SÉMINAIRE DE PROBABILITÉS (STRASBOURG)

DAVID J. ALDOUS
MARTIN T. BARLOW

On countable dense random sets

Séminaire de probabilités (Strasbourg), tome 15 (1981), p. 311-327
<http://www.numdam.org/item?id=SPS_1981__15__311_0>

On Countable Dense Random Sets
by

D. J. Aldous and M. T. Barlow

We shall discuss point processes whose realisations consist typically of a countable dense set of points. In particular, we discuss when such a process may be regarded as Poisson.

The most primitive way to describe a point process on $[0,\infty)$ is as a subset B of $\Omega \times [0,\infty)$, where the section B_ω represents the times of the "points" in realisation ω. In the locally finite case, there are the more familiar descriptions using the counting process

$$N_t(\omega) = \#(B_\omega \cap [0,t]) \qquad \text{(as in [BJ])}$$

or using the random measure

$$\xi(\omega,D) = \#(B_\omega \cap D) \qquad \text{(as in [K])}$$

Our point processes will generally not be locally finite, so we cannot use these familiar descriptions: we revert to describing a process as a subset B.

We first describe an (obvious) construction of a countable dense Poisson process. Let Θ be a countable infinite set. Let (F_t) be a filtration (all filtrations are assumed to satisfy the usual conditions). Suppose $\{S_i^\theta : i \geq 1, \theta \in \Theta\}$ are optional times such that each counting process $N_t^\theta = \sum_i 1_{(S_i^\theta \leq t)}$ is a Poisson process of rate 1 with respect to (F_t), and suppose the process N^θ are independent. Let ξ be the random measure on $\Theta \times [0,\infty)$ whose realisation $\xi(\omega)$ has the set of atoms $\{(\theta, S_i^\theta(\omega)) : i \geq 1, \theta \in \Theta\}$. Then ξ describes a uniform Poisson process on $\Theta \times [0,\infty)$, with respect to (F_t). But we can also think of ξ as a marked point process on the line. That is, each realisation is an a.s. countable dense set $\{S_i^\theta(\omega) : i \geq 1, \theta \in \Theta\}$ of points in $[0,\infty)$, and each point is marked by some θ. The corresponding unmarked process can be described by

(1) $$B = \{(\omega,t) : S_i^\theta(\omega) = t \text{ for some } i,\theta\} = \{(\omega,t) : \xi(\omega,\Theta\times\{t\})=1\} .$$

Think of B as a σ-finite Poisson process. We are concerned with the converse procedure: given a set B, when can we assign marks θ to the points of B to construct a uniform Poisson process ξ satisfying (1)? To allow external randomisation in assigning marks, we make the following definitions:

(2) <u>Definition</u>. (G_t) is an <u>extension</u> of (F_t) if for each t

(i) $G_t \supset F_t$

(ii) G_t and F_∞ are conditionally independent given F_t

(3) <u>Definition</u> B is a <u>σ-finite Poisson process</u> with respect to (F_t) if

(i) B is (F_t)-optional

(ii) There exists a uniform Poisson process ξ on $\Theta \times [0,\infty)$ with respect to some extension (G_t) of (F_t) such that (1) holds.

Theorem 4 below gives a more intrinsic description of σ-finite Poisson processes. First we recall some notation. An optional time T has <u>conditional intensity</u> $a(\omega,s)$ if T has compensator $A_t = \int_0^t a(s)ds$. We may assume $a(\omega,s)$ is previsible by [D.V. 19]. Replacing (F_t) by an extension does not alter the conditional intensity of an (F_t)-optional time T.

Recall also the notation

$$\begin{aligned} T_D &= T \quad \text{on } D \\ &= \infty \quad \text{elsewhere.} \end{aligned}$$

Let λ be Lebesgue measure on $[0,\infty)$.

(4) THEOREM. Let (F_t) be a filtration. Let B be an optional set whose sections B_ω are a.s. countable. The following are equivalent

(a) B is a σ-finite Poisson process

(b) There exists a family (T^n) such that

(5) T^n is optional; the graphs $[T^n]$ are disjoint; $B = \cup[T^n]$ a.s.;

(6) T^n has a conditional intensity, say $a_n(\omega,s)$;

(7) $\sum_n a_n(\omega,s) = \infty$ a.e. $(P \times \lambda)$

(b') Every family (T^n) satisfying (5) also satisfies (6) and (7)

(c) For every previsible set C

$$\{\omega : C_\omega \cap B_\omega = \emptyset\} = \{\omega : \lambda(C_\omega) = 0\} \quad \text{a.s.}$$

Remark Families satisfying (5) certainly exist, by the section theorem and transfinite induction [D. VI. 33] .

The next result comes out of the proof of Theorem 4.

(8) PROPOSITION. Let μ be a probability measure on $[0,\infty)$ which is equivalent to Lebesgue measure .

(a) Let (Y_i) be i.i.d. with law μ , and let (F_t) be the smallest filtration making each Y_i optional - that is, the filtration generated by the processes $1_{[Y_i,\infty)}$. Then $B = \cup[Y_i]$ is a σ-finite Poisson process with respect to (F_t).

(b) Conversely, let B be a σ-finite Poisson process with respect to some (F_t) . Then there exist times (Y_i) such that $B = \cup[Y_i]$ a.s., (Y_i) are i.i.d. with law μ , and (Y_i) are optional with respect to some extension of (F_t) .

Before the proofs, here is an amusing example.

Example There exists a process X_t and filtrations (F_t) , (G_t) such that X is optional with respect to each of (F_t) and (G_t) , but X is not optional with respect to $F_t \cap G_t$.

To construct the example, let (Y_i),B,(F_t) be as in part (a) of Proposition 8, and let $X = 1_B$. Let Π be the set of finite permutations $\pi = (\pi(1), \pi(2), \ldots)$ of $(1, 2, \ldots)$. Since Π is countable we can construct a random element π^* of Π such that $P(\pi^* = \pi) > 0$ for each $\pi \in \Pi$. Take π^* independent of $\underset{\sim}{Y} = (Y_1, Y_2, \ldots)$. Define $\underset{\sim}{V} = (V_1, V_2, \ldots) = (Y_{\pi^*(1)}, Y_{\pi^*(2)}, \ldots)$. Let (F_t) be the smallest filtration making each V_i optional. Since $X_t = \Sigma 1_{(Y_i = t)} = \Sigma 1_{(V_i = t)}$, plainly X is both (F_t)- and (G_t)-optional. But $F_\infty \cap G_\infty$ is trivial! For let $D \in F_\infty \cap G_\infty$. Then there exist measurable functions f, g such that

$$1_D = f(\underset{\sim}{Y}) = g(\underset{\sim}{V}) \quad \text{a.s.}$$

So $f(\underset{\sim}{Y}) = h(\underset{\sim}{Y}, \pi^*)$ a.s., where $h(y_1 y_2, \ldots ; \pi) = g(y_{\pi(1)}, y_{\pi(2)}, \ldots)$ But π^* is independent of $\underset{\sim}{Y}$ with support Π, so

$$f(\underset{\sim}{Y}) = h(\underset{\sim}{Y}, \pi) \quad \text{a.s., each } \pi \in \Pi .$$

So, putting $G = \{g = 1\}$,

$$D = \{(Y_{\pi(1)}, Y_{\pi(2)}, \ldots) \in G\} \quad \text{a.s., each } \pi \in \Pi .$$

Thus D is exchangeable, and so is trivial by the Hewitt-Savage zero-one law.

We now start the proof of Theorem 4. The lemma below shows that (b) and (b') are equivalent.

(9) LEMMA. Let (T^n) be optional times whose graphs $[T^n]$ are disjoint. Let $(\hat{T}^m)$ be a similar family, and suppose $U[T^n] = U[\hat{T}^m]$. Suppose T^n has conditional intensity a_n.

Then $\hat{T}^m$ has a conditional intensity, $\hat{a}_m$ say, and $\Sigma\hat{a}_m = \Sigma a_n$ a.e. $(P \times \lambda)$.

<u>Proof</u> Put $U_{m,n} = T^n{}_{(T^n=\hat{T}^m)}$. Then $U_{m,n}$ has a conditional intensity, $a_{m,n}$ say. It is easy to verify

$$a_n = \sum_m a_{m,n} \quad \text{a.e.}$$

$\hat{a}_m \equiv \sum_n a_{m,n}$ is the conditional intensity of $\hat{T}^m$, where the sum is a.e. finite because

$$E\int \sum_{n=1}^{N} a_{m,n}(s)\, ds = \sum_{n=1}^{N} P(U_{m,n} < \infty) \le P(T^m < \infty) \le 1 .$$

Hence $\Sigma a_n = \Sigma\Sigma a_{m,n} = \Sigma\hat{a}_m < \infty$ a.e.

Lemmas 10 and 13 show that conditions (b') and (c) are equivalent.

(10) LEMMA. For B as in theorem 4, the following are equivalent

(i) $\{\omega : C_\omega \cap B_\omega = \emptyset\} \supset \{\omega : \lambda(C_\omega) = 0\}$ a.s., each previsible C .

(ii) Each family (Y^n) satisfying (5) also satisfies (6).

<u>Proof: (ii) implies (i)</u> Let C be previsible. Put $T = \inf\{t : \lambda(C_\omega \cap [0,t]) > 0\}$. Then T is optional, so $C' = C \cap [0,T]$ is previsible. Now $\lambda(C'_\omega) = 0$ a.s. We must prove

(11) $C'_\omega \cap B_\omega = \emptyset$ a.s.

Let (T^n) satisfy (5) and (6). Then

$$P(T^n \in C'_\omega) = E\int 1_{C'}\, dl_{[T^n,\infty)}$$

$$= E\int 1_{C'}(s)\ a_n(s)\ ds$$

$$= 0 \ .$$

Since $B = \cup[T^n]$, (11) follows.

<u>(i) implies (ii)</u>. Let T be optional, $[T] \subset B$. Let A_t be the compensator of T . From the proof of the Lebesgue decomposition theorem, we can write $A_t = \hat{A}_t + \int_0^t a(s)ds$, where there exists a progressive set D such that

(12) $\lambda(D_\omega) = 0$ a.s.; the measure $d\hat{A}(\omega)$ is carried on D_ω a.s.

Let $C = \{{}^p(1_D) > 0\}$; then C is previsible and since

$$\hat{A}_t \geq \int_0^t 1_C(s)d\hat{A}_s \geq \int_0^t {}^p(1_D)(s)d\hat{A}_s = \int_0^t 1_D(s)d\hat{A}_s = \hat{A}_t \ ,$$

and

$$\int_0^t {}^p(1_D)(s)ds = \int_0^t 1_D(s)ds = 0 \ ,$$

C satisfies (12). However

$$E\hat{A}_\infty = E\int 1_C(s)d\hat{A}_s$$

$$= E\int 1_C(s)dA_s$$

$$= P(T \in C_\omega) = 0 \ \text{ by (11).}$$

So $\hat{A} \equiv 0$.

(13) LEMMA. For B as in Theorem 4, the following are equivalent.

(i) $\{\omega : C_\omega \cap B_\omega \neq \emptyset\} \supset \{\omega : \lambda(C_\omega) > 0\}$ a.s., each previsible C.

(ii) Each family (T^n) satisfying (5) and (6) also satisfies (7).

<u>Proof</u>. <u>(ii) implies (i)</u> Let C be previsible. Define optional times :

$$T = \inf \{t : \lambda(C_\omega \cap [0,t]) > 0\}$$

$$S = \inf \{t : t \in B_\omega \cap C_\omega\}.$$

It is sufficient to prove

(14) $S \leq T$ a.s.

Consider the previsible set $C' = C \cap (T,S]$

Let (T^n) satisfy (5), (6). By definition of S , the sets $\{\omega : T^n \in C'_\omega\}$ are disjoint. So $\sum_n P(T^n \in C'_\omega) \leq 1$. But

$$\begin{aligned}\Sigma\, P(T^n \in C'_\omega) &= \Sigma E\int 1_{C'}\, d1_{[T^n,\infty)} \\ &= \Sigma E\int 1_{C'}(s)a_n(s)ds \\ &= \Sigma E\int 1_{C'}(s)\ \Sigma a_n(s)\ ds\ .\end{aligned}$$

But $\Sigma a_n = \infty$ a.e., and so $\lambda(C'_\omega) = 0$ a.s. But by definition of T we have $\lambda(C'_\omega) > 0$ on $\{T < S\}$. This proves 14.

<u>(i) implies (ii)</u> Let (T_n) satisfy (5), (6). Fix $N < \infty$. Consider the previsible set $H = \{(\omega,s): \Sigma a_n \le N-1\}$. We must prove $P \times \lambda(H) = 0$. Suppose not : then for some $\varepsilon > 0$ we have

$$P(\Omega_0) \geq \varepsilon \text{ , where } \Omega_0 = \{\omega : \lambda(H_\omega) > \varepsilon\}$$

Define optional times

$$S_i = \inf\{t : \lambda(H_\omega \cap [0,t]) > i\varepsilon/N\} \qquad i = 0,\dots,N .$$

Consider the previsible sets

$$H^i = H \cap (S_{i-1}, S_i] \qquad i = 1,\dots,N$$

$$\bar{H} = H \cap (S_0, S_n] .$$

By construction, $\lambda(H^i_\omega) = \varepsilon/N$ on Ω_0. So by (i), $B_\omega \cap H^i_\omega$ is a.s. non-empty on Ω_0. So

$$E \sum_n 1_{(T_n \in \bar{H}_\omega)} = E \sum_{in}\sum 1_{(T_n \in H^i_\omega)} \gtrsim N\, P(\Omega_0) \gtrsim N\varepsilon$$

But $E \sum_n 1_{(T_n \in \bar{H}_\omega)} = E\sum_n \int 1_{\bar{H}}\, dI_{[T_n,\infty)}$

$$= E \int 1_{\bar{H}}(s) \cdot \Sigma a_n(s)\, ds$$

$$\le (N-1)\, \varepsilon$$

because $\Sigma a_n \le N-1$ on H, and $\lambda(\bar{H}_\omega) \le \varepsilon$ by construction. This contradiction establishes the result.

It remains to prove that (b) and (a) are equivalent. Recall from [BJ] that optional times $0<S_1<S_2< \dots$ form a Poisson process of rate 1 with respect to (F_t) iff S_n has conditional intensity $1_{(S_{n-1} < s \le S_n)}$. If moreover this condition holds for each family $(S_i^\theta)_{i\geq 1}$, $\theta\in\Theta$, and if the graphs $\{[S_i^\theta] : i\geq 1, \theta\in\Theta\}$ are disjoint, then the families $\{(S_i^\theta)_{i\geq 1} : \theta\in\Theta\}$ are independent.

The proof that (a) implies (b) is easy. The family (S_i^θ) in (1) plainly satisfies the conditions of (b) with respect to the extension (G_t). Because (b) implies (b'), we deduce that any (G_t)-optional family satisfying (5) will also satisfy (6) and (7) with respect to (G_t). Now, as remarked before, there exists a family satisfying (5) with respect to (F_t); and since conditional intensities are unchanged by extension, this family satisfies (6) and (7) with respect to (F_t).

The proof that (b) implies (a) is harder. There are only two ideas. First, we show how to construct S_1 with $[S_1] \subset B$ such that S_1 has exponential law (Lemma 19). Then we can proceed inductively to construct a uniform Poisson process (S_i^θ). Finally, we must show that $\bigcup_{i,\theta} [S_i^\theta]$ exhausts B.

Here is a straightforward technical lemma.

(14) LEMMA. Let (Q_i) be optional times with conditional intensities a_i. Suppose $Q_i \to \infty$ a.s. and $[Q_i]$ are disjoint. Let $T= \min(Q_i)$. Then

$T_{(T=Q_i)}$ has conditional intensity $a_i 1_{(s\le T)}$

T has conditional intensity $\Sigma a_i\, 1_{(s\le T)}$.

Here is an informal description of the external randomisation. Suppose

(15) T is optional, with conditional intensity a,

$p(\omega,s)$ is previsible, $0\le p\le 1$.

Then we can define Q such that:

$$\text{if } T = t \text{ then } Q = t \text{ with probability } p(\omega,t)$$
$$= \infty \text{ otherwise}$$

It is intuitively obvious that Q has conditional intensity p.a. Here is the formal construction and proof.

(16) LEMMA. Let T,a,p be as in (15), on a filtration $(\hat{F}_t)$. Let U be uniform on $[0,1]$, independent of $\hat{F}_\infty$. Define

$$Q = T \text{ if } U \le p(T) \equiv p(\omega,T(\omega))$$
$$= \infty \quad \text{otherwise.}$$

Let G_t be the usual augmentation of $G_t^0 = \sigma(\hat{F}_t, Q_{(Q\le t)})$. Then (G_t) is an extension of $(\hat{F}_t)$, and Q is (G_t)-optional with conditional intensity p.a.

<u>Proof</u> $Q_{(Q\le t)} \in \sigma(\hat{F}_t, U)$, and hence $G_t^0 \subset \sigma(\hat{F}_t, U)$, so (G_t) is indeed an extension of $(\hat{F}_t)$. Plainly Q is (G_t)-optional. To prove the final assertion, let $S<\infty$ be a (G_t)-optional time. It is sufficient to prove

(17) $$P(Q \le S) = E\int_0^S a(s)p(s)ds \, .$$

We assert

(18) $R = S_{(S<T)}$ is (F_t)-optional.

For $\{R<u\} = \bigcup_{\substack{t<u \\ t \text{ rational}}} \{S<t<T\}$, and $\{S<t<T\}$ is in F_t since $G_t \cap \{T>t\} = F_t \cap \{T>t\}$. To prove (17), note that $\{Q\leq S\} = \{T\leq S, Q<\infty\} = \{T\leq R, Q<\infty\} = \{T\leq R, T<\infty, U\leq p(T)\}$. So

$$\begin{aligned} P(Q\leq S) &= P(T\leq R, T<\infty, U\leq p(T)) \\ &= E(1_{(T\leq R, T<\infty)}\, P(U\leq p(T) \mid F_\infty)) \\ &= E\, 1_{(T\leq R, T<\infty)}\, p(T) \quad \text{by the independence of } U \\ &= E \int 1_{(s\leq R)} p(s)\, d1_{[T,\infty)} \\ &= E \int 1_{(s\leq R)} p(s)\, a(s)\, ds\,. \end{aligned}$$

(17) now follows, as $[S,R] \subset [T,\infty)$, and $a = 0$ on this set.

(19) LEMMA. Let $(\hat{F}_t)$ be an extension of (F_t). Suppose (T^n) satisfies condition (b) with respect to $(\hat{F}_t)$. Let $S_0<\infty$ be $(\hat{F}_t)$-optional. Then there exists an extension (G_t) of $(\hat{F}_t)$ and a (G_t)-optional time S with conditional intensity $1_{(S_0<s\leq S)}$ such that $[S] \subset \bigcup [T^n]$.

<u>Proof</u> Define $$\phi(x) = \begin{cases} 1 & x \geq 1 \\ x & 0 \leq x \leq 1 \\ 0 & x \leq 0 \end{cases}$$

Define inductively

$$p_1(\omega,s) = \phi\left(\frac{1}{a_1(\omega,s)}\right) 1_{(s>S_0)}$$

$$p_j = \phi\left(\frac{1 - \sum_1^{j-1} a_i\, p_i}{a_j}\right) 1_{(s>S_0)}$$

Then p_j is predictable, $0 \leq p_j \leq 1$, and

$$(20) \quad \sum_1^N a_j p_j = \left(1 \wedge \sum_1^N a_j\right) \cdot 1_{(s>S_0)}$$

By Lemma 16 we can construct extensions (G_t^j) of (F_t) and (G_t^j)-optional times Q_j such that

$[Q_j] \subset [T^j]$,

Q_j has conditional intensity $p_j a_j$.

Then

$$\sum_j P(Q_j < t) = \sum_j E\int_0^t p_j(s) a_j(s)\, ds$$

$$= E\int_0^t \Sigma a_j(s) p_j(s)\, ds$$

$$\leq t \quad \text{by (20)}.$$

By the Borel-Cantelli lemma, $Q_j \to \infty$ a.s.

Set $S = \min(Q_j)$, and let (G_t) be the filtration generated by $(G_t^j, j \geq 1)$. By Lemma 14, S has conditional intensity $\Sigma a_j p_j \ 1_{(s \leq S)}$, and by (20) this equals $1_{(S_0 < s \leq S)}$.

For later use, note that, by Lemma 14, $S_{(S=T^n)}$ has conditional intensity $p_n a_n \ 1_{(s \leq S)}$. In other words, using (20),

(21) $T^n_{(T^n=S)}$ has conditional intensity

$$[(1 \wedge \sum_1^N a_i) - (1 \wedge \sum_1^{N-1} a_i)] \ 1_{(S_0 < s \leq S)} .$$

We can now prove (b) implies (a). Let $(T^{1,n})$ satisfy condition (b). By Lemma 19 we can construct extensions $G_t^1, G_t^2, \ldots$ of F_t and (G_t^i)-optional times S_i^1 such that $[S_i^1] \subset B$ and such that S_i^1 has conditional intensity $1_{(S_{i-1}^1 < s \leq S_i^1)}$. Let F^1 be the filtration generated by $(G^i : i \geq 1)$. Then $(S_i^1)_{i \geq 1}$ is a Poisson process of rate 1 with respect to F^1.

Now let $T^{2,n} = T^{1,n}_{(T^{1,n} \neq S_i^1 \text{ for any } i)}$.

We assert that $(T^{2,n})$ satisfies (b) with respect to (F_t^1), for a certain set B'. We need only check (7). Write $a_{k,n}$ for the conditional intensity of $T^{k,n}$. Write

$$R_{n,i} = T^{1,n}_{(T^{1,n} = S_i^1)}$$

$$R_n = T^{1,n}_{(T^{1,n} = S_i^1 \text{ for some } i)} .$$

Then

(22) R_n has conditional intensity $a_{1,n} - a_{2,n} \geq 0$.
But $U[R_n] = \bigcup_{n,i}[R_{n,i}] = \bigcup_i[S_i^1]$, so by Lemma 9

$$\sum_n (a_{1,n} - a_{2,n}) = \sum_i 1_{(S_{i-1}^1 < s \le S_i^1)} = 1 \quad \text{a.e.}$$

Thus condition (7) extends from $(T^{1,n})$ to $(T^{2,n})$.

Now we may apply Lemma 19 again to construct an extension F^2 and F^2-optional times (S_i^2) with $[S_i^2] \subset \bigcup_n [T^{2,n}]$ and such that $(S_i^2)_{i\geq 1}$ is again a Poisson process of rate 1.

Continuing, we obtain a uniform Poisson process $(S_i^k : i,k\geq 1)$ on $\{1,2,\dots\} \times [0,\infty)$. By construction $\bigcup_{i,k}[S_i^k] \subset B$, but we must show there is a.s. equality. Thus we must show that, for each n ,

(23) $$P(T^{k,n} < \infty) = E\int a_{k,n}(s)\, ds \to 0 \quad \text{as } k \to \infty \quad .$$

Define

$$R_n^k = T^{k,n}_{(T^{k,n} = S_i^k \text{ for some } i)}$$

As at (22), R_n^k has conditional intensity $a_{k,n} - a_{k+1,n}$.
But from (21), R_n^k has conditional intensity $(1\wedge\sum_1^n a_{k,j}) - (1\wedge \sum_1^{n-1} a_{k,j})$.
So

(24) $$E\int (a_{k,N} - a_{k+1,N})\, ds = E\int (1\wedge\sum_1^N a_{k,j}) - (1\wedge \sum_1^{N-1} a_{k,j})\, ds$$

Now $a_{k,m} \downarrow a_{\infty,n}$, say, as $k \to \infty$. Suppose, inductively, that (23) holds for $n<N$. As $k \to \infty$ the left side of (24) tends to 0 , and the right side tends to $E\int (1 \wedge a_{\infty,N})ds$ by the inductive hypothesis. Thus $a_{\infty,N} = 0$ a.e, so (23) holds for N .

Proof of Proposition 8. Put $f(t) = \frac{F'(t)}{1-F(t)}$, where F is the distribution function of μ .
From [BJ] , if Y has conditional intensity $f(s)1_{(s\le Y)}$ then Y has law μ : conversely, if Y has law μ then Y has conditional intensity $f(s)1_{(s\le Y)}$ with respect to the smallest filtration making Y optional. Thus the random variables (Y_i) in part (a) of Proposition 8 satisfy condition (b) of Theorem 4, so $U[Y_i]$ is indeed a σ-finite Poisson process.

Part (b) is similar to , but simpler than, the proof that (b) implies (a) in Theorem 4. Let B be a σ-finite Poisson process, and let $(T^{1,n})$ satisfy condition (b) of Theorem 4. Lemma 19 showed how to construct an optional time S with conditional intensity $1_{(s\le S)}$. Essentially the same argument shows we can construct Y_1 with conditional intensity $f(s)1_{(s\le Y_1)}$, and hence with law μ . Put $T^{2,n} = T^{1,n}_{(T^{1,n}\ne Y_1)}$, and continue. We obtain i.i.d. variables (Y_k) , with $U[Y_k] \subset B$: arguing as at (23), we show that there is a.s. equality.

Acknowledgements. This work arose from conversations with T.C. Brown and A.D. Barbour at the 1980 Durham Conference on Stochastic Integration.

References

Brémaud, P., Jacod, J. : Processus ponctuels et martingales: résultats récents sur la modelisation et le filtrage. Adv. Appl. Prob. **9**, 362-416 (1977)

Dellacherie, C. : Capacités et processus stochastiques. Springer 1972

Kallenberg, O. : Random measures. Academic Press 1976

Department of Statistics
University of California, Berkeley
Berkeley, California 94720
U.S.A.

Statistical Laboratory
16 Mill Lane
Cambridge CB2 1SB