SHIGETOKU KAWABATA
TOSHIO YAMADA

On some limit theorems for solutions of stochastic differential equations

Séminaire de probabilités (Strasbourg), tome 16 (1982), p. 412-441
<http://www.numdam.org/item?id=SPS_1982__16__412_0>

On some limit theorems for solutions of stochastic differential equations

Shigetoku Kawabata and Toshio Yamada
Department of Applied Science
Kyushu University, Fukuoka 812, Japan

Introduction.

We consider the following Itô type stochastic differential equations;

$$dx(t) = \sigma(t,x(t))dB(t) + b(t,x(t))dt$$

$$dx_n(t) = \sigma_n(t,x_n(t))dB(t) + b_n(t,x_n(t))dt, \quad n = 1,2,\ldots$$

What we will do in the present paper is to propose some sufficient conditions which guarantee stability properties for solutions of the above equations when the coefficients σ_n and b_n tend to σ and b respectively.

This problem has been discussed by Stroock and Varadhan in chapter 11 of their book ([6]), by the method of the martingale problem. In the case where the martingale problem for σ and b has a unique solution, they showed that the stability properties are guaranteed in the law sense when σ_n and b_n tend to σ and b respectively in a suitable sense.

In this paper we will treat the problem in the case where the path-wise uniqueness holds for σ and b, and show that under some convergence conditions for σ_n and b_n, the stability properties hold in the path-wise sense.

In the formulation of our conditions, the operator $\mathcal{L}$ which was introduced by Okabe and Shimizu with the idea of applying it to pathwise uniqueness problem will play an important role. ([5]).

Preliminaries.

We consider the following Itô's stochastic differential equations;

$$(1) \quad x^i(t) = x^i(0) + \sum_{j=1}^{d} \int_0^t \sigma_j^i(s,x(s))\,dB^j(s) + \int_0^t b^i(s,x(s))\,ds \quad (1 \le i,j \le d)$$

$$(2) \quad x_n^i(t) = x_n^i(0) + \sum_{j=1}^{d} \int_0^t \sigma_{n,j}^i(s,x_n(s))\,dB^j(s) + \int_0^t b_n^i(s,x_n(s))\,ds \quad (1 \le i,j \le d)$$

$$n = 1,2,\ldots$$

We suppose in this paper that the coefficients in the above equations satisfy the following conditions (A), (B) and (C).

(A) $\sigma_j^i(s,x)$, $\sigma_{n,j}^i(s,x)$, $b^i(s,x)$ and $b_n^i(s,x)$ $(1 \le i,j \le d)$, $n = 1,2,\ldots$ are Borel measurable functions defined on $[0,\infty) \times R^d$.

(B) There exists a positive constant $K > 0$, such that

$$(3) \quad \|\sigma(s,x)\|^2 + \|b(s,x)\|^2 \le K(1 + \|x\|^2) \ (*)$$

and

$$(4) \quad \|\sigma_n(s,x)\|^2 + \|b_n(s,x)\|^2 \le K(1 + \|x\|^2) \ (*)$$

hold, where $\sigma(s,x) = (\sigma_j^i(s,x))$, $\sigma_n(s,x) = (\sigma_{n,j}^i(s,x))$, $b(s,x) = (b^i(s,x))$ and $b_n(s,x) = (b_n^i(s,x))$.

(*) $\|A\|$ stands for $\sqrt{\sum_{i=1}^{m} \sum_{j=1}^{n} (a_j^i)^2}$ where $A = (a_j^i)$ is m×n-matrix, and $\|A\|^p$ for $\sum_{i=1}^{m} \sum_{j=1}^{n} |a_j^i|^p$ $(p > 1)$

(C) Define $D_r = \{x \in R^d, \|x\| \leq r\}$. Then for any $r > 0$ and $T > 0$, the relation $\lim_{n\to\infty} \sup_{(t,x)\in[0,T]\times D_r} \{\|\sigma_n(t,x) - \sigma(t,x)\| + \|b_n(t,x) - b(t,x)\|\} = 0$ holds.

By a probability space with an increasing family of Borel fields which is denoted by $(\Omega, \mathcal{F}, P: \mathcal{F}_t)$ we mean a probability space $(\Omega, \mathcal{F}, P)$ with a system $\{\mathcal{F}_t\}_{t\in[0,\infty)}$ of sub Borel fields of $\mathcal{F}$ such that $\mathcal{F}_s \subset \mathcal{F}_t$ if $s < t$.

Definition 1

By a solution of the equation (1), we mean a probability space with an increasing family of Borel fields $(\Omega, \mathcal{F}, P: \mathcal{F}_t)$ and a family of stochastic processes $\mathcal{X} = \{x(t) = (x^1(t), \ldots x^d(t)), B(t) = (B^1(t), \ldots B^d(t))\}$ defined on it such that

(i) with probability one, $x(t)$ and $B(t)$ are continuous in t and $B(0) = 0$,

(ii) they are adapted to $\mathcal{F}_t$,

(iii) $B(t)$ is a system of $\mathcal{F}_t$-martingale such that

$$E[(B^i(t) - B^i(s))(B^j(t) - B^j(s))/\mathcal{F}_s] = \delta_{ij}\cdot(t-s), \quad (1 \leq i,j \leq d)$$

(iv) $\mathcal{X}(t)$ satisfies

$$x^i(t) = x^i(0) + \sum_{j=1}^{d} \int_0^t \sigma_j^i(s,x(s))dB^j(s) + \int_0^t b^i(s,x(s))ds, \quad (1 \leq i \leq d)$$

where the integral by dB is understood in the sense of Itô's integral.

One defines a solution of the equation (2) in the similar way as in the definition 1.

Now, we introduce the operator $\mathcal{L}$ which is defined by

$$(5)\quad (\mathcal{L}V)(t,x,y) = \frac{\partial V}{\partial t} + \sum_{i=1}^{d} \frac{\partial V}{\partial x_i} b^i(t,x) + \sum_{i=1}^{d} \frac{\partial V}{\partial y_i} b_n^i(t,y)$$

$$+ \frac{1}{2}\{ \sum_{i,j=1}^{d} \frac{\partial^2 V}{\partial x_i \partial x_j}(\sum_{k=1}^{d} \sigma_k^i(t,x)\sigma_k^j(t,x)) + 2 \sum_{i,j}^{d} \frac{\partial^2 V}{\partial x_i \partial y_j}(\sum_{k=1}^{d} \sigma_k^i(t,x)\sigma_{n,k}^j(t,y))$$

$$+ \sum_{i,j=1}^{d} \frac{\partial^2 V}{\partial y_i \partial y_j}(\sum_{i=1}^{d} \sigma_{n,k}^i(t,y)\sigma_{n,k}^j(t,y))\},$$

where $V(t,x,y)$ is defined on $[0,\infty) \times R^d \times R^d$.

§2 Some limit theorems.

Theorem 1. Let p be a positive integer $p \geq 1$. Let $(\Omega, \mathcal{F}, P: \mathcal{F}_t)$ be a probability space with an increasing family of Borel fields. Suppose we are given the following;

(i) a solution of the equation (1) $\mathcal{X}(t) = \{x(t), B(t)\}$ defined on $(\Omega, \mathcal{F}, P: \mathcal{F}_t)$, such that

$$(6)\quad E[\| x(0) \|^{2p}] < + \infty,$$

(ii) a solution of the equation (2) $\mathcal{X}_n(t) = \{x_n(t), B(t)\}$ for each $n = 1,2,\dots,$ defined on the same $(\Omega, \mathcal{F}, P: \mathcal{F}_t)$ such that

$$(7)\quad \sup_n E[\| x_n(0) \|^{2p}] < + \infty.$$

Let $T > 0$ and $r > 0$ be two positive constants. Suppose that there exists a sequence of functions $V_{T,r}$, $V_{T,r}^m$, $m = 1,2,\dots,$ defined on $[0,T] \times D_r \times D_r$, continuously differentiable in t and twice continuously differentiable in (x,y), such that

(V1) $V_{T,r}^m(t,x,y) \geq 0$ for $(t,x,y) \in [0,T] \times D_r \times D_r$, (V2) $V^m_{T,r}(t,x,y)$ converges uniformly to the function $V_{T,r}(t,x,y)$ as m tends to infinity.

(V3) there exist two constants $C_1(T,r)$ and $C_2(T,r)$ such that

$$C_1(T,r)\|x-y\| \le V_{T,r}(t,x,y) \le C_2(T,r)\|x-y\| \quad \text{for } (t,x,y) \in [0,T] \times D_r \times D_r \tag{8}$$

(V4) there exists a non-decreasing sequence of integers $\{m_n\}$ $n = 1,2,\ldots$ such that $\lim_{n\to\infty} m_n = \infty$ and

$$\overline{\lim_{n\to\infty}}\, E\left[\int_0^{t\wedge\tau_r^{(n)}} (\mathcal{L}V^{m_n}_{T,r})(s,x(s),x_n(s))ds\right] \le 0 \tag{9}$$

where $\tau_r^{(n)} = \inf\{t : \max(\|x(t)\|, \|x_n(t)\| \ge r)\}$.

Then, the relation

$$\lim_{n\to\infty} E[\|x_n(0) - x(0)\|] = 0 \tag{10}$$

implies

$$\lim_{n\to\infty} E[\|x^r(t) - x^r_n(t)\|] = 0 \quad \text{for all } t \in [0,T], \tag{11}$$

where $x^r(t) = x(t\wedge\tau_r^{(n)})$ and $x^r_n(t) = x_n(t\wedge\tau_r^{(n)})$. ∎

Proof.

By Itô's formula we have

$$V^{m_n}_{T,r}(t\wedge\tau_r^{(n)}, x^r(t), x^r_n(t)) = V^{m_n}_{T,r}(0, x^r(0), x^r_n(0))$$
$$+ \text{ a martingale of zero mean}$$
$$+ \int_0^{t\wedge\tau_r^{(n)}} (\mathcal{L}V^{m_n}_{T,r})(s, x^r(s), x^r_n(s))ds.$$

Hence, we get

$$(12)\quad E[V_{T,r}^{m_n}(t\wedge\tau_r^{(n)}, x^r(t), x_n^r(t))]$$
$$= E[V_{T,r}^{m_n}(0, x^r(0), x_n^r(0))] + E[\int_0^{t\wedge\tau_r^{(n)}} (\mathcal{L} V_{T,r}^{m_n})(s, x^r(s), x_n^r(s))ds].$$

Now, we will show that

$$(13)\quad \lim_{n\to\infty} [V_{T,r}^{m_n}(0, x^r(0),\ x_n^r(0))] = 0 \quad \text{holds.}$$

By the condition (V2), we can choose for any positive $\varepsilon > 0$ the integer $N(\varepsilon)$ so that

$$(14)\quad V_{T,r}(t,x,y) - \varepsilon \le V_{T,r}^{m_n}(t,x,y) \le V_{T,r}(t,x,y) + \varepsilon$$
$$\text{for}\quad n \ge N(\varepsilon) \quad\text{and}\quad (t,x,y) \in [0,T] \times D_r \times D_r.$$

Combine the relation (14) with the condition (V3). Then we can see that $V_{T,r}^{m_n}(s, x^r(0),\ x_n^r(0)) \le C_2(T,r)\| x^r(0) - x_n^r(0)\| + \varepsilon$. Therefore the condition (10) implies that

$$(15)\quad 0 \le \overline{\lim_{n\to\infty}}\, E[V_{T,r}^{m_n}(0, x^r(0), x_n^r(0))] \le \varepsilon.$$

Since ε is an arbitrary positive number, we can conclude that the relation (13) holds.

By (V3), (V4), (12) and (13), we have

$$0 \le C_1(T,r)\overline{\lim_{n\to\infty}}\, E[\| x^r(t) - x_n^r(t)\|]$$
$$\le \overline{\lim_{n\to\infty}}\, E[V_{T,r}^{m_n}(t\wedge\tau_r^{(n)}, x^r(t), x_n^r(t))]$$
$$= \overline{\lim_{n\to\infty}}\, E[\int_0^{t\wedge\tau_r^{(n)}} (\mathcal{L}V_{T,r}^{m_n})(s, x(s), x_n(s))ds] \le 0.$$

This relation implies immediately (11). Q.E.D.

Theorem 2. Suppose we are given a solution of the equation (1) $\mathcal{X}(t) = \{x(t), B(t)\}$ and a sequence of solutions of the equation (2) $\mathcal{X}_n = \{x_n(t), B(t)\}$ $n=1,2,\ldots$ so that they are defined on a same probability space with an increasing family of Borel fields $(\Omega, \mathcal{F}, P; \mathcal{F}_t)$ and they satisfy (6) and (7), for $p = 1$.

Suppose further that for any $T > 0$ and $r > 0$, there exists a sequence of functions $V_{T,r}(t,x,y)$, $V^m_{T,r}(t,x,y)$, $m=1,2,\ldots$ such that they satisfy the conditions (V1), (V2), (V3) and (V4).

Then the relation

$$\lim_{n\to\infty} E[\| x_n(0) - x(0) \|] = 0$$

implies

$$\lim_{n\to\infty} E[\| x_n(t) - x(t) \|] = 0 \quad \text{for all } t \in [0,\infty). \tag{16}$$

$\blacksquare$

For the proof of the Theorem 2, we shall prepare several lemmas.

Lemma 1. Let p be a positive integer $p \geq 1$.

Under the condition (B), the following inequalities hold;

$$E[\| x(t) \|^{2p}] \leq K(p,T)(1 + E[\| x(0) \|^{2p}]) \quad \text{for } t \in [0,T] \tag{17}$$

$$E[\| x_n(t) \|^{2p}] \leq K(p,T)(1 + E[\| x_n(0) \|^{2p}]) \quad \text{for } t \in [0,T] \tag{18}$$

where $K(p,T)$ is a positive constant which depends on p, T and K in the condition (B).

The assertions in Lemma 1 are well known (See e.g. ([3])), so we omit the proof.

The following lemma can be derived easily from Lemma 1.

Lemma 2. Under the condition (B), the relations (6) and (7) imply that the system of random variables

$$\{\| \sigma(t,x(t))\|^p, \| \sigma_n(t,x_n(t))\|^p, \| b(t,x(t))\|^p, \| b_n(t,x_n(t))\|^p, \quad n = 1,2,\dots \quad t \in [0,T]\}$$

is uniformly integrable with respect to $([0,T]\times\Omega,\ dt \otimes dP)$.

Lemma 3. Under the condition (B), (6), and (7), there exists a positive constant $L(T) < +\infty$ such that

$$(19)\quad E[\sup_{0\le t\le T} \| x(t)\|^2] \le L(T)$$

and

$$(20)\quad E[\sup_{0\le t\le T} \| x_n(t)\|^2] \le L(T)$$

hold.

Proof. We will show (19).

It is easy to choose a positive constant $C(d)$ depending on d such that

$$\| x(t)\|^2 \le C(d)\| x(0)\|^2$$

$$+ C(d) \sum_{i,k} (\int_0^t \sigma_k^i(s,x(s))dB^k(s))^2 + C(d)\sum_i (\int_0^t b^i(s,x(s))ds)^2.$$

Using the Doob's inequality, we get from the above

$$(21)\quad E[\sup_{0\le t\le T} \| x(t)\|^2] \le C(d)E[\| x(0)\|^2]$$

$$+ 2C(d) \sum_{i,k} E[(\int_0^T (\sigma_k^i(s,x(s))^2 ds)]$$

$$+ C(d)\sum_i E[(\int_0^T |b^i(s,x(s))|ds)^2]$$

$$= C(d)E[\| x(0)\|^2] + J_1 + J_2, \quad \text{say.}$$

By the condition (B), we have for J_1

$$J_1 \leq 2C(d)d^2\int_0^T K(1 + E[\| x(s)\|^2])ds.$$

Hence by lemma 1

$$(22)\quad J_1 \leq 2C(d)d^2\int_0^T K\{1 + K(2,T)(1 + E(\| x(0)\|^2))\}ds < +\infty$$

holds.

On the other hand, we will evaluate J_2. We have

$$(23)\quad J_2 \leq C(d)\cdot d\cdot T\cdot E[\int_0^T (b^i(s,x(s))^2 ds]$$

$$\leq C(d)\cdot d\cdot T\cdot \int_0^T K(1 + E(\| x(s)\|^2))ds$$

$$\leq C(d)\cdot d\cdot T\cdot \int_0^T K(1 + K(2,T)(1 + E(\| x(0)\|^2))ds$$

$$< +\infty.$$

By (21), (22) and (23), it is easy to choose a constant $L(T)$ such that (19) holds. By the similar way one can show (20) with the same constant $L(T)$ as in (19). Q.E.D.

We are now in a position to prove the Theorem 2.

Proof of the Theorem 2. Put

$$\Omega_r = \{\omega : \sup_{0\leq t\leq T} \| x(t)\| < r\}$$

and

$$\Omega_{n,r} = \{\omega : \sup_{0\le t\le T} \| x_n(t) \| < r\}.$$

Then, by lemma 3 we have

$$(24)\quad P(\Omega_r^c) = P(\{\omega : \sup_{0\le t\le T} \| x(t) \|^2 \ge r^2\}) \le \frac{E[\sup_{0\le t\le T} \| x(t) \|^2]}{r^2} \le \frac{L(T)}{r^2},$$

and

$$(25)\quad P(\Omega_{n,r}^c) \le \frac{L(T)}{r^2}.$$

On the other hand, by lemma 1, we know that the system of random functions

$$(26)\quad \{\| x(t) \| , \| x_n(t) \| \quad n = 1,2,\ldots, \quad t \in [0,T]\}$$

is uniformly integrable.

Now, we have that

$$E[\| x_n(t) - x(t) \|] \le E[\| x_n(t) - x(t) \| , \Omega_{n,r} \cap \Omega_r]$$

$$+ E[\| x_n(t) - x(t) \| , \Omega_{n,r}^c] + E[\| x_n(t) - x(t) \| , \Omega_r^c]$$

$$= E[\| x_n^r(t) - x^r(t) \|] + E[\| x_n(t) - x(t) \| , \Omega_{n,r}^c]$$

$$+ E[\| x_n(t) - x(t) \| , \Omega_r^c].$$

Let $\varepsilon > 0$ be an arbitrary positive number. Use (24), (25) and the fact that the system of random variables (26) is uniformly integrable. Then, there exist a positive number $r > 0$ and an integer $N > 0$ such that

$$E[\| x_n^r(t) - x^r(t) \|] < \frac{\varepsilon}{3} \qquad \text{for} \quad n \geq N,$$

$$E[\| x_n(t) - x(t) \| : \Omega_{n,r}^c] < \frac{\varepsilon}{3}, \quad n = 1,2,\ldots$$

and

$$E[\| x_n(t) - x(t) \| , \Omega_r^c] < \frac{\varepsilon}{3}, \quad n = 1,2,\ldots$$

where we have used the Theorem 1.

Hence we can conclude that $\lim_{n\to\infty} E[\| x_n(t) - x(t) \|] = 0$ holds for all t. Q.E.D.

In the following theorem we suppose that the coefficients of the equations (1) and (2) satisfy the following condition (A') in place of the condition (A).

Condition (A'). $\sigma_j^i(s,x)$, $\sigma_{n,j}^i(s,x)$, $b^i(s,x)$ and $b_n^i(s,x)$ $(1 \leq i,j \leq d,\ n=1,2,\ldots)$ are continuous in (s,x).

Theorem 3. Let T be a positive number. Suppose that we are given a solution of the equation (1) $\mathcal{X}(t) = (x(t),B(t))$ and a sequence of solutions of (2) $\mathcal{X}_n(t) = (x_n(t),B(t))$ $n=1,2,\ldots$ so that they are defined on a same probability space with an increasing family of Borel fields $(\Omega,\mathcal{F},P:\mathcal{F}_t)$ and they satisfy (6) and (7) for some interger $p \geq 2$.

Suppose further that for any $r > 0$ there exists a sequence of $V_{T,r}(t,x,y)$ and $V^m_{T,r}(t,x,y)$ $m=1,2,\ldots$ such that they satisfy the conditions (V1), (V2), (V3) and (V4).

Then, the relation

$$\lim_{n\to\infty} E[\| x_n(0) - x(0)\|^p] = 0$$

implies

$$(27)\quad \lim_{n\to\infty} E[\sup_{0\le t\le T} \| x_n(t) - x(t)\|^p] = 0. \qquad \blacksquare$$

For the proof of the Theorem 3, we prepare the following lemma.

Lemma 4. Under the conditions in Theorem 3, the relations

$$(28)\quad \lim_{n\to\infty} E[\| \sigma_n(t,x_n(t)) - \sigma(t,x(t))\|^p] = 0$$

and

$$(29)\quad \lim_{n\to\infty} E[\| b_n(t,x_n(t)) - b(t,x(t))\|^p] = 0$$

hold for $t \in [0,T]$.

Proof. We will show the relation (28) by the method of the reduction to absurdity.

Suppose that there exist a sub sequence $\{n_q\}$ of $\{n\}$ and a positive number $\gamma > 0$ such that

$$(30)\quad \lim_{n_q\to\infty} E\{\| \sigma_{n_q}(t,x_{n_q}(t)) - \sigma(t,x(t))\|^p] = \gamma.$$

Since we know by the Theorem 2 that $\lim_{n_q\to\infty} E[\| x_{n_q}(t) - x(t)\|] = 0$ holds, we can choose a sequence $\{n_k\} \subset \{n_q\}$ such that $x_{n_k}(t)$ converges to

$x(t)$ a.s.

We have

$$\| \sigma_{n_k}(t,x_{n_k}(t)) - \sigma(t,x(t)) \|$$

$$\leq \| \sigma_{n_k}(t,x_{n_k}(t)) - \sigma(t,x_{n_k}(t)) \|$$

$$+ \| \sigma(t,x_{n_k}(t)) - \sigma(t,x(t)) \| = J_1^{n_k} + J_2^{n_k}, \text{ say.}$$

By the condition (C) $J_1^{n_k}$ tends to zero as n_k goes to infinity. On the other hand by the condition (A') $J_2^{n_k}$ also tends to zero when n_k goes to infinity. Hence we have that $\| \sigma_{n_k}(t,x_{n_k}(t) - \sigma(t,x(t)) \|$ tends to zero a.s. as n_k goes to infinity.

Therefore using the lemma 2, we must conclude that

$$\lim_{n_k \to \infty} E[\| \sigma_{n_k}(t,x_{n_k}(t)) - \sigma(t,x(t)) \|^p] = 0.$$

But this relation is contradictory to (30).

By the similar way one can prove the relation (29), so we omit the proof of this part. Q.E.D.

Proof of the Theorem 3. There exists a constant $C(p,d)$ which depends on p and d such that

$$(31) \quad \| x_n(t) - x(t) \|^p \leq C(p,d)\{\| x_n(0) - x(0) \|^p$$

$$+ \sum_{i,k} \left| \int_0^t \{\sigma^i_{n,k}(s,x_n(s)) - \sigma^i_k(s,x(s))\} dB^k(s) \right|^p$$

$$+ \sum_i \left| \int_0^t \{b^i_n(s,x(s)) - b^i(s,x(s))\} ds \right|^p\}$$

$$= C(p,d)\{\| x_n(0) - x(0)\|^p + L_1(t) + L_2(t)\}, \quad \text{say.}$$

By the Burkholder type inequality* and Hölder's inequality, we have for $L_1(t)$ that

$$E[\sup_{0\le t\le T} L_1(t)] \le \sum_{i,k} E[(\int_0^T |\sigma^i_{n,k}(s,x_n(s)) - \sigma^i_k(s,x(s))|^2 ds)^{\frac{p}{2}}]$$

$$\le \sum_{i,k} T^{\frac{p-2}{2}} E[\int_0^T |\sigma^i_{n,k}(s,x_n(s)) - \sigma^i_k(s,x(s))|^p ds],$$

where we have used the fact that $p \ge 2$. Hence there exists a constant $C_2(p,d)$ such that

$$(32)\quad E[\sup_{0\le t\le T} L_1(t)] \le C_2(p,d)E[\int_0^T \| \sigma^n(s,x_n(s)) - \sigma(s,x(s))\|^p ds]$$

On the other hand we will evaluate $L_2(t)$. We have

$$E[\sup_{0\le t\le T} L_2(t)] \le \sum_i E[(\int_0^T |b^i_n(s,x_n(s)) - b^i(s,x(s))|ds)^p]$$

$$\le \sum_i E[T^{(p-1)}(\int_0^T |b^i_n(s,x_n(s) - b^i(s,x(s))|^p ds)].$$

Therefore one can choose a constant $C_3(p,d)$ such that

$$(33)\quad E[\sup_{0\le t\le T} L_2(t)] \le C_3(p,d)E[\int_0^T \| b_n(s,x_n(s)) - b(s,x(s))\|^p ds]$$

Then by (31), (32) and (33) we have

$$E[\sup_{0\le t\le T} \| x_n(t) - x(t)\|^p] \le C(p,d)E[\| x_n(0) - x(0)\|^p]$$

$$+ C(p,d)C_2(p,d)E[\int_0^T \| \sigma^n(s,x(s)) - \sigma(s,x(s))\|^p ds]$$

---

* See for example, pp 54-55 in ([4]).

$$+ C(p,d)C_3(p,d)E[\int_0^T \| b_n(s,x_n(s)) - b(s,x(s)) \|^p ds].$$

Thus, by lemma 4, the relation $\lim_{n\to\infty} E[\| x_n(0) - x(0)\|^p] = 0$ implies

(27) $$\lim_{n\to\infty} E[\sup_{0\le t\le T} \| x_n(t) - x(t)\|^p] = 0.$$ Q.E.D.

§3 Examples.

Example 1.

Consider the following one dimensional stochastic differential equations;

(1') $$x(t) = x(0) + \int_0^t \sigma(s,x(s))dB(s) + \int_0^t b(s,x(s))ds$$

and

(2') $$x_n(t) = x_n(0) + \int_0^t \sigma_n(s,x(s))dB(s) + \int_0^t b_n(s,x(s))ds \qquad n = 1,2,\ldots$$

Assume that the coefficients in (1') and (2') satisfy the conditions (A'), (B) and (C). Suppose further that the coefficient satisfy the following conditions (A.1) and (A.2).

(A.1) For any $T > 0$ and $r > 0$ there exists a non negative increasing function $\rho_{T,r}(u)$ defined on $[0,\infty)$ such that

$$|\sigma(t,x) - \sigma(t,y)| \le \rho_{T,r}(|x - y|) \qquad \text{for } (t,x,y) \in [0,T] \times D_r \times D_r$$

$$|\sigma_n(t,x) - \sigma_n(t,y)| \le \rho_{T,r}(|x - y|) \qquad \text{for } (t,x,y) \in [0,T] \times D_r \times D_r \quad n = 1,2,\ldots$$

and

(34) $$\int_{+0} \frac{du}{\rho_{T,r}^2(u)} = +\infty.$$

(A.2) There exists a positive constant $K_1 > 0$ such that

$$|b(t,x) - b(t,y)| \leq K_1|x - y| \quad \text{for } (t,x,y) \in [0,T] \times R^1 \times R^1$$

and

$$|b_n(t,x) - b_n(t,y)| \leq K_1|x - y| \quad \text{for } (t,x,y) \in [0,T] \times R^1 \times R^1,\ n = 1,2,\ldots$$

Suppose that we are given a solution of the equation (1') $\mathcal{X}(t) = (x(t),B(t))$ and a sequence of solutions of the equation (2') $\mathcal{X}_n(t) = (x_n(t),B(t))$ $n=1,2,\ldots$ such that they are defined on a same probability space with an increasing family of Borel fields $(\Omega, \mathcal{F}, P: \mathcal{F}_t)$ and they satisfy (6) and (7) for some integer $p \geq 2$.

Then, the relation $\lim_{n\to\infty} E[\| x_n(0) - x(0)\|^p] = 0$ implies

$$\lim_{n\to\infty} E[\sup_{0\leq t\leq T} \| x_n(t) - x(t)\|^p] = 0. \qquad \blacksquare$$

Remark.

Under the conditions (A'), (B), (A.1) and (A.2), it is well known that for a given probability space with an increasing family of Borel fields $(\Omega, \mathcal{F}, P: \mathcal{F}_t)$ and a given $\mathcal{F}_t$-Brownian motion $B(t)$, there exist a solution of the equation (1') $\mathcal{X}(t) = \{x(t),B(t)\}$ and a sequence of solutions of the equation (2) $\mathcal{X}_n(t) = \{x_n(t),B(t)\}$ both defined on the given probability space. (cf. [1] and [7]).

Proof. Let two constants $T > 0$ and $r > 0$ be fixed.

Choose the sequence $\{a_m\}_{m=1,2,\ldots} \subseteq (0,1)$ so that $a_m \downarrow 0$ as $m \to \infty$ and

$$\int_{a_m}^{a_{m-1}} \frac{du}{\rho_{T,r}^2(u)} = m, \qquad \text{for } m \geq 1.$$

Define a sequence of continuous functions $\{\phi''_m(u)\}_{m=1,2,\ldots}$ such that

$$\phi''_m(u) = \begin{cases} 0, & 0 \le u \le a_m \\ \text{between } 0 \text{ and } 2/m\rho^2_{T,r}(u), & a_m \le u \le a_{m-1} \\ 0, & a_{m-1} < u \end{cases}$$

and

$$\int_0^\infty \phi''_m(u)\,du = 1.$$

Set $\phi_m(x) = \int_0^{|x|} dv \int_0^v \phi''_m(u)\,du$. Then we have

(35) $|u| - a_m \le \phi_m(u) \le |u|$.

Set $V^m_{T,r}(t,x,y) = e^{-kt}\phi_m(x - y)$ and $V_{T,r}(t,x,y) = e^{-kt}|x - y|$, where k is a positive constant so that $k > K_1$.

Then it is seen clearly that the functions $V_{T,r}$ and $V^m_{T,r}$, $m=1,2,\ldots$ satisfy the conditions (V1) and (V2). Using (35), we can show that the condition (V3) is satisfied by them.

Put $\varepsilon_n = \sup_{(t,x)\in[0,T]\times D_r}\{|\sigma_n(t,x) - \sigma(t,x)| + |b_n(t,x) - b(t,x)|\}$. Then, we know by the condition (C) that $\lim_{n\to\infty} \varepsilon_n = 0$.

Choose a non decreasing sequence of integers $\{m_n\}_{n=1,2,\ldots}$ so that $\lim_{n\to\infty} m_n = \infty$ and

(36) $\varepsilon_n^2 \max_{a_{m_n} \le u \le a_{m_n - 1}} \dfrac{1}{\rho^2_{T,r}(u)} \le 1.$

Now, we will show that the functions $\{V^{m_n}_{T,r}\}_{m=1,2,\ldots}$ satisfy the condition (V4).

First, we have

$$(37)\quad E\left[\int_0^{t\wedge\tau_r^{(n)}} (\mathcal{L}V_{T,r}^{m_n})(s,x_n^r(s),x^r(s))ds\right]$$

$$= E\left[\int_0^{t\wedge\tau_r^{(n)}} e^{-ks}\{-k\phi_{m_n}(x^r(s) - x_n^r(s))\right.$$

$$+ \phi'_{m_n}(x^r(s) - x_n^r(s))(b(s,x^r(s)) - b_n(s,x_n^r(s)))$$

$$+ \frac{1}{2}\phi''_{m_n}(x^r(s) - x_n^r(s))(\sigma^2(s,x^r(s)) - 2\sigma(s,x^r(s))\sigma_n(s,x_n^r(s))$$

$$\left. + \sigma_n^2(s,x^r(s)))\}ds\right]$$

$$= E\left[\int_0^{t\wedge\tau_r^{(n)}} e^{-ks}\{-k\phi_{m_n}(x^r(s) - x_n^r(s)) + I_1(s) + I_2(s)\}ds\right], \quad \text{say.}$$

By the condition (A.2) we have for I_1

$$(38)\quad E\left[\int_0^{t\wedge\tau_r^{(n)}} e^{-ks}|I_1(s)|ds\right]$$

$$\leq E\left[\int_0^{t\wedge\tau_r^{(n)}} e^{-ks}|\phi'_{m_n}||b(s,x^r(s)) - b_n(s,x_n^r(s))|ds\right]$$

$$\leq E\left[\int_0^{t\wedge\tau_r^{(n)}} |b(s,x_n^r(s)) - b_n(s,x_n^r(s))|ds\right]$$

$$+ E\left[\int_0^{t\wedge\tau_r^{(n)}} |b(s,x_n^r(s)) - b(s,x^r(s))|ds\right]$$

$$\leq \varepsilon_n T + K_1 E\left[\int_0^{t\wedge\tau_r^{(n)}} |x_n^r(s) - x^r(s)|ds\right]$$

On the other hand, using the condition (A.1) and (36), we have for I_2

$$(39)\quad E\Big[\int_0^{t\wedge\tau_r^{(n)}} e^{-ks}|I_2(s)|ds\Big]$$

$$\le \frac{1}{2} E\Big[\int_0^{t\wedge\tau_r^{(n)}} \phi''_{m_n}(x^r(s) - x_n^r(s))\{\sigma_n(s,x_n^r(s)) - \sigma(s,x^r(s))\}^2 ds\Big]$$

$$\le E\Big[\int_0^{t\wedge\tau_r^{(n)}} \phi''_{m_n}(x^r(s) - x_n^r(s))\{\sigma_n(s,x_n^r(s)) - \sigma(s,x_n^r(s))\}^2 ds\Big]$$

$$+ E\Big[\int_0^{t\wedge\tau_r^{(n)}} \phi''_{m_n}(x^r(s) - x_n^r(s))\{\sigma(s,x_n^r(s)) - \sigma(s,x^r(s))\}^2 ds\Big]$$

$$\le E\Big[\int_0^T \frac{2\varepsilon_n^2}{m_n} \operatorname*{Max}_{a_{m_n}\le|x^r(s)-x_n^r(s)|\le a_{m_n-1}} \rho_{T,r}^{-2}(|x^r(s) - x_n^r(s)|)ds\Big]$$

$$+ E\Big[\int_0^T \frac{2}{m_n} \rho_{T,r}^{-2}(|x^r(s) - x_n^r(s)|)\rho_{T,r}^2(|x^r(s) - x_n^r(s)|)ds\Big]$$

$$\le \frac{4T}{m_n}.$$

By (37), (38) and (39), we observe that

$$(40)\quad E\Big[\int_0^{t\wedge\tau_r^{(n)}} (\mathcal{L} V_{T,r}^{m_n})(s,x^r(s),x_n^r(s))ds\Big]$$

$$\le \varepsilon_n T + K_1 E\Big[\int_0^{t\wedge\tau_r^{(n)}} |x^r(s) - x_n^r(s)|ds\Big]$$

$$- kE\Big[\int_0^{t\wedge\tau_r^{(n)}} \phi_{m_n}(x^r(s) - x_n^r(s))ds\Big] + \frac{4T}{m_n}$$

$$\le \varepsilon_n T + (K_1 - k)E\Big[\int_0^{t\wedge\tau_r^{(n)}} |x^r(s) - x_n^r(s)|ds\Big] + a_m kT$$

$$+ \frac{4T}{m_n},$$ where we have used (35).

Since we have chosen k so that $k > K_1$, we get from (40) that

$$\text{(V.4)} \quad \overline{\lim_{n\to\infty}}\, E\left[\int_0^{t\wedge\tau_r^{(n)}} (\mathcal{L} V_{T,r}^{m_n})(s, x^r(s), x_n^r(s))ds\right] \le 0 \quad \text{for } t \in [0,T].$$

Hence we can apply the Theorem 3 to this example. Q.E.D.

Example 2.

Consider the following one dimensional stochastic differential equations;

$$\text{(1")} \quad x(t) = x(0) + \int_0^t \sigma(x(s))dB(s) + \int_0^t b(x(s))ds,$$

and

$$\text{(2")} \quad x_n(t) = x_n(0) + \int_0^t \sigma_n(x_n(s))dB(s) + \int_0^t b_n(x_n(s))ds, \quad n = 1,2,\ldots.$$

Assume that the coefficients in (1") and (2") satisfy the conditions (A), (B) and (C). Suppose further that the coefficients satisfy the following conditions (B.1), (B.2) and (B.3).

(B.1) There exists a positive constant M such that

$$\sup_x |b(x)| < M$$

$$\sup_{x,n} |b_n(x)| < M$$

(B.2) There exists a non negative increasing function $\rho(u)$ defined on $[0,\infty)$ such that

$$|\sigma(x) - \sigma(y)| \le \rho(|x - y|), \qquad x,y \in R^1$$

$$|\sigma_n(x) - \sigma_n(y)| \leq \rho(|x - y|), \quad x,y \in R^1 \quad n = 1,2,\ldots$$

and

$$\int_{0+} \frac{du}{\rho^2(u)+u^2} = +\infty.$$

(B.3) There exists a positive constant $\delta > 0$ so that

$$\delta \leq \sigma(x) \leq M \qquad x \in R^1$$

and

$$\delta \leq \sigma_n(x) \leq M \qquad x \in R^1, \quad n = 1,2,\ldots$$

Suppose that we are given a solution of the equation (1") $\mathcal{X}(t) = \{x(t),B(t)\}$ and a sequence of solutions of the equation (2") $\mathcal{X}_n(t) = \{x_n(t),B(t)\}$ $n=1,2,\ldots$ such that

(i) they are defined on a same probability space with an increasing family of Borel fields $(\Omega,\mathcal{F},P:\mathcal{F}_t)$

(ii) $x(0) = x_n(0) = a$, a.s. $n = 1,2,\ldots.$

Then $\lim_{n\to\infty} E[|x_n(t) - x(t)|] = 0$ holds for $t \in [0,\infty)$. ∎

Remark. Under the conditions (A), (B), (B.1), (B.2) and (B.3), it is known that for a given probability space with an increasing family of Borel fields $(\Omega,\mathcal{F},P:\mathcal{F}_t)$ and a given $\mathcal{F}_t$-Brownian motion $B(t)$ there exist a solution of the equation (1") and a sequence of solutions of the equation (2") such that they are defined on the given probability space. (cf. (5) and (7)).

Let $P(t,x,dy)$ and $P_n(t,x,dy)$ be the transition probability measure of the process $x(t)$ and $x_n(t)$ respectively. It is well known that under the conditions in example 2, there exist $P(t,x,y)$ and $P_n(t,x,y)$ such that

$$P(t,x,dy) = P(t,x,y)dy$$

and

$$P_n(t,x,dy) = P_n(t,x,y)dy \quad \text{hold. (cf. (6)).}$$

For the proof of the claim of example 2, we shall prepare the following lemma.

Lemma 5. For any fixed $a \in R^1$ the system of functions

$$\{P(t,a,y), P_n(t,a,y),\ t \in [0,T],\ n=1,2,\ldots\}$$

is uniformly integrable with respect to $([0,T] \times R^1,\ dtdy)$.

Proof. By the Theorem 9.2.6 in ([6]), we have for each $q \geq 1$

$$(41)\quad \left(\int_{R^1} |P_n(s,a,y)|^q dy\right)^{1/q} \leq C(s \wedge 1)^{-\nu}$$

where (i) $\nu = \frac{3(q-1)}{2q}$ and (ii) C is a constant which depends δ, M, q and $\rho(u)$.

Put $q = 1+\varepsilon$, $0 < \varepsilon < \frac{2}{3}$. Then we have from (41) that

$$\int_0^T \int_{R^1} |P_n(s,a,y)|^{1+\varepsilon} dyds \leq C^{(1+\varepsilon)} \int_0^T (s \wedge 1)^{-\frac{3}{2}\varepsilon} ds.$$

Since $-\frac{3}{2}\varepsilon > -1$ we get from the above that

$$\int_0^T \int_{R^1} |P_n(s,a,y)|^{1+\varepsilon} dyds < C^{1+\varepsilon}\int_0^T (s \wedge 1)^{-\frac{3}{2}\varepsilon} ds < +\infty .$$

where the right hand side of the inequality does not depend on n.

This implies immediately that the system $\{P(t,a,y), P_n(t,a,y)\}$ is uniformly integrable. Q.E.D.

We are now in a position to prove the claim in Example 2.

Set

$$C_m(x) = m\int_x^{x+\frac{1}{m}} b(y)dy, \quad m = 1,2,\ldots$$

and

$$C_{n,m}(x) = m\int_x^{x+\frac{1}{m}} b_n(y)dy, \quad m = 1,2,\ldots.$$

Clearly the functions $C_m(x)$ and $C_{n,m}(x)$ are bounded and continuous.

Define

$$f_m(x) = -2\int_0^x \frac{C_m(y)}{\sigma^2(y)} dy.$$

Then $f_m(x)$ is continuously differentiable function.

We will show the following inequalities.

$$(42)\quad C^{-1}(r)|x - y| \le \left|\int_x^y e^{f_m(u)} du\right| \le C(r)|x - y|, \quad \text{for } x,y \in D_r$$

where $C(r)$ is a positive constant which depends on r.

To this end, we note that

$$(43)\quad |C_m(x)| = m\left|\int_x^{x+\frac{1}{m}} b(y)dy\right| \le M, \quad \text{for } x \in R^1$$

and

$$(44)\quad |f_m(x)| \leq 2 \sup_x \left| \int_0^x \frac{C_m(y)}{\sigma^2(y)}\, dy \right| \leq \frac{2}{\delta^2} \sup_x \left| \int_0^x C_m(y)\, dy \right| \leq \frac{2M}{\delta^2}\, r$$

for $x \in D_r$, where we have used (B.1) and (B.3).

Since $\left| \int_x^y e^{f_m(u)}\, du \right| = |x - y| \, |e^{f_m(\xi)}|$, $y \leq \xi \leq x$, we have from (43) and (44) that

$$(45)\quad e^{\frac{-2rM}{\delta^2}} |x - y| \leq \left| \int_y^x e^{f_m(u)}\, du \right| \leq e^{\frac{2rM}{\delta^2}} |x - y|.$$

Put $C(r) = e^{\frac{2rM}{\delta^2}}$. Then (45) implies (42).

Let $\tilde{\rho}(u) = (\rho^2(u) + u^2)^{\frac{1}{2}}$. Then by the condition (B.2), we have $\int_{0+} \frac{1}{\tilde{\rho}^2(u)}\, du = +\infty$.

Choose the sequence $\{a_m\}_{m=0,1,\ldots} \subseteq (0.1)$ so that $a_m \downarrow 0$ and

$$\int_{a_m}^{a_{m-1}} \frac{1}{\tilde{\rho}^2(u)}\, du = m, \qquad m = 1,2,\ldots.$$

Define a sequence of continuous functions $\{\phi''_m(u)\}$ $m = 1,2,\ldots$ such that

$$\phi''_m(u) = \begin{cases} 0, & 0 \leq u \leq a_m \\ \text{between } 0 \text{ and } 2/m\tilde{\rho}^2(u), & a_m < u < a_{m-1} \\ 0, & a_{m-1} \leq u \end{cases}$$

and

$$\int_0^\infty \phi''_m(u)\, du = 1.$$

Set $\phi_m(x) = \int_0^{|x|} dv \int_0^v \phi''_m(u)du.$

Define $V^m_{T,r}(t,x,y) = \phi_m(C(r)\int_y^x e^{f_m(u)} du)$ for $(t,x,y) \in [0,T] \times D_r \times D_r$.

We will show that the sequence of functions $V^m_{T,r}(t,x,y)$ $m = 1,2,\ldots$ satisfy the condition (V.4).

To this end, we put

$$\varepsilon_n = \sup_{x\in D_r} \{|\sigma_n(x) - \sigma(x)| + |b_n(x) - b(x)|\}.$$

Choose a non decreasing sequence of integers $\{m_n\}_{n=1,2,\ldots}$ so that

(i) $\lim_{n\to\infty} m_n = +\infty$

and

(ii) $\varepsilon_n^2 \max_{a_{m_n} \le u \le a_{m_n-1}} \frac{1}{\tilde{\rho}^2(u)} \le 1$

We have

$$(46)\quad (\mathcal{L}V^{m_n}_{T,r})(t,x,y) = \frac{\partial V^{m_n}}{\partial x}(b(x) - C_{m_n}(x))$$
$$+ \frac{\partial V^{m_n}_{T,r}}{\partial y}(b_n(y) - C_{n,m_n}(x))$$
$$+ \frac{\partial V^{m_n}_{T,r}}{\partial y}\{C_{n,m_n}(y) - C_{m_n}(y)\frac{\sigma_n^2(y)}{\sigma^2(y)}\}$$
$$+ \frac{1}{2}\phi''_{m_n}(C(r)\int_y^x e^{f_{m_n}(u)} du)C^2(r)\{e^{f_{m_n}(x)}\sigma(x) - e^{f_{m_n}(y)}\sigma_{m_n}(y)\}^2$$
$$= I_1^{m_n} + I_2^{m_n} + I_3^{m_n} + I_4^{m_n}, \text{ say.}$$

We will treat the term $I_4^{m_n}$. Note that

$$|e^{f_{m_n}(x)} - e^{f_{m_n}(y)}| \leq |x - y||f'_m(\xi)|e^{f_{m_n}(\xi)}$$

$$\leq 2|x - y||\frac{C_{m_n}(\xi)}{\sigma^2(\xi)}|e^{f_{m_n}(\xi)} \leq 2|x - y| \frac{M}{\delta^2} C(r), \quad x,y \in D_r.$$

Then we have

$$(47) \quad |e^{f_{m_n}(x)} \sigma(x) - e^{f_{m_n}(y)} \sigma_n(y)|$$

$$\leq e^{f_{m_n}(x)} |\sigma(x) - \sigma(y)| + \sigma(y)|e^{f_{m_n}(x)} - e^{f_{m_n}(y)}|$$

$$+ e^{f_{m_n}(y)} |\sigma(y) - \sigma_n(y)|$$

$$\leq C(r)\rho(|x - y|) + 2 \frac{M^2}{\delta^2} C(r)|x - y| + C(r)|\sigma(y) - \sigma_n(y)|.$$

Hence there exists a positive number $\tilde{C}(r)$ such that

$$(48) \quad |e^{f_{m_n}(x)} \sigma(x) - e^{f_{m_n}(y)} \sigma_n(y)|^2 \leq \tilde{C}(r)(|x - y|^2 + \rho^2(|x - y|) + \varepsilon_n^2).$$

Thus we have from (48)

$$(49) \quad I_4^{m_n}$$

$$\leq \frac{1}{2} C^2(r)\frac{2}{m_n} \max_{a_{m_n} \leq |C(r)\int_y^x e^{f_{m_n}(u)} du| \leq a_{m_n-1}} \frac{1}{\tilde{\rho}^2(C(r)\int_y^x e^{f_{m_n}(u)} du)}$$

$$\times \tilde{C}(r)(\tilde{\rho}^2(|x - y|) + \varepsilon_n^2)$$

$$\leq \frac{1}{2} C^2(r) \frac{1}{m_n} \max_{a_{m_n} \leq |C(r)\int_y^x e^{f_{m_n}(u)} du| \leq a_{m_n-1}} \frac{1}{\tilde{\rho}^2(C(r)\int_y^x e^{f_{m_n}(u)} du)}$$

$$\times \tilde{C}(r)\{\tilde{\rho}^2(|C(r)\int_y^x e^{f_{m_n}(u)} du|) + \varepsilon_n^2\}$$

$$\leq \frac{3}{2}\,\frac{C^2(r)\tilde{C}(r)}{m_n}\,.$$

Now, we are going to evaluate the term $I_3^{m_n}$. Since there exists a positive constant $C_2(r)$ so that

$$(50)\quad \left|\frac{\partial V_{T,r}^{m_n}}{\partial x}(t,x,y)\right| \leq C_2(r) \qquad \text{and}$$

$$\left|\frac{\partial V_{T,r}^{m_n}}{\partial y}(t,x,y)\right| \leq C_2(r),\ n = 1,2,\ldots \qquad (x,y) \in D_r \times D_r,$$

we have for $I_3^{m_n}$ that

$$(51)\quad |I_3^{m_n}| \leq C_2(r)|C_{n,m_n}(y) - C_{m_n}(y)|$$

$$+ C_2(r)\left|C_{m_n}(y)\frac{\sigma^2(y) - \sigma_n^2(y)}{\sigma^2(y)}\right|$$

$$\leq C_2(r)|C_{n,m_n}(y) - C_m(y)| + C_2(r)\frac{2M^2}{\delta^2}|\sigma(y) - \sigma_n(y)|$$

$$\leq C_2(r)|C_{n,m_n}(y) - C_m(y)| + C_2(r)\frac{2M^2}{\delta^2}\,\varepsilon_n$$

Thus, by (46), (49) and (51) we have

$$(52)\quad \overline{\lim_{n\to\infty}}\, E\left[\int_0^{t\wedge\tau_r^{(n)}} (\mathcal{L}V_{T,r}^{m_n})(s,x^r(s),x_n^r(s))ds\right]$$

$$\leq \overline{\lim_{n\to\infty}}\, E\left[\int_0^{t\wedge\tau_r^{(n)}} \frac{\partial V_{T,r}^{m_n}}{\partial x}(x^r(s),x_n^r(s))(b(x^r(s)) - C_{m_n}(x^r(s)))ds\right]$$

$$+ \overline{\lim_{n\to\infty}}\, E\left[\int_0^{t\wedge\tau_r^{(n)}} \frac{\partial V_{T,r}^{m_n}}{\partial y}(x^r(s),x_n^r(s))(b_n(x_n^r(s)) - C_{n,m_n}(x_n^r(s)))ds\right]$$

$$+ \overline{\lim_{n\to\infty}}\left(T\cdot\frac{\frac{2}{3}C(r)\tilde{C}(r)}{m_n} + TC_2(r)\frac{2M^2}{\delta^2}\,\varepsilon_n\right)$$

$$+ C_2(r)\overline{\lim_{n\to\infty}} E\Big[\int_0^{t\wedge\tau_r^{(n)}} |C_{n,m_n}(x_n^r(s)) - C_{m_n}(x_n^r(s))|ds\Big].$$

Notincing that

$$(53)\quad |C_{n,m_n}(x) - C_{m_n}(x)| \le m_n\Big|\int_x^{x+\frac{1}{m_n}} (b_n(y) - b(y))dy\Big| \le \varepsilon_n$$

we get from (52)

$$(54)\quad \overline{\lim_{n\to\infty}} E\Big[\int_0^{t\wedge\tau_r^{(n)}} (\mathcal{L}V_{T,r}^{m_n})(x^r(s),x_n^r(s))ds\Big]$$

$$\le C_2(r)\overline{\lim_{n\to\infty}} E\Big[\int_0^{t\wedge\tau_r^{(n)}} |b(x^r(s)) - C_{m_n}(x^r(s))|ds\Big]$$

$$+ C_2(r)\overline{\lim_{n\to\infty}} E\Big[\int_0^{t\wedge\tau_r^{(n)}} |b_n(x_n^r(s)) - C_{n,m_n}(x_n^r(s))|ds\Big]$$

$$\le C_2(r)\overline{\lim_{n\to\infty}} \int_{R^1}\int_0^T |b(y) - C_{m_n}(y)|P(s,a,y)dy$$

$$+ C_2(r)\overline{\lim_{n\to\infty}} \int_{R^1}\int_0^T |b_n(y) - C_{n,m_n}(y)|P_n(s,a,y)dy.$$

Use the fact that

$$|b_n(y) - C_{n,m_n}(y)| \le |b_n(y) - b(y)| + |b(y) - C_{m_n}(y)|$$

$$+ |C_{m_n}(y) - C_{n,m_n}(y)| \le 2\varepsilon_n + |b(y) - C_{m_n}(y)|.$$

Then we get from (54)

$$(55)\quad \overline{\lim_{n\to\infty}} E\Big[\int_0^{t\wedge\tau_r^{(n)}} (\mathcal{L}V_{T,r}^{m_n})(x^r(s),x_n^r(s))ds\Big]$$

$$\le C_2(r)\overline{\lim_{n\to\infty}} \int_{R^1}\int_0^T |b(y) - C_{m_n}(y)|P(s,a,y)dy$$

$$+ C_2(r)\overline{\lim_{n\to\infty}} \int_{R^1} \int_0^T |b(y) - C_{m_n}(y)| P_n(s,a,y)dy$$

$$+ 2C_2(r)T\cdot\varepsilon_n.$$

Note that

(i) $|b(y) - C_{m_n}(y)| \leq 2M$

and

(ii) $C_{m_n}(y)$ converges to $b(y)$ a.e..

Then, using the lemma 5 we obtain from (55) that

$$\overline{\lim_{n\to\infty}} E[\int_0^{t\wedge\tau_r^{(n)}} (\mathcal{L} V_{T,r}^{m_n})(x^r(s), x_n^r(s))ds] \leq 0 \quad \text{holds.}$$

Clearly the functions $V_{T,r}$ and $V_{T,r}^m$ $m=1,2,\ldots$ satisfy the conditions (V.1), (V.2) and (V.3). Hence we can apply the Theorem 1 and 2 to this example. Q.E.D.

References

(1) Ikeda, N., Watanabe, S. : Stochastic Differential Equations and Diffusion Processes, North-Holland/Kodansha, Amsterdam, Oxford, New York, Tokyo, (1981).

(2) Itô, K. : On stochastic differential equations, Mem. Amer. Math. Soc. 4, (1951).

(3) Maruyama, G. : Continuous Markov processes and stochastic equations, Rend. Circ. Mat. Palermo, Ser. 2, T. 4, 48-90 (1955).

(4) Meyer, P. A., Priouret, P., Spitzer, F. : Ecole d'Eté de Probabilités de Saint-Flour 1973, Lecture Notes in Math. Vol. 390, Springer-Verlag, Berlin, Heidelberg, New York, (1974).

(5) Okabe, Y., Shimizu, A. : On the pathwise uniqueness of solutions of stochastic differential equations, J. Math. Kyoto Univ. Vol. 15, No. 2, 455-466 (1975).

(6) Stroock, D. W., Varadhan, S. R. S. : Multidimensional Diffusion Processes, Springer-Verlag, Berlin, Heidelberg, New York, (1979).

(7) Yamada, T., Watanabe, S. : On the uniqueness of solutions of stochastic differential equations, J. Math. Kyoto Univ. Vol. 11, No. 1, 155-167 (1971).