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Spatial Trajectories

R.V. Chacon, Y. Le Jan, and J.B. Walsh

Introduction

It was W. Feller who suggested thinking of the paths of a stochastic process in terms of a speedometer and a road map. The road map tells one where the path is going and the speedometer tells him how fast. In certain questions of Markov processes, it is useful to study these road maps alone, without looking at the speedometer. When we talk of a path observed without reference to a speedometer, we will call it a trajectory. More formally, a trajectory is an equivalence class of paths, where two paths are equivalent if they trace out the same points in the same order, or, to continue with Feller's metaphor, if they follow the same routes.

One technical question which comes up almost immediately is whether or not the Borel field on the space of trajectories has good measure-theoretic properties. This was answered (it does) in connection with a study of time-changes of Markov processes (see [1] and [2]) but only under the assumption that the processes had no holding points. This condition on the holding points turns out to be unnecessary, although it does greatly simplify the mathematics.

We will show here that if the paths are right continuous and have left limits, then the trajectory field is a separable subfield of the σ-field of a Blackwell space (which passes for good behavior in this permissive age) and that the trajectories are determined by a countable number of intrinsic stopping times.

§1 Equivalence and Intrinsic times

Let $\mathcal{D}$ be the set of all right-continuous functions with left limits, from $\mathbb{R}_+$ to a locally compact metric space E. We say two functions f and g in $\mathcal{D}$ are equivalent if there exist functions F and G which are right continuous and increasing from $\mathbb{R}_+$ to $\mathbb{R}_+$ such that

(i) $f = g \circ F$ and $g = f \circ G$

(ii) F and G are inverses, i.e. $F(t) = \inf\{s \geq 0: G(s) > t\}$ and $G(t) = \inf\{s \geq 0: F(s) > t\}$.

We write $f \sim g$ and say that f and g are equivalent via F and G.

Equivalence can be described in terms of time-changes: f and g are time changes of each other. The equivalence classes of $\mathcal{D}$ generated by this relation are called trajectories; two equivalent functions determine the same trajectory.

The functions F and G above are not necessarily continuous and in fact may not even be uniquely determined by f and g. If, however, f and g have no "flat spots", then F and G are continuous, strictly increasing, and unique, which simplifies the situation enormously. In a way, the rather formidable technical complications encountered below in the proof of Theorem 2.1 are all due to the possibility of flat spots.

There are several elementary properties of time changes which we will need to deal with these flat spots and with possibly discontinuous F and G. Let $f \sim g$ via F and G. Then the reader can verify straightforwardly that

(1.1) $F(G(t)-) \leq t \leq F(G(t))$;

(1.2) g is constant on the closed interval $[F(t-),F(t)]$;(*)

(1.3) if $f(t-) \neq f(t)$ and if $u = F(t-)$, then

$(f(t-),f(t)) = (g(u-),g(u))$.

Let $\mathcal{E}$ be the σ-field on $\mathcal{D}$ generated by the cylinder sets, and let $\mathcal{E}^*$ be its universal completion, that is, if $\mathcal{E}^\mu$ is the completion of $\mathcal{E}$ with respect to the measure μ, then $\mathcal{E} = \cap \mathcal{E}^\mu$, where μ ranges over all probability measures on $\mathcal{E}$.

Following Courrège and Priouret [3], if $t \geq 0$ we define the <u>stopping operator</u> $\alpha_t: \mathcal{D} \to \mathcal{D}$ by $(\alpha_t f)(s) = f(s \wedge t)$.

<u>Definition</u>. A function $T: \mathcal{D} \to [0,\infty]$ is a <u>stopping time</u> if

(i) T is $\mathcal{E}^*$-measurable

(ii) if $T(f) < t$, then $T(\alpha_t f) = T(f)$.

This definition agrees with the usual one [3]. Since we won't be dealing with measures in this article, we won't be overly concerned with (i); it is (ii) which expresses the usual condition that T is determined by the past. We will be particularly interested in intrinsic times, which are stopping times, which, like first hitting times, are determined by the road map, not the speedometer.

<u>Definition</u>. A stopping time T is <u>intrinsic</u> if, whenever $f \sim g$ via F and G, then

(1.4) $T(g) \leq F(T(f))$ and $T(f) \leq G(T(g))$.

<u>Example</u>. There need not be equality in (1.4): let $f(t) \equiv t$,

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*Indeed G is equal to t and therefore g equal to $f(t)$ on the half-open interval $[F(t-), F(t))$. If G jumps at $F(t)$, F is constant between t and $G(F(t))$. Thus $g(F(t)) = f(G(F(t)) = f(t)$.

$g(t) = (t-1)^+$ $(a^+ = a \vee 0)$; then f and g are equivalent via $F(t) = t + 1$ and $G(t) = (t-1)^+$. Let S be the first hit of zero, which is intrinsic. Then $S(f) = S(g) = 0$, so $S(g) < 1 = F(S(f))$.

Here are some elementary properties of intrinsic times.

Proposition 1.1. Let S and T be intrinsic times and let $f \sim g$ via F and G. Then

(i) $T(f) < \infty \iff T(g) < \infty$, and $F(T(f)-) \leq T(g) \leq F(T(f))$;

(ii) $f(T(f)) = g(T(g))$;

(iii) $T(g)$ does not fall into the interior of an interval of constancy of g. In particular, $T(g)$ equals either $F(T(f)-)$ or $F(T(f))$;

(iv) if $S(g) < T(g)$, then $S(f) \leq T(f)$, and there may be equality;

(v) the class of intrinsic times is closed under finite suprema, infima, and under monotone convergence;

(vi) $T \equiv 0$ is intrinsic, but other constant times are not;

(vii) first jump and hitting times are intrinsic. More generally, if A and B are Borel subsets of $E \times E$ and if A does not intersect the diagonal, then D and τ are intrinsic, where

$$D(f) = \inf\{t > S(f) : (f(t-), f(t)) \in A\}$$

$$\tau(f) = \inf\{t > S(f) : (f(S(t)), f(t)) \in B\} .$$

Proof. (i) Since $T(f) \leq G(T(g))$ and $T(g) \leq F(T(f))$, $T(f)$ and $T(g)$ are finite together. These inequalities together with (1.1) imply that

$$F(T(f)-) \le F(G(T(g))-) \le T(g) \le F(T(f)) .$$

(ii) $f(T(f)) = g(F(T(f)))$, while g is constant on $[F(T(f)-), F(T(f))]$, which contains $T(g)$ by (i) .

(iii) g is equivalent to itself, and if it is constant on $[a,b]$, this equivalence can be realized via the function G and its inverse, where

$$G(t) = \begin{cases} t & \text{if } t < a \text{ or } t > b \\ b & \text{if } a \le t \le b . \end{cases}$$

If $T(g) > a$, then $b \le G(T(g)-) \le T(g)$, so $T(g)$ can't be in the open interval (a,b) . The second statement follows since g is constant on $[F(T(f)-), F(T(f))]$ by (1.1).

(iv) $S(f) \le G(S(g)) \le G(T(g)-) \le T(f)$ by (i). To see there can be equality, consider the example above and let T be the first hit of $(0,\infty)$. Then $S(f) = T(f) = 0$, even though $S(g) = 0$ and $T(g) = 1$.

(v) This is clear.

(vi) A strictly positive constant does not satisfy (iii).

(vii) The universal measurability of D and τ is well-known. If $S(f) < t < b$ and if $f(t-) \neq f(t)$, then $S(g) \le F(S(f)) \le F(t-) \le F(b)$ and, if $u = F(t-)$, then $f(t-) = g(u-)$ and $f(t) = g(u)$ by (1.3). Thus if $(f(t-),f(t)) \in A$ then $(g(u-),g(u)) \in A$, so that $D(g) \le u \le F(b)$. Letting $b \downarrow D(f)$, we see $D(g) \le F(D(f))$, so D is intrinsic. The proof for τ is similar.

§2 Trajectories

Let $\mathcal{T}$ be the sub-σ-field of $\mathcal{E}$ generated by the trajectories, i.e. a set $\Lambda \in \mathcal{T}$ iff $\Lambda \in \mathcal{E}$ and if $f \in \Lambda$ and $g \sim f$ imply $g \in \Lambda$. We call $\mathcal{T}$ the σ-field of <u>spatial</u> <u>events</u>.

Let d be a distance on E and for each $n \geq 1$ define two sequences of intrinsic times by induction (the fact that they are intrinsic is a consequence of Prop. 1.1 (vii)):

$$\tau_{n\,0}(f) = 0$$

$$\tau_{n\,k+1}(f) = \inf\{t > \tau_{nk}(f) : d(f(t), f(\tau_{nk}(f))) > 1/n\}$$

and

$$D_{n\,1}(f) = \inf\{t > 0 : d(f(t-), f(t)) > 1/n\}$$

$$D_{n\,k+1}(f) = \inf\{t > D_{nk}(f) : d(f(t-), f(t)) > 1/n\} .$$

For each n,k, set

$$Q_{nk}(f) = \begin{cases} 1 & \text{if } \exists\, a < D_{nk}(f) \ni f \text{ is} \\ & \text{constant on } (a, D_{nk}(f)) \\ 0 & \text{otherwise.} \end{cases}$$

This brings us to the central results of this paper.

<u>Theorem 2.1</u>. Let $f, g \in \mathcal{D}$. Then $f \sim g$ iff

(i) $f(\tau_{nk}(f)) = g(\tau_{nk}(g))$ for all n,k, and

(ii) $Q_{nk}(f) = Q_{nk}(g)$.

<u>Remarks</u>. 1° It is implicit in (i) that $\tau_{nk}(f) < \infty \iff \tau_{nk}(g) < \infty$.

2° If f is constant on $[t,\infty)$ for some t, one of the D_{nk} will be infinite. Thus the Q_{nk} also tell whether f is constant on an infinite interval.

3° One could replace the τ_{nk} above by any set of intrinsic times $\{T_j\}$ with the property that if f is not constant on a given interval $[a,b]$, then there exists j such that $a \leq T_j(f) \leq b$.

4° The only thing at all surprising about Thm. 2.1 is that the Q_{nk} should be necessary. Here is an example to indicate why they are.

Let

$$f(x) = \begin{cases} x & \text{if } x < \pi/4 \\ x+1 & \text{if } x \geq \pi/4 \end{cases}, \quad g(x) = \begin{cases} x & \text{if } x < \pi/4 \\ \pi/4 & \text{if } \pi/4 \leq x < \pi/4 + 1 \\ x & \text{if } x \geq \pi/4 + 1 \end{cases}.$$

None of the $\tau_{nk}(f)$ or $\tau_{nk}(g)$ can equal $\pi/4$, so that $f(\tau_{nk}(f)) = g(\tau_{nk}(g))$ for all n,k. In spite of this, f and g are not equivalent, for g takes on the value $\pi/4$ while f does not. This is reflected in the fact that $Q_{11}(f) = 0$ while g, which is constant for an interval preceeding its unique jump, has $Q_{11}(g) = 1$.

Corollary 2.2. The σ-field of spatial events is separable.

Proof. As a measurable space, $(\mathcal{D},\mathcal{E})$ is isomorphic to an analytic subspace of the unit interval ([4] ch. IV, §19). By Blackwell's theorem ([4] Ch. III, §26) a separable sub-σ-field $\mathcal{S}$ of $\mathcal{T}$ equals $\mathcal{T}$ iff it contains the atoms of $\mathcal{T}$, i.e. the trajectories. Now if $\mathcal{S}$ is the σ-field generated by the $Q_{nk}(f)$ and $f(\tau_{nk}(f))$, then $\mathcal{S} \subset \mathcal{T}$, since these are $\mathcal{E}$-measurable functions which are constant on trajectories. Furthermore, $\mathcal{S}$ is generated by a countable family of functions, so it is a separable σ-field. Each trajectory is an atom of $\mathcal{S}$ by Theorem 2.1. Thus $\mathcal{S}$ and $\mathcal{T}$ have the same atoms, so $\mathcal{S} = \mathcal{T}$.

Q.E.D.

§3. The Proof of Theorem 2.1.

The τ_{nk} and D_{nk} are intrinsic, so if $f \sim g$, (i) holds by Prop. 1.1 (ii). Let $D = D_{mn}$ for some m and n, and suppose $f \sim g$ via the functions F and G. Since D is intrinsic, $D(f) \in [G(D(g)-), G(D(g))]$. But f is constant on this interval so we couldn't have $D(f) > G(D(g)-)$ or ... no jump at $D(f)$. Thus $D(f) = G(D(g)-)$. Then if f is constant on some interval $[a,D(f))$, $g(t) = f(G(t))$ must be constant on some interval $[D(g)-\varepsilon, D(g))$, and (ii) must hold.

The proof of the converse involves the verification of a large - but finite - number of details. We will arrange this into a sequence of statements with proofs. Thus suppose f and g satisfy (1) and (2). To show that $f \sim g$, we must produce the pair F and G of functions required by the definitions. Our first step in this direction is to construct a sequence of functions $\{f_n\}$, all equivalent to g, which converge uniformly to f.

Fix n, and define a continuous, strictly increasing function F_n as follows:

(a) $F_n(\tau_{nk}(f)) = \tau_{nk}(g)$ if $\tau_{nk}(f) < \infty$;

(b) F_n is linear between the $\tau_{nk}(f)$;

(c) if $\tau_{nk}(f) < \infty$ while $\tau_{n\,k+1}(f) = \infty$, set

$$F_n(\tau_{nk}(f)+t) = \tau_{nk}(g) + t \quad \text{for} \quad t \geq 0 .$$

Let G_n be the inverse function of F_n, and define f_n by $f_n(t) = g(F_n(t))$. Then clearly

1° $f_n \sim g$ via F_n, G_n .

2° $\tau_{nk}(f_n) = \tau_{nk}(f)$ and $f_n(\tau_{nk}(f)) = f(\tau_{nk}(f))$, $k = 0,1,2,\ldots$

Furthermore, each t must fall into some interval $[\tau_{nk}(f), \tau_{n\ k+1}(f))$, so by 2° and the triangle inequality

3° $||f_n - f||_\infty \leq 2/n$.

Thus the sequence f_n converges uniformly to f . Let us look at the convergence properties of F_n and G_n . We first note the following, which is an easy consequence of the definition of the τ_{nk} and the triangle inequality.

4° Let $f,h \in \mathcal{D}$ and let $p \geq 3q$ be integers. If $||f - h||_\infty < 1/p$ and if $\tau_{q\ k+1}(f) < \infty$, there exists a j such that $\tau_{pj}(h) \in [\tau_{qk}(f), \tau_{q\ k+1}(f)]$.

5° For each t, the sequences $\{F_n(t)\}$ and $\{G_n(t)\}$ are bounded.

Proof. We will prove this for the F_n . The proof for G_n is similar.

Consider first the case where f is constant on some interval $[a,\infty)$. Then g and f_n must be constant on intervals $[b,\infty)$ and $[a_n,\infty)$ respectively by (ii) and Remark 2° above.

F_n was defined to be linear after the last finite $\tau_{nk}(f_n)$, which is less than b, so that

$$F_n(t) \leq b + t - \tau_{nk}(f_n) \leq b + t .$$

Next, consider the case in which f is not constant on $[t,\infty)$.

There must exist q,k such that $t < \tau_{qk}(f) < \tau_{q\ k+1}(f) < \infty$. By 3° and 4°, for all large enough n, say $n \geq n_0$, there exists j such that, if $p = 3q$, $\tau_{qk}(f) \leq \tau_{pj}(f_n) < \infty$. A priori, j may depend on n, but we claim there is j_0 such that $\tau_{p\ j_0}(f_n) > t$ for all $n \geq n_0$. Indeed, if $\tau_{pj}(f) < \infty$ for only finitely many j, choose j_0 to be the largest j for which this is finite. Since this is also the largest j for which $\tau_{pj}(f_n) < \infty$ for all n, this must work. On the other hand, if $\tau_{pj}(f) < \infty$ for all j, there must still be a j_0 such that $\tau_{p\ j_0}(f_n) > t$ for all n. Suppose not. Then there exists a subsequence (n_j) such that for all j $\tau_{pj}(f_{n_j}) \leq t$. But let $r \geq 3p$ and apply 4° again: for large enough j, there is at least one $\tau_{ri}(f)$ in each interval $[\tau_{pk}(f_{n_j}), \tau_{p\ k+1}(f_{n_j})]$. Consequently, if $\tau_{pj}(f_{n_j}) \leq t$, then $\tau_{ri}(f) \leq t$ for at least $j/2$ values of i. As j is arbitrary, we have $\tau_{ri}(f) \leq t$ for all i, which is impossible.

But now, if $t < \tau_{p\ j_0}(f_n)$ for all large enough n,

$$F_n(t) \leq F_n(\tau_{p\ j_0}(f_n)) = \tau_{p\ j_0}(g) < \infty,$$

proving 5°.

Since the $F_n(t)$ are bounded we may assume, by taking a subsequence if necessary, that

6° there exist increasing functions F and G such that $F_n(t) \rightarrow F(t)$ and $G_n(t) \rightarrow G(t)$ for all $t \geq 0$.

For the remainder of the proof we will simplify notation by arranging the jump times $\{D_{mn}\}$ in a single sequence $\{D_i\}$ and writing

$$d_i = D_i(f), \qquad \delta_i = D_i(g) .$$

When F and G enter symmetrically, or nearly so, we will give the proof for F alone.

For each i, let A_i be the maximal interval of constancy of f of the form $A_i = [a_i, d_i)$ and B_i the maximal interval of constancy of g of the form $B_i = [b_i, \delta_i)$. (This defines a_i and b_i.) Let $A = \bigcup_i A_i$ and $B = \bigcup_i B_i$. The A_i and B_i may be empty (corresponding to $a_i = d_i$ nad $b_i = \delta_i$ resp.) but by hypothesis (ii), $A_i = \phi$ iff $B_i = \phi$.

7° For each i, if n is large enough, $D_i(f_n) = d_i$ and $F(d_i) = F_n(d_i) = \delta_i$. Similarly $G(\delta_i) = G_n(\delta_i) = d_i$.

Proof. By uniform convergence $\Delta f_n(t) \overset{\text{def}}{=} f_n(t) - f_n(t-)$ converges to $\Delta f(t)$ for each t. Consequently $D_i(f_n) = D_i(f)$ for large enough n. As $D_i(f_n) \doteq \tau_{nk}(f_n)$ for some k (again if n is large enough) we get that $F_n(D_i(f_n)) = \delta_i$ and 7° follows since $F(d_i) = \lim_{n\to\infty} F_n(d_i) = \delta_i$. The proof for G is similar.

8° a) If $s \in \mathbb{R}_+ - A$ and $t \in \mathbb{R}_+ - B$, then $f(s) = g(F(s))$ and $g(t) = f(G(t))$.

b) If $s \in A_i$ and $t \in B_i$, then $f(s) = f(d_i-) = g(d_i-) = g(t)$.

Proof. $f(s) = g(F(s))$ if g is continuous at t (by 1° and 6°) or if $s = d_i$ (by 7°). Evidently it can fail only if $F(s) = \delta_i$ for some i. Then $F_n(s) < \delta_i$ for large n, for if not the right continuity of g would give $g(F(s)) = \lim g(F_n(s)) = f(s)$. Thus -

as $F_n(d_i) = \delta_i$ for large n - we must have $s < d_i$, and evidently $f(s) = f(d_i-) = g(\delta_i-)$. But note that the same must hold for each $t' \in [t,d_i)$, so that $[t,d_i) \subset A_i$, the maximal interval of constancy of f.

9° $F(A_i) \subset \overline{B}_i$ and $G(B_i) \subset \overline{A}_i$.

Proof. Let $a_i \leq t < d_i$. Then $F_n(t) < F_n(d_i) = \delta_i$, so $F_n(t)$, and hence $F(t)$, is bounded above by δ_i. Now we claim that $F(t) \geq b_i$. Suppose not. Then, for some $\varepsilon > 0$, $F_n(a_i) \leq b_i - \varepsilon$ for all large n. As F_n is continuous and $F_n(d_i) = \delta_i$, the range of $\{f_n(t): a_i \leq t < d_i\}$ contains the set $K = \{g(t): b - \varepsilon \leq t < \delta_i\}$. Since f_n converges uniformly to f, we find that the closure of the range: $\{f(t),\ a_i \leq t < d_i\}$ also contains K. But, as $[b_i,\delta_i)$ is a maximal interval of constancy, K is not a singleton, while $\{f(t),\ a_i \leq t < d_i\} = \{f(\delta_i)\}$ is one, which is a contradiction. This proves 9°.

The equations 8° may not hold for s and t in A and B, and we are forced to modify F and G there. We do this in two steps. First define

$$\overline{F}(t) = \begin{cases} F(t) & \text{if } t \in \mathbb{R}_+ - A \\ b_i + \dfrac{\delta_i - b_i}{d_i - a_i}(t-a_i) & \text{if } t \in A_i \end{cases}$$

$$\overline{G}(t) = \begin{cases} G(t) & \text{if } t \in \mathbb{R}_+ - B \\ a_i + \dfrac{d_i - a_i}{\delta_i - b_i}(t-b_i) & \text{if } t \in B_i\ . \end{cases}$$

10° $\overline{F}$ is increasing, maps A_i one to one and onto B_i, maps $\mathbb{R}_+ - A$ into $\mathbb{R}_+ - B$, and $f(t) = g(\overline{F}(t))$ for all t. The corresponding statements hold for $\overline{G}$.

Proof. $\overline{F}$ is increasing on each A_i by construction and on $\mathbb{R}_+ - A$ since it equals F there. It is not hard to verify that $s < a_i \Rightarrow \overline{F}(s) < b_i$ and $t > d_i \Rightarrow \overline{F}(t) \geq \delta_i$, which allows us to conclude that $\overline{F}$ is increasing on $\mathbb{R}_+$. It also shows that if $t \notin A_i$, $\overline{F}(t) \notin B_i$. Since $\overline{F}$ maps A_i one-to-one and onto B_i by construction, we can conclude that $\overline{F}$ maps $\mathbb{R}_+ - A$ into $\mathbb{R}_+ - B$ as well. Finally, $f(t) = g(\overline{F}(t)$ for all $t \in \mathbb{R}_+ - A$ by 8(a), and for all $t \in A$ by 8(b).

Now $\overline{F}$ and $\overline{G}$ may not be right-continuous, so define

$$\hat{G}(t) = \overline{G}(t+) \quad \text{and} \quad \hat{F}(t) = \overline{F}(t+) .$$

By 10° and the right-continuity of f and g

11° $f(t) = g(\hat{F}(t))$ and $g(t) = f(\hat{G}(t))$ for all $t \geq 0$.

The proof of the theorem will be complete once we show that $\hat{F}$ and $\hat{G}$ are inverses. We begin by noting that, as F_n and G_n are inverses for all n, 6° implies straightforwardly that $F(t+)$ and $G(t+)$ are too, i.e.

12° $F(t+) = \inf\{s: G(s+) > t\}$ for all $t \geq 0$.

13° a) $\hat{F}(t) = \overline{F}(t)$ if $t \in A$ and $\hat{G}(t) = \overline{G}(t)$ if $t \in B$.

b) $\hat{F}(t) = F(t+)$ if $t \in \mathbb{R}_+ - A$ and $\hat{G}(t) = G(t+)$ if $t \in \mathbb{R}_+ - B$.

Proof. a) is trivial since $\overline{F}$ is already right continuous on A. Since $\overline{F} = F$ on $\mathbb{R}_+ - A$, (b) is clear except possibly when $t \in \mathbb{R}_+ - A$ is a limit from the right of points in A. But in this case, it must be a limit from the right of the d_i, so

$$\hat{F}(t) = \lim_{d_i \downarrow t} \overline{F}(d_i-) = \lim_{d_i \downarrow t} F(d_i) = F(t+) .$$

14° $\hat{F}(t) = \inf\{s: \overline{G}(s) > t\}$ for all $t \in \mathbb{R}_+$.

Proof. This holds for $t \in B_i$ by 13°(b) and the definitions of $\overline{F}$ and $\overline{G}$, hence it holds on all of B. Thus we can restrict our attention to t in $\mathbb{R}_+ - B$. Consider

$$H(t) = \inf\{s: \hat{G}(s) > t\}$$

and

$$F(t+) = \inf\{s: G(s+) > t\} .$$

By 12° and 13°, $F(t+) = \hat{F}(t)$ for $t \in \mathbb{R}_+ - A$, so we must show that $H(t) = F(t+)$ for all $t \in \mathbb{R}_+ - A$. This will follow if we can show that

a) $t \in \mathbb{R}_+ - A$ and $G(s+) \leq t \Rightarrow \hat{G}(s) \leq t$

b) $t \in \mathbb{R}_+ - A$ and $G(s+) > t \Rightarrow \hat{G}(s) > t$.

But if $s \in \mathbb{R}_+ - B$, then $G(s+) = \hat{G}(s)$ and a) and b) both follow trivially, so suppose $s \in B_i$ for some i. Then $G(s)$ and $G(s+)$ are both in $[a_i, d_i]$ by 9°. Thus $G(s+) \leq t$ and $t \in \mathbb{R}_+ - A$ imply $t \geq d_i$. But $\hat{G}(s) \in A_i$ by 10° and 13° so $\hat{G}(s) \leq t$ as well, verifying (a). To verify (b), note that $G(s+) > t$ and $t \in \mathbb{R}_+ - A$

implies $t < a_i$, hence $\hat{G}(s) > a_i$ by 10° and 13° again. This completes the proof of the theorem.

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