text{(33)} \qquad \operatorname{Im} h(iz) = p_2 \beta + \int \frac{\beta}{|r+z|^2} J(dr).$$

Now, we already know from the proof of Theorem 25 that $\operatorname{Im}(h(iz))$ defines a nonnegative harmonic function in the first quadrant. From the analytic point of view, the fact that (33) holds - equivalently, the fact that p_4 has the form (32) - exactly corresponds to the fact that $\operatorname{Im}(h(iz))$ extends to a nonnegative harmonic function on the whole of $\mathbb{H}^+$, and that J 'reflected in 0' is the Poisson representing measure of this function. See §1.2 of Dym-McKean [2]. We are unable to give a direct proof of the described extension property of $\operatorname{Im}(h(iz))$.

(b) Theorem 31 is 'similar in spirit' to a number of known results. For example, see Kingman [5] and work of Reuter and others cited therein.

(c) As remarked earlier, some kind of explanation (rather than verification) of Theorem 31 is provided in Paper II.

2.3. Obtaining (p_1, p_2, J) from (γ, G): discrete case. On taking $z = \theta^2$, we see that the following Lemma states that if $\gamma = 0$ and G consists of atoms of masses G_i at points $\sqrt{(\mu_i)}$ $(0 \le i \le n)$, then (30) holds where $p_1 = p_2 = 0$ and J consists of atoms of masses J_i at $\sqrt{(\nu_i)}$ $(0 \le i \le n)$.

(34) LEMMA. Suppose that $G_i > 0$ $(0 \le i \le n)$ and that

$$0 = \mu_0 < \mu_1 < \ldots < \mu_n .$$

Then there exist strictly positive constants J_i $(0 \le i \le n)$ and ν_i $(0 \le i \le n)$ with

$$\mu_0 < \nu_0 < \mu_1 < \nu_1 < \ldots < \mu_n < \nu_n \tag{35}$$

such that for all z in $\mathbb{C}$ (with the obvious interpretation at various poles)

$$\sum_i \frac{G_i}{z + \mu_i} = \frac{\sum_i \frac{J_i}{z + \nu_i}}{z \sum_k \frac{J_k}{(z + \nu_k)\sqrt{(\nu_k)}}} . \tag{36}$$

Proof of Lemma 34. First, assume that (36) holds. Let $z \to -\nu_j$ in (36) to obtain:

$$\sum_i \frac{G_i}{\mu_i - \nu_j} = - \frac{1}{\sqrt{(\nu_j)}} .$$

Hence, the values ν_j must be roots of the equation:

$$\sum_i \frac{G_i}{x - \mu_i} = \frac{1}{\sqrt{x}} . \tag{37}$$

But, on sketching the graphs of the two sides of (37), we see that (37) has exactly $(n+1)$ roots $\nu_0, \nu_1, \ldots, \nu_n$ within $(0, \infty)$, and that the order-relations (35) hold.

On putting $z = -\mu_i (i \neq 0)$ in (36), we obtain:

$$(38) \qquad \sum_k \frac{J_k}{(\nu_k - \mu_i)\sqrt{(\nu_k)}} = 0 \qquad \text{for } i \neq 0.$$

We are entitled to augment (38) by the 'normalisation' condition:

$$(39) \qquad \sum_k \frac{J_k}{\sqrt{(\nu_k)}} = 1.$$

Some elementary manipulations on determinants allow us to solve (38) and (39) explicitly to obtain:

$$(40) \qquad \frac{J_k}{\sqrt{(\nu_k)}} = \frac{\prod_{j\neq 0} (\nu_k - \mu_j)}{\prod_{j\neq k} (\nu_k - \nu_j)} ,$$

and it is immediate from (35) that $J_k/\sqrt{(\nu_k)} > 0$, so that $J_k > 0$.

Now, we can multiply (36) by

$$\left[\sum_k \frac{J_k}{(z+\nu_k)\sqrt{(\nu_k)}}\right]\left[\prod_{i=0}^{n} (z+\mu_i)\right]\left[\prod_{j=0}^{n} (z+\nu_j)\right] .$$

Then (36) asserts the equality of two polynomials P and Q (say) where P-Q is of degree at most $2n + 1$. But what we have proved is that if we <u>define</u> $\nu_0, \nu_1, \ldots, \nu_n$ to be the roots of (37) satisfying (35) and <u>define</u> the constants J_k via (40), then the polynomials P and Q agree at all $(2n+2)$ points listed at (35). Hence the polynomials P and Q are identical, and the lemma is proved. □

2.4. <u>Obtaining</u> (p_1, p_2, J) <u>from</u> (γ, G): <u>general case</u>. Now let (γ, G) be any pair satisfying (19). For $\theta > 0$, we can write

$$\gamma + \int_{[0,\infty)} \frac{G(dr)}{r^2 + \theta^2} = \int_{[0,\infty]} \frac{r^2 + 1}{r^2 + \theta^2} \hat{G}(dr)$$

where

(41) $$\hat{G}(dr) = (r^2 + 1)^{-1} G(dr) \quad \text{on} \quad (0,\infty), \quad \hat{G}\{\infty\} = \gamma.$$

Since we are ignoring the case when $\gamma = \infty$, the measure $\hat{G}$ is a bounded measure on $[0,\infty]$. In the sense of weak* convergence of bounded measures on $[0,\infty]$, we can approximate $\hat{G}$ by measures $\hat{G}^{(n)}$ each consisting of an atom at 0 together with a finite number of atoms within $(0,\infty)$. From Lemma 34, we know that

(42) $$\int_{[0,\infty]} \frac{r^2 + 1}{r^2 + \theta^2} \hat{G}^{(n)}(dr) = \frac{\int_{[0,\infty]} \frac{r(r+1)}{r^2 + \theta^2} \hat{J}^{(n)}(dr)}{\int_{[0,\infty]} \frac{\theta^2(r+1)}{r^2 + \theta^2} \hat{J}^{(n)}(dr)}$$

for some atomic measure $\hat{J}^{(n)}$ on $[0,\infty]$ which we can take to be a probability measure. If $\hat{J}$ is any weak* limit of $\hat{J}^{(n)}$ as $n \to \infty$, we have

$$\gamma + \int_{[0,\infty)} \frac{G(dr)}{r^2 + \theta^2} = \frac{\int_{[0,\infty]} \frac{r(r+1)}{r^2 + \theta^2} \hat{J}(dr)}{\int_{[0,\infty]} \frac{\theta^2(r+1)}{r^2 + \theta^2} \hat{J}(dr)}$$

$$= \frac{p_2 + \int_{(0,\infty)} \frac{J(dr)}{r^2 + \theta^2}}{p_1 + \theta^2 \int_{(0,\infty)} \frac{J(dr)}{r(r^2 + \theta^2)}} ,$$

where

$$J(dr) = r(r+1)\hat{J}(dr) \quad \text{on} \quad (0,\infty),$$
$$p_1 = \hat{J}\{0\}, \quad p_2 = \hat{J}\{\infty\}.$$

<u>2.5</u>. <u>Obtaining</u> (γ, G) <u>from</u> (p_1, p_2, J). Because the weak*-convergence argument goes through smoothly, it is enough to deal with the case when $p_1 = p_2 = 0$ and J consists of finitely many atoms. So, assume that J_i $(0 \le i \le n)$ and $\nu_i (0 \le i \le n)$ <u>are strictly positive numbers, and that</u>

$$\nu_1 < \nu_2 < \dots < \nu_n .$$

We must show that (36) holds where $\mu_0 = 0$ and μ_i $(1 \le i \le n)$ and G_i $(0 \le i \le n)$ are strictly positive.

Clearly, we define $\mu_1, \mu_2, \dots, \mu_n$ to be the unique numbers satisfying the order relations (35) which are roots of the equation:

$$\Sigma \frac{J_k}{(\nu_k - x)\sqrt{(\nu_k)}} = 0.$$

We put $\mu_0 = 0$. On comparing the residues of the two sides of (36) at $z = -\mu_i$, and using l'Hopital's rule in the usual way, we see that we must take

$$(43) \qquad G_i = \frac{\sum_k \frac{J_k}{\nu_k - \mu_i}}{\sum_k \frac{J_k\sqrt{(\nu_k)}}{(\nu_k - \mu_i)^2}} \qquad (i = 1,2,\dots,n)$$

But for $\mu > 0$,

$$\sum_{\nu_k > \mu} \frac{J_k}{\nu_k - \mu} > \sqrt{\mu} \sum_{\nu_k > \mu} \frac{J_k}{(\nu_k - \mu)\sqrt{(\nu_k)}},$$

$$-\sum_{\nu_k > \mu} \frac{J_k}{\nu_k - \mu} < -\sqrt{\mu} \sum_{\nu_k < \mu} \frac{J_k}{(\nu_k - \mu)\sqrt{(\nu_k)}}.$$

Hence for $i = 1,2,\ldots,n$,

$$\sum_k \frac{J_k}{\nu_k - \mu_i} > \sqrt{(\mu_i)} \sum \frac{J_k}{(\nu_k - \mu_i)\sqrt{(\nu_k)}} = 0,$$

and $G_i > 0$. Of course,

(44) $$G_0 = (\Sigma J_i/\nu_i)(\Sigma J_k/\nu_k^{3/2}) > 0.$$

To show that (36) must hold if the G_i $(0 \le i \le u)$ are defined via (43) and (44), we can apply the 'polynomial' argument at the end of §2.3, or else appeal to the Mittag-Leffler theorem.

The proof of Theorem 31 is now complete.

2.6. Notes on equation (21). The Greenwood-Pitman paper [3] explains very clearly the probabilistic significance of equation (20) viewed as a Wiener-Hopf factorization of $\phi \circ \Lambda^{-1}$, and equation (21) makes up one part of the Greenwood-Pitman path decomposition.

The partial result provided by equation (21) also admits a direct proof by our martingale method. If m consists only of a finite number of atoms within $(-\infty,0]$, then we can find a bounded function g_θ on $\mathbb{R}$ such that

$$N_t^\theta \equiv \exp(-\tfrac{1}{2}\theta^2 t) g_\theta(B_t) \text{ defines a martingale } N^\theta ;$$

and we can prove (21) by applying the optional-sampling theorem to N^θ. A weak*-convergence argument completes the proof.

Paper II points to a simple proof of (21), and to some substantial generalisations.

3. The key uniqueness theorem

Let f_θ, M^θ, p_1, p_2, J, etc. have their now-familiar significance.

Suppose that B starts at x where $x < 0$. By applying the optional-sampling theorem to the martingale M^θ of (10) at time τ_t^+, we obtain:

$$(45) \qquad E^x f_\theta(Y_t^+) = e^{-\frac{1}{2}\theta^2 t} f_\theta(x) \qquad (\forall\theta > 0).$$

We shall prove that <u>the</u> P^x <u>law of</u> Y_t^+ <u>is uniquely determined by (45)</u>.

Now, if $p_1 = 0$, then the P^x law of Y_t^+ is a <u>probability</u> measure, while, if $p_1 > 0$, then the P^x law of Y_t^+ is a measure of total mass less than 1. The desired uniqueness result is therefore an immediate consequence of the following theorem.

(46) THEOREM.

(i) <u>Suppose that</u> μ_1 <u>and</u> μ_2 <u>are probability measures on the open interval</u> $(0,\infty)$ <u>such that</u>

$$(47) \qquad \int_{(0,\infty)} f_\theta(y)\mu_1(dy) = \int_{(0,\infty)} f_\theta(y)\mu_2(dy), \qquad \forall\theta > 0.$$

<u>Then</u> $\mu_1 = \mu_2$.

(ii) <u>Suppose that</u> $p_1 > 0$. <u>If</u> μ_1 and μ_2 <u>are finite measures on</u> $(0,\infty)$ <u>such that (47) holds, then</u> $\mu_1 = \mu_2$.

Notes.

(a) Observe that part (i) would be false if the interval $(0,\infty)$ were replaced by $[0,\infty)$. For if $p_1 = p_2 = 0$ and p_4 is a probability measure, then in the functional notation for measures, we have $p_4(f_\theta) = \delta_0(f_\theta)$, $\forall\theta > 0$, where δ_0 is the unit mass at 0.

(b) The case when $f(0) = \infty$ must of course be interpreted as the cosine-transform theorem. We continue to ignore that case.

Proof of (i). Suppose that μ_1 and μ_2 are probability measures on $(0,\infty)$ such that (47) holds. Because of the Wiener-Hopf factorization (30), we may rewrite (47) as follows:

$$\int F_\theta(y)\mu_1(dy) = \int F_\theta(y)\mu_2(dy), \qquad \forall\theta > 0, \tag{48}$$

where

$$F_\theta(y) = \left[\theta p_2 + \int \frac{\theta r K(dr)}{r^2 + \theta^2}\right]\cos\theta y + \left[p_1 + \int \frac{\theta^2 K(dr)}{\theta^2 + r^2}\right]\sin\theta y,$$

$K(dr)$ being the $r^{-1}J(dr)$ of our previous notation. Recall that

$$\int (r+1)^{-1} J(dr) < \infty.$$

As before, define

$$\mathbb{H} \equiv \{z \in \mathbb{C}: \operatorname{Im}(z) \geq 0\}, \qquad \mathbb{H}^+ \equiv \{z \in \mathbb{C}: \operatorname{Im}(z) > 0\},$$

$$h(z) \equiv p_1 - ip_2 z - \int \frac{izK(dr)}{r - iz}.$$

Then h is analytic in $\mathbb{H}^+$ and continuous on $\mathbb{H}$. Moreover, if $z = a + ib$ (of course, a no longer has the significance it had at (2)), then

(49) $$\mathrm{Re}\,h(z) = p_1 + ap_2 + \int \frac{b(r+b) + a^2}{(r+b)^2 + a^2} K(dr) > 0 \quad \text{on} \quad \mathbb{H}\setminus\{0\},$$

so that $h \neq 0$ on $\mathbb{H}\setminus\{0\}$.

For $j = 1,2$, define $\tilde{\mu}_j : \mathbb{H} \to \mathbb{C}$ by the equation:

(50) $$\tilde{\mu}_j(z) \equiv \int (1 - e^{izy})\mu_j(dy) = 1 - \int e^{izy}\mu_j(dy),$$

and for $z \in \mathbb{H}\setminus\{0\}$, define:

(51) $$\Psi_j(z) \equiv \tilde{\mu}_j(z)/h(z).$$

Then Ψ_j is analytic in $\mathbb{H}^+$ and continuous on $\mathbb{H}\setminus\{0\}$.

Now, equation (48) states that

(52) $$\mathrm{Im}\Psi_1(\theta) = \mathrm{Im}\Psi_2(\theta), \qquad \forall\theta > 0.$$

Moreover, it is trivially true that

(53) $$\mathrm{Im}\Psi_1(i\theta) = \mathrm{Im}\Psi_2(i\theta), \qquad \forall\theta > 0,$$

because both sides of (53) are zero. We would like to conclude that

(54) $$\mathrm{Im}\Psi_1(z) = \mathrm{Im}\Psi_2(z) \quad \text{for all } z \text{ in the first quadrant.}$$

To do this, we need to establish appropriate growth conditions. But let us assume for the moment that (54) is proved. Then

$$\Psi_1(z) = \Psi_2(z) + c \quad \text{in the first quadrant,}$$

where c is a real constant. Thus, from (50) with z equal to (or, if you prefer, tending to) $i\theta$ with $\theta > 0$, we obtain

$$(55) \qquad \int (1 - e^{-\theta y}) \mu_1(dy) = \int (1 - e^{-\theta y}) \mu_2(dy) + c[p_1 + p_2\theta + \int (1 - e^{-\theta y}) p_4(dy)].$$

By examining what happens when $\theta \to \infty$, it is trivial to show that $c = 0$. [Note that it is here that we need the fact that the μ_j are probability measures on $(0,\infty)$ not $[0,\infty)$.] Hence $\mu_1 = \mu_2$.

We must now prove (54). To ensure that a function g, which is harmonic in the open first quadrant and continuous on the closed first quadrant except perhaps at 0, is determined by its values on the edges of the quadrant, it is enough to show that g is bounded near 0 and that $g(z) = 0(|z|)$ as $|z| \to \infty$. We apply this principle not to the function $\Psi_j(z)$ but to the function $\Psi_j(1/z)$, that is, to the function $\Psi_j \circ z^{-1}$ defined on the fourth quadrant. Translating back to the first quadrant, we see that to prove (54), we need to establish:

(56) $\Psi_j(z)$ is bounded near ∞ (within the first quadrant),

(57) $z\Psi_j(z) \to 0$ as $z \to 0$ (within the first quadrant).

Note that $|\tilde{\mu}_j| \le 2$ on $\mathbb{H}$. From (49), $|h| \ge p_1$ on $\mathbb{H}$, so that if $p_1 > 0$ then (see (50)) $|\Psi_j| \le 2p_1^{-1}$ on $\mathbb{H}$, and (56) and (57) follow.

It remains to prove (56) and (57) when $p_1 = 0$. [As usual, we ignore the case when ($p_2 \ne 0$ and) the measure K is zero. The theorem is classical in that case.] From (49),

$$|h(z)| \ge \int \frac{b(r+b) + a^2}{(r+b)^2 + a^2} K(dr)$$

$$\ge \frac{1}{2}\int \frac{a^2 + b^2}{r^2 + a^2 + b^2} K(dr) \ge \frac{1}{2}\int \frac{1}{r^2 + 1} K(dr)$$

when $|z| \geq 1$; and (56) follows. Next,

$$z\Psi_j(z) = \tilde{\mu}_j(z) \Big/ \left[- ip_2 - i\int \frac{K(dr)}{r - iz}\right].$$

But $\tilde{\mu}_j(z) \to 0$ as $z \to 0$, and if $a^2 + b^2 \leq \varepsilon^2$, then

$$\left|p_2 + \int \frac{K(dr)}{r - i\theta}\right| \geq p_2 + \int \frac{(b + r)K(dr)}{(r + b)^2 + a^2}$$

$$\geq p_2 + \frac{1}{2}\int \frac{rK(dr)}{r^2 + \varepsilon^2}$$

The result (57) follows, and the proof of part (i) of the theorem is complete. □

<u>Proof of (ii)</u>. Assume that $p_1 > 0$. Suppose that μ_1 and μ_2 are finite measures on $(0,\infty)$. Using the modified definitions:

$$\tilde{\mu}_j(z) \equiv \int e^{izy}\mu_j(dy) \qquad (j = 1,2),$$

we transfer the proof of part (i) in the obvious way, the bound $|\Psi_j| \leq p_1^{-1}$ making everything easy. □

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