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Some extensions of Ito's formula

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In recent studies of stochastic differential equations on manifold, stochastic calculus to differential geometric objects are often considered. In this note we shall discuss three types of formulas for stochastic calculus which may be considered as extensions of Ito's formula. The first formulas (Theorem 1.1 and 1.2) are concerned with the composition of stochastic flows of diffeomorphisms defined by stochastic differential equation. A similar formula is obtained by Bismut [1].

The second formula (Theorem 2.4) is for the stochastic parallel displacement of tensor fields introduced by K. Itô [3]. The third one (Theorem 3.3) is concerned with the stochastic transformation of tensor fields induced by flows of diffeomorphisms defined by stochastic differential equation. The theorem is due to S. Watanabe [9]. A special case of the formula is also discussed in Kunita [7].

1. Ito's formula for the composition of processes.

Let (Ω,F,P) be a complete probability space equipped with a right continuous increasing family F_t, $t \geq 0$ of sub σ-fields of F.

Theorem 1.1. Let $F_t(x)$, $t \geq 0$, $x \in R^d$ be a stochastic process continuous in (t,x) a.s., satisfying

(i) For each $t > 0$, $F_t(\cdot)$ is a C^2-map from R^d into R^1 a.s.

(ii) For each x, $F_t(x)$ is a continuous semimartingale represented as

$$(1.1)\qquad F_t(x) = F_0(x) + \sum_{j=1}^{m} \int_0^t f_s^j(x)dN_s^j$$

where $N_s^1, \ldots, N_s^m$ are continuous semimartingales, $f_s^j(x)$, $s \geq 0$, $x \in R^d$ are stochastic processes continuous in (s,x) such that

(a) For each $s > 0$, $f_s^j(x)$ are C^1-maps from R^d into R^1.

(b) For each x, $f_s^j(x)$ are adapted processes.

Let now $M_t = (M_t^1, \ldots, M_t^d)$ be continuous semimartingales. Then we have

$$(1.2)\qquad F_t(M_t) = F_0(M_0) + \sum_{j=1}^{m} \int_0^t f_s^j(M_s)dN_s^j + \sum_{i=1}^{d}\int_0^t \frac{\partial F_s}{\partial x_i}(M_s)dM_s^i$$

$$+ \sum_{i=1}^{d} \sum_{j=1}^{m} \int_0^t \frac{\partial f_s^j}{\partial x_i}(M_s)d\langle N^j, M^i\rangle_s {}^{1)}$$

$$+ \frac{1}{2} \sum_{i,j=1}^{d} \int_0^t \frac{\partial^2 F_s}{\partial x_i \partial x_j}(M_s)d\langle M^i, M^j\rangle_s .$$

Proof. Fix a time t and let $\Delta_n = \{0=t_0<t_1< \ldots <t_n=t\}$ be a partition of $[0, t]$. Then

$$F_t(M_t) - F_0(M_0) = \sum_{k=0}^{n-1} (F_{t_{k+1}}(M_{t_k}) - F_{t_k}(M_{t_k}))$$

$$+ \sum_{k=0}^{n-1} (F_{t_{k+1}}(M_{t_{k+1}}) - F_{t_{k+1}}(M_{t_k}))$$

$$= I_1^{(n)} + I_2^{(n)}.$$

It holds

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1) $\langle N^j,M^i\rangle_t$ is a continuous process of bounded variation such that $\tilde{N}_t^j\tilde{M}_t^i - \langle N^j,M^i\rangle_t$ is a local martingale, where $\tilde{N}_t^j(\tilde{M}_t^i)$ is the local martingale part of $N_t^j(M_t^i)$. See Kunita-Watanabe [8].

$$I_1^{(n)} = \sum_{k=0}^{n-1} \sum_{j=1}^{m} \int_{t_k}^{t_{k+1}} f_s^j(x)\,dN_s^j \Big|_{x=M_{t_k}} = \sum_{j=1}^{m} \sum_{k=0}^{n-1} \int_{t_k}^{t_{k+1}} f_s^j(M_{t_k})\,dN_s^j.$$

Let Δ_n, $n = 1,2,\ldots$ be a sequence of partions such that $|\Delta_n| \longrightarrow 0$. Then

$$\lim_{n\to\infty} I_1^{(n)} = \sum_{j=1}^{m} \int_0^t f_s^j(M_s)\,dN_s^j.$$

The second member is computed as follow.

$$I_2^{(n)} = \sum_{i=1}^{d} \sum_{k=0}^{n-1} \frac{\partial}{\partial x_i} F_{t_{k+1}}(M_{t_k})(M_{t_{k+1}}^i - M_{t_k}^i)$$

$$+ \frac{1}{2} \sum_{i,j=1}^{d} \sum_{k=0}^{n-1} \frac{\partial^2}{\partial x_i \partial x_j} F_{t_{k+1}}(\xi_k)(M_{t_{k+1}}^i - M_{t_k}^i)(M_{t_{k+1}}^j - M_{t_k}^j)$$

$$= J_1^{(n)} + J_2^{(n)},$$

where ξ_k are random variables such that $|\xi_k - M_{t_k}| \leq |M_{t_{k+1}} - M_{t_k}|$.

We have

$$J_1^{(n)} = \sum_{i=1}^{d} \sum_{k=0}^{n-1} \frac{\partial}{\partial x_i} F_{t_k}(M_{t_k})(M_{t_{k+1}}^i - M_{t_k}^i)$$

$$+ \sum_{i=1}^{d} \sum_{k=0}^{n-1} \left(\frac{\partial}{\partial x_i} F_{t_{k+1}}(M_{t_k}) - \frac{\partial}{\partial x_i} F_{t_k}(M_{t_k})\right)(M_{t_{k+1}}^i - M_{t_k}^i).$$

The first member converges to

$$\sum_{i=1}^{d} \int_0^t \frac{\partial F_s}{\partial x_i}(M_s)\,dM_s^i$$

The second member is written as

$$\sum_{i=1}^{d}\sum_{j=1}^{m}\sum_{k=0}^{n-1}\int_{t_k}^{t_{k+1}}\frac{\partial f_s^j}{\partial x_i}(x)dN_s^j\Big|_{x=M_{t_k}}\times(M_{t_{k+1}}^i-M_{t_k}^i)$$

$$=\sum_{i=1}^{d}\sum_{j=1}^{m}\sum_{k=0}^{n-1}\left(\int_{t_k}^{t_{k+1}}\frac{\partial f_s^j}{\partial x_i}(M_{t_k})dN_s^j\right)(M_{t_{k+1}}^i-M_{t_k}^i)$$

This converges to

$$\sum_{i=1}^{d}\sum_{j=1}^{m}\int_0^t\frac{\partial f_s^j}{\partial x_i}(M_s)d\langle N^j,M^i\rangle_s.$$

It is easily seen that $J_2^{(n)}$ converges to

$$\frac{1}{2}\sum_{i,j=1}^{d}\int_0^t\frac{\partial^2 F_s}{\partial x_i\partial x_j}(M_s)d\langle M^i,M^j\rangle_s.$$

Summing up these calculations, we arrive at the formula (1.2).

In order to establish Ito formula for Stratonovich integral, we need a stronger assumption.

Theorem 1.2. Let $F_t(x)$, $t \geq 0$, $x \in R^d$ be a stochastic process continuous in (t, x) a.s., satisfying

(i) For each $t > 0$, $F_t(\cdot)$ is a C^3-map from R^d into R^1 for a.s. ω.

(ii) For each x, $F_t(x)$ is a continuous semimartingale represented as

$$(1.3)\qquad F_t(x) = F_0(x) + \sum_{j=1}^{m}\int_0^t f_s^j(x)\circ dN_s^j,\ ^{1)}$$

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1) The symbol $\circ$ denotes Stratonovich integral.

where $N_s^1, \ldots, N_s^m$ are continuous semimartingales, $f_t^j(x)$ are stochastic processes satisfying conditions (i) and (ii) of Theorem 1.1, that is, they are continuous in (t, x) a.s., C^2-maps from R^d into R^1 for each $t > 0$ a.s., and are represented as

$$(1.4) \qquad f_t^j(x) = f_0^j(x) + \sum_{k=1}^{\ell} \int_0^t g_s^{jk}(x) dO_s^k ,$$

where $O_t^1, \ldots, O_t^\ell$ are continuous semimartingales and $g_s^{jk}(x)$ are continuous in (s, x), satisfying conditions (a) and (b) of Theorem 1.1.

Let now $M_t = (M_t^1, \ldots, M_t^d)$ be continuous semimartingales. Then we have

$$(1.5) \qquad F_t(M_t) = F_0(M_0) + \sum_{j=1}^{m} \int_0^t f_s^j(M_s) \circ dN_s^j + \sum_{i=1}^{d} \int_0^t \frac{\partial F_s}{\partial x_i}(M_s) \circ dM_s^i .$$

Proof. Using Ito integral, $F_t(x)$ of (1.3) is written as

$$F_t(x) = F_0(x) + \sum_{j=1}^{m} \int_0^t f_s^j(x) dN_s^j + \frac{1}{2} \sum_{j,k} \int_0^t g_s^{jk}(x) d\langle O^k, N^j \rangle_s .$$

Hence by Theorem 1.1,

$$\begin{aligned}(1.6) \qquad F_t(M_t) = F_0(M_0) &+ \sum_j \int_0^t f_s^j(M_s) dN_s^j + \frac{1}{2} \sum_{j,k} \int_0^t g_s^{jk}(M_s) d\langle O^k, N^j \rangle_s \\ &+ \sum_i \int_0^t \frac{\partial F_s}{\partial x_i} (M_s) dM_s^i + \sum_{i,j} \int_0^t \frac{\partial f_s^j}{\partial x_i} (M_s) d\langle N^j, M^i \rangle_s \\ &+ \frac{1}{2} \sum_{i,j} \int_0^t \frac{\partial^2 F_s}{\partial x_i \partial x_j} (M_s) d\langle M^i, M^j \rangle_s .\end{aligned}$$

We shall apply Theorem 1.1 to $f_t^j(x)$ in the place of $F_t(x)$. Then we see that $f_t^j(M_t)$ is a continuous semimartingale whose martingale part equals

$$\sum_i \int_0^t \frac{\partial f_s^j}{\partial x_i}(M_s)d\tilde{M}_s^i + \sum_k \int_0^t g_s^{jk}(M_s)d\tilde{0}_s^k ,$$

where $\tilde{M}_s^i$ and $\tilde{0}_s^k$ are martingale parts of M_s^i and 0_s^k, respectively. Therefore we have

$$\int_0^t f_s^j(M_s)\circ dN_s^j = \int_0^t f_s^j(M_s)dN_s^j + \frac{1}{2}<f^j(M), N^j>_t$$

$$= \int_0^t f_s^j(M_s)dN_s^j + \frac{1}{2}\sum_i \int_0^t \frac{\partial f_s^j}{\partial x_i} d<M^i, N^j>_s$$

$$+ \frac{1}{2}\sum_k \int_0^t g_s^{jk}(M_s)d<0^k, N^j>_s .$$

Similarly, $\frac{\partial F_s}{\partial x_i}(M_s)$ is a continuous semimartingale whose martingale part is

$$\sum_j \int_0^t \frac{\partial f_s^j}{\partial x_i}(M_s)d\tilde{N}_s^j + \sum_j \int_0^t \frac{\partial^2 F_s}{\partial x_i \partial x_j}(M_s)d\tilde{M}_s^j ,$$

where $\tilde{N}_t^j$ are martingale parts of N_t^j. Then we have

$$\int_0^t \frac{\partial F_s}{\partial x_i}(M_s)\circ dM_s^i = \int_0^t \frac{\partial F_s}{\partial x_i}(M_s)dM_s^i + \frac{1}{2}<\frac{\partial F}{\partial x_i}(M), M^i>_t$$

$$= \int_0^t \frac{\partial F_s}{\partial x_i}(M_s)dM_s^i + \frac{1}{2}\sum_j \int_0^t \frac{\partial f_s^j}{\partial x_i}(M_s)d<N^j, M^i>_s$$

$$+ \frac{1}{2} \sum_{j} \int_0^t \frac{\partial^2 F_s}{\partial x_i \partial x_j}(M_s)\, d\langle M^j, M^i\rangle_s \ .$$

Hence the right hand side of (1.6) equals that of (1.5). The proof is complete.

In [6] and [7], the author used the above formula (1.5) without proof for the study of the composition of flows of diffeomorphisms defined by stochastic differential equations. We shall briefly discuss the problem.

Let M be a connected, σ-compact C^∞-manifold of dimension d. Given C^∞-vector fields $X_1,\dots,X_r$ on M and continuous semimartingales $M_t^1,\dots,M_t^r$, $t \geq 0$, consider a stochastic differential equation

$$d\xi_t = \sum_{j=1}^{r} X_j(\xi_t) \circ dM_t^j \ . \tag{1.7}$$

The solution starting at x at time 0 is denoted by $\xi_t(x)$. Under some conditions on vector fields $X_1,\dots,X_r$, ξ_t defines a flow of diffeomorphisms of M a.s. See [7]. We assume it throughout this note.

Now let $F_t(x)$, $t \geq 0$, $x \in M$ be a real valued stochastic process continuous in (t,x) a.s., satisfying conditions (i) and (ii) of Theorem 1.2, where we replace R^d by M. Then we have

$$F_t(\xi_t) = F_0(\xi_0) + \sum_{j=1}^{m} \int_0^t f_s^j(\xi_s) \circ dN_s^j + \sum_{j=1}^{r} \int_0^t X_j F_s(\xi_s) \circ dM_s^j. \tag{1.8}$$

Here, $X_j F_s(x)$ is the derivation of $F_s(x)$ (s; fixed) by X_j. In fact, let $(x^1,\dots,x^d)$ be a local coordinate and let $X_j^i(x)$, $i=1,\dots,d$ be components of X_j, i.e., $X_j = \sum_i X_j^i \frac{\partial}{\partial x^i}$. Then $\xi_t = (\xi_t^1,\dots,\xi_t^d)$ are continuous semimartingales represented as

$$(1.9) \qquad d\xi_t^i = \sum_j X_j^i(\xi_t)\circ dM_t^j\,, \qquad i=1,\dots,d.$$

Apply formula (1.5) to $F_t(\xi_t)$. Then the third term of the right hand side of (1.5) is

$$\sum_{i=1}^{d}\int_0^t \frac{\partial F_s}{\partial x_i}(\xi_s)\circ d\xi_s^i = \sum_j\sum_i \int_0^t X_j^i(\xi_s)\frac{\partial F_s}{\partial x_i}(\xi_s)\circ dM_s^j$$

$$= \sum_j \int_0^t X_jF_s(\xi_s)\circ dM_s^j.$$

This shows the formula (1.8).

Let now $Y_1,\dots,Y_m$ be other C^∞-vector fields on M and η_t be a solution of stochastic differential equation

$$(1.10) \qquad d\eta_t = \sum_{k=1}^{m} Y_k(\eta_t)\circ dN_t^k\,.$$

Then the solution $\eta_t(x)$ starting at x is a C^∞-map under some conditions on $Y_1,\dots,Y_s$. Using a local coordinate $(x^1,\dots,x^d)$, $\eta_t = (\eta_t^1,\dots,\eta_t^d)$ satisfies

$$\eta_t^i(\xi_t) = x^i + \sum_{k=1}^{m}\int_0^t Y_k^i(\eta_s\circ\xi_s)\circ dN_s^k + \sum_{j=1}^{r}\int_0^t X_j\eta_s^i(\xi_s)\circ dM_s^j\,.$$

Then the composed process $\zeta_t = \eta_t\circ\xi_t$ satisfies

$$(1.11) \qquad d\zeta_t = \sum_{k=1}^{m} Y_k(\zeta_t)\circ dN_t^k + \sum_{j=1}^{r}\eta_{t*}(X_j)(\zeta_t)\circ dM_t^j\,.$$

Here $\eta_{t*}(X_j)$ is a stochastic vector field defined by

$$\eta_{t*}(X_j)_x = (\eta_{t*})_{\eta_t^{-1}(x)}(X_j)_{\eta_t^{-1}(x)},$$

where η_{t*} is the differential of the map η_t. See [6] and [7] for other problems of decompositions.

2. Ito's formula for stochastic parralel displacement of tensor fields.

As an application of extended Ito's formula established in Section 1, we shall discuss an Ito's formula for stochastic parallel displacement of tensor fields along curves obtained by a stochastic differential equation. Stochastic parallel displacement along Brownian curves on Riemannian manifold was introduced by K. Itô [3], [4]. Our definition is close to [4]. See Ikeda-Watanabe [2] for other approaches by Eelles-Elworthy and Malliavin, where stochastic moving frames play an important role.

We shall recall some facts on parallel displacement needed later. Let M be a connected, σ-compact C^∞-manifold of dimension d, where an affine connection is defined. Denote by $T_x(M)$ the tangent space at the point x of M. Suppose we are given a smooth curve $\phi_s(x)$, $s \geq 0$ starting at x at time 0. Let u_t be a tangent vector belonging to $T_{\phi_t(x)}(M)$ and let u_0 be the parallel displacement of u_t along the curve $\phi_s(x)$, $0 \leq s \leq t$ from the point $\phi_t(x)$ to x. Then the map $\pi_{tx} : u_t \longrightarrow u_0$ defines an isomorphism from $T_{\phi_t(x)}(M)$ to $T_x(M)$.

Given a vector field Y on M, we denote by Y_x the restriction of Y to the point x, which is an element of $T_x(M)$. For each $t > 0$, a vector field $\pi_t Y$ is defined by $(\pi_t Y)_x = \pi_{tx} Y_{\phi_t(x)}$, $\forall x \in M$. The one parameter family of vector fields $\pi_t Y$, $t \geq 0$ satisfies

$$(2.1) \qquad \frac{d}{dt}(\pi_t Y)_x = (\pi_t \nabla_{\dot\phi} Y)_x , \qquad \forall x \in M ,$$

where $\nabla_{\dot\phi} Y$ is the covariant derivative of Y along the curve ϕ_t. If $\phi_t(x)$ is a solution of an ordinary differential equation:

$$\dot\phi_t = \sum_{j=1}^{r} X_j(\phi_t)u_j(t), \qquad \phi_0 = x,$$

where $X_1,\ldots,X_r$ are vector fields on M and $u_1(t),\ldots,u_r(t)$ are smooth scalar functions, then equation (2.1) becomes

$$(2.2) \qquad \frac{d}{dt}(\pi_t Y)_x = \sum_{j=1}^{r} (\pi_t \nabla_{X_j} Y)_x u_j(t).$$

The inverse map π_{tx}^{-1} defines another vector field $\pi_t^{-1}Y$ as $(\pi_t^{-1}Y)_x = \pi^{-1}_{t\phi_t^{-1}(x)} Y_{\phi_t^{-1}(x)}$,[1] which is the parallel displacement of $Y_{\phi_t^{-1}(x)}$ along the curve ϕ_s, $0 \leq s \leq t$ from $\phi_t^{-1}(x)$ to x. It holds

$$(2.3) \qquad \frac{d}{dt}(\pi_t^{-1}Y)_x = -\sum_{j=1}^{r} (\nabla_{X_j} \pi_t^{-1}Y)_x u_j(t).$$

Let $T_x(M)^*$ be the cotangent space at x (dual of $T_x(M)$). The dual π_{tx}^* is an isomorphism from $T_x(M)^*$ to $T_{\phi_t(x)}(M)^*$ such that $\langle\pi_{tx}^*\theta, Y\rangle = \langle\theta, \pi_{tx}Y\rangle$ holds for any $\theta \in T_x(M)^*$ and $Y \in T_{\phi_t(x)}(M)$. Given a 1-form θ (covariant vector field), $\pi_t^*\theta$ is a 1-form defined by $(\pi_t^*\theta)_x = \pi^*_{t\phi_t^{-1}(x)} \theta_{\phi_t^{-1}(x)}$. $\pi_t^{*-1}\theta$ is defined similarly.

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1) It is assumed that ϕ_t is a one to one map from M into itself for any $t \geq 0$.

A tensor field K of type (p,q) is, by definition, an assignment of a tensor K_x of $T^p_q(x)$ to each point x of M, where

$$T^p_q(x) = T_x(M) \otimes \dots \otimes T_x(M) \otimes T_x(M)^* \otimes \dots \otimes T_x(M)^*$$

($T_x(M)$; p times and $T_x(M)^*$; q times). Hence for each x, K_x is a multilinear form on the product space

$$T_x(M)^* \times \dots \times T_x(M)^* \times T_x(M) \times \dots \times T_x(M).$$

Thus, for given 1-forms $\theta^1,\dots,\theta^p$ and vector fields $Y_1,\dots,Y_q$,

$$K_x(\theta^1,\dots,\theta^p, Y_1,\dots,Y_q) \;(\equiv K_x(\theta^1_x,\dots,\theta^p_x, Y_{1x},\dots,Y_{qx}))$$

is a scalar field. In the sequel, we assume that it is a C^∞-function.[1)]

The parallel displacement $\pi_t K$ of the tensor field K along the curve ϕ_s is defined by the relation

$$(2.4) \qquad (\pi_t K)_x(\theta^1,\dots,\theta^p, Y_1,\dots,Y_q) = K_{\phi_t(x)}(\pi_t^*\theta^1,\dots,\pi_t^*\theta^p, \pi_t^{-1}Y_1,\dots,\pi_t^{-1}Y_q).$$

If K is a vector field, it coincides clearly with the usual parallel displacement mentioned above. If K is a 1-form, it coincides with $\pi_t^{*-1} K$. Hence we can write the above relation as

$$(2.4') \qquad (\pi_t K)_x(\theta^1,\dots,\theta^p, Y_1,\dots,Y_q) = K_{\phi_t(x)}(\pi_t^{-1}\theta^1,\dots,\pi_t^{-1}\theta^p, \pi_t^{-1}Y_1,\dots,\pi_t^{-1}Y_q).$$

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1) K is a C^∞-tensor field.

Let X be a complete vector field and ϕ_t, the one parameter group of transformations generated by X. Then the covariant derivative $\nabla_X K$ of tensor field K is defined by

$$(2.5)\qquad (\nabla_X K)_x(\theta^1,\ldots,\theta^p, Y_1,\ldots,Y_q) = \frac{d}{dt}(\pi_t K)_x(\theta^1,\ldots,\theta^p, Y_1,\ldots,Y_q)\Big|_{t=0}.$$

The following relation is easily checked.

$$(2.6)\qquad (\nabla_X K)_x(\theta^1,\ldots,\theta^p, Y_1,\ldots,Y_q)$$

$$= X(K_x(\theta^1,\ldots,\theta^p, Y_1,\ldots,Y_q))$$

$$- \sum_{k=1}^{p} K_x(\theta^1,\ldots,\nabla_X\theta^k,\ldots,\theta^p, Y_1,\ldots,Y_q)$$

$$- \sum_{\ell=1}^{q} K_x(\theta^1,\ldots,\theta^p, Y_1,\ldots,\nabla_X Y_\ell,\ldots, Y_q).$$

Now let $\xi_t(x)$ be the stochastic flow of diffeomorphisms defined by the equation (1.7). The curves $\xi_s(x)$, $0 \le s \le t$ are not smooth a.s., so that the argument of the parallel displacement mentioned above is not applied directly. We shall define the stochastic parallel displacement following the idea of Ito [4]. We begins with defining the stochastic parallel displacement of vector fields along $\xi_s(x)$, $0 \le s \le t$ from $\xi_t(x)$ to x.

A stochastic analogue of equation (2.2) is as follow.

$$(2.7)\qquad (\pi_t Y)_x = Y_x + \sum_{j=1}^{r}\int_0^t (\pi_s \nabla_{X_j} Y)_x \circ dM_s^j\ , \quad \forall x \in M.$$

Here, π_t is a stochastic linear map acting on the space of vector fields such that $\pi_t(fY)_x = f(\xi_t(x))(\pi_t Y)_x$ for scalar function f. Let $(x^1,\dots,x^d)$ be a local coordinate and let $\partial_k = \frac{\partial}{\partial x^k}$. Then equation (2.7) is written as

$$(2.8)\qquad (\pi_t\partial_k)_x = (\partial_k)_x + \sum_j \sum_{\alpha,\ell}\int_0^t X_j^\alpha(\xi_s(x))\Gamma_{\alpha k}^\ell(\xi_s(x))(\pi_s\partial_\ell)_x \circ dM_s^j,$$
$$k = 1,\dots,d,$$

where $X_j = \sum_\alpha X_j^\alpha\,\partial_\alpha$ and $\Gamma_{\alpha k}^\ell$ is the Christoffel symbol. It may be considered as an equation on the tangent space $T_x(M)$. The equation has a unique solution $(\pi_t\partial_k)_x$, $k=1,\dots,d$ for any x. Define $(\pi_t Y)_x = \sum_i Y^i(\xi_t(x))(\pi_t\partial_i)_x$ if $Y = \sum Y^i\partial_i$. Then it is a unique solution of (2.7). We shall call $(\pi_t Y)_x$ the parallel displacement of $Y_{\xi_t(x)}$ along the curve $\xi_s(x)$, $0 \le s \le t$ from $\xi_t(x)$ to x. Denote the linear map $Y_{\xi_t(x)} \longrightarrow (\pi_t Y)_x$ as π_{tx}.

Lemma 2.1. π_{tx} is an isomorphism from $T_{\xi_t(x)}(M)$ to $T_x(M)$ a.s.

Proof. Using the above local coordinate, we shall write

$$(\pi_t\partial_i)_x = \sum_j \pi_t^{ij}(x)(\partial_j)_x,\quad p_j^{k\ell}(x) = \sum_\alpha X_j^\alpha(x)\Gamma_{\alpha k}^\ell(x).$$

From (2.8), the matrix $\Pi_t(x) = (\pi_t^{ij}(x))$ satisfies

$$(2.9)\qquad \Pi_t(x) = I + \sum_{j=1}^r \int_0^t P_j(\xi_s(x))\,\Pi_s(x)\circ dM_s^j\ ,$$

where $P_j = (p_j^{k\ell})$ and I is the identity. Consider the adjoint matrix equation of (2.9):

$$(2.10) \qquad \Sigma_t(x) = I - \sum_{j=1}^{r}\int_0^t \Sigma_s(x)P_j(\xi_s(x))\circ dM_s^j .$$

Then Ito's formula implies $d\Sigma_t(x)\Pi_t(x) = 0$. This proves $\Sigma_t(x)\Pi_t(x) = I$ so that $\Pi_t(x)$ has the inverse $\Sigma_t(x)$. The proof is complete.

Now the inverse map $\pi_{tx}^{-1} : T_x(M) \longrightarrow T_{\xi_t(x)}(M)$ defines the stochastic parallel displacement from x to $\xi_t(x)$. Obviously we have $\pi_{tx}^{-1}(\partial_k)_x = \sum_\ell \sigma_t^{k\ell}(x)(\partial_\ell)_{\xi_t(x)}$, where $\Sigma_t = (\sigma_t^{k\ell})$. The components of the vector $\pi_{tx}^{-1}(\partial_k)_x$ satisfies by (2.10)

$$(2.11) \qquad \sigma_t^{k\ell}(x) = \delta_{k\ell} - \sum_{j=1}^{r}\int_0^t \sum_{i,\alpha} X_j^\alpha(\xi_s(x))\Gamma_{\alpha i}^\ell(\xi_s(x))\sigma_s^{ki}(x)\circ dM_s^j.$$

In [2] and [4], the above equation is employed for defining the stochastic parallel displacement. Actually, if $\sigma_t^{k\ell}$ is a solution of (2.11), $\sum_\ell \sigma_t^{k\ell}(\partial_\ell)_{\xi_t(x)}$ is defined as the stochastic parallel displacement of $(\partial_k)_x$ along $\xi_s(x)$, $0 \le s \le t$ from x to $\xi_t(x)$: Then equation (2.8) is induced from it as the inverse. A reason that we adopt (2.7) as the definition is that all $(\pi_t Y)_x$ are elements of the fixed tangent space $T_x(M)$. While $\pi_{tx}^{-1}Y_x$ are moving in various tangent spaces $T_{\xi_t(x)}(M)$ as t and ω vary. In fact we may consider that (2.11) is an equation for stochastic moving frames represented by local coordinate $(x^1,\dots,x^d,$ $\sigma^{11},\dots,\sigma^{1d},\dots,\sigma^{d1},\dots,\sigma^{dd})$ (c.f. [2]).

Given a vector field Y, we denote by $(\pi_t^{-1}Y)_x$ the stochastic parallel displacement of Y along ξ_s, $0 \le s \le t$ from $\xi_t^{-1}(x)$ to x. Then it holds $(\pi_t^{-1}Y)_x = \pi^{-1}_{t\xi_t^{-1}(x)} Y_{\xi_t^{-1}(x)}$.

Proposition 2.2. It holds

$$(2.12)\qquad (\pi_t^{-1}Y)_x = Y_x - \sum_{j=1}^{r}\int_0^t (\nabla_{X_j}\pi_s^{-1}Y)_x \circ dM_s^j.$$

Proof. It is known that the inverse map ξ_t^{-1} satisfies

$$d\xi_t^{-1}(x) = -\sum_j \xi_{t*}^{-1}(X_j)(\xi_t^{-1}(x))\circ dM_t^j.$$

(Kunita [7], Proposition 5.1). Apply Theorem 1.2 to Σ_t. Then

$$(2.13)\qquad \sigma_t^{k\ell}(\xi_t^{-1}(x)) = \delta_{k\ell} - \sum_{j=1}^{r}\int_0^t \sum_{i,\alpha} X_j^\alpha(x)\Gamma_{\alpha i}^\ell(x)\sigma_s^{ki}(\xi_s^{-1}(x))\circ dM_s^j$$
$$- \sum_{j=1}^{r}\int_0^t \xi_{s*}^{-1}(X_j)\sigma_s^{k\ell}(\xi_s^{-1}(x))\circ dM_s^j.$$

Noting $\xi_{s*}^{-1}(X_j)f(\xi_s^{-1}(x)) = X_j(f\circ\xi_s^{-1})(x)$, we see that $\kappa_t^{k\ell} \equiv \sigma_t^{k\ell}\circ\xi_t^{-1}$ satisfies

$$(2.14)\qquad \kappa_t^{k\ell} = \delta_{k\ell} - \sum_{j=1}^{r}\int_0^t \sum_\alpha X_j^\alpha(\sum_i \Gamma_{\alpha i}^\ell\kappa_s^{ki} + \partial_\alpha(\kappa_s^{k\ell}))\circ dM_s^j.$$

Since $\pi_t^{-1}\partial_k = \sum_\ell \kappa_t^{k\ell}\partial_\ell$, the above equality shows

$$\pi_t^{-1}\partial_k = \partial_k - \sum_{j=1}^{r}\int_0^t \nabla_{X_j}\pi_s^{-1}\partial_k\circ dM_s^j.$$

This proves the proposition.

The dual π_t^* of π_t is defined as before. It is acting on the space of 1-forms. It holds

$$\langle\pi_t^*\theta, Y\rangle_{\xi_t(x)} = \langle\theta, \pi_t Y\rangle_x$$

for any 1-form θ and vector field Y. We shall obtain equations for $\pi_t^*\theta$ and $\pi_t^{*-1}\theta$.

Proposition 2.3. It holds

$$(2.15) \qquad (\pi_t^*\theta)_x = \theta_x - \sum_{j=1}^{r}\int_0^t (\nabla_{X_j}\pi_s^*\theta)_x \circ dM_s^j ,$$

$$(2.16) \qquad (\pi_t^{*-1}\theta)_x = \theta_x + \sum_{j=1}^{r}\int_0^t (\pi_s^{*-1}\nabla_{X_j}\theta)_x \circ dM_s^j .$$

Proof. Set $F_t(x) = \langle\theta,\pi_t Y\rangle_x$. We shall calculate $F_t(\xi_t^{-1}(x))$, using Theorem 1.2. It holds

$$F_t(\xi_t^{-1}(x)) - F_0(x) = -\sum_{j=1}^{r}\int_0^t \xi_{s*}^{-1}(X_j)(F_s)(\xi_s^{-1}(x))\circ dM_s^j$$
$$+ \sum_{j=1}^{r}\int_0^t \langle\theta,\pi_s\nabla_{X_j}Y\rangle_{\xi_s^{-1}(x)} \circ dM_s^j .$$

Note that

$$\xi_{s*}^{-1}(X_j)(F_s)(\xi_s^{-1}(x)) = X_j(\langle\theta,\pi_s Y\rangle_{\xi_s^{-1}(x)}).$$

Since $\langle\nabla_{X_j}\theta,Y\rangle + \langle\theta,\nabla_{X_j}Y\rangle = X_j(\langle\theta,Y\rangle)$ holds by (2.6), the above formula leads to

$$\langle\theta,\pi_t Y\rangle_{\xi_t^{-1}(x)} - \langle\theta,Y\rangle_x = -\sum_j\int_0^t \langle\nabla_{X_j}\pi_s^*\theta,Y\rangle_x \circ dM_s^j .$$

This proves (2.15). (2.16) is proved similarly.

The stochastic parallel displacement of tensor field K is defined similarly as before: $\pi_t K$ is a tensor field such that

$$(2.17)\qquad (\pi_t K)_x(\theta^1,\dots,\theta^p, Y_1,\dots,Y_q) = K_{\xi_t(x)}(\pi_t^*\theta^1,\dots,\pi_t^*\theta^p, \pi_t^{-1}Y_1,\dots,\pi_t^{-1}Y_q).$$

We shall obtain an Ito's formula for $\pi_t K$, which is an extension of formulas (2.7) and (2.16).

Theorem 2.4. It holds

$$(2.18)\qquad \pi_t K = K + \sum_{j=1}^{r}\int_0^t \pi_s \nabla_{X_j} K \circ dM_s^j$$

$$= K + \sum_{j=1}^{r}\int_0^t \pi_s \nabla_{X_j} K dM_s^j + \frac{1}{2}\sum_{j,k}\int_0^t \pi_s \nabla_{X_j}\nabla_{X_k} K d\langle M^j, M^k\rangle_s .$$

Proof. Apply Ito's formula to the multilinear form K_x. Noting the relation (2.12) and (2.15), we have

$$K_x(\pi_t^*\theta^1,\dots,\pi_t^*\theta^p, \pi_t^{-1}Y_1,\dots,\pi_t^{-1}Y_q) - K_x(\theta^1,\dots,\theta^p, Y_1,\dots,Y_q)$$

$$= -\sum_{j=1}^{r}\{\sum_{k=1}^{p}\int_0^t K_x(\pi_s^*\theta^1,\dots,\nabla_{X_j}\pi_s^*\theta^k,\dots,\pi_s^{-1}Y_1,\dots,\pi_s^{-1}Y_q)\circ dM_s^j$$

$$+ \sum_{\ell=1}^{q}\int_0^t K_x(\pi_s^*\theta^1,\dots,\pi_s^{-1}Y_1,\dots,\nabla_{X_j}\pi_s^{-1}Y_\ell,\dots,\pi_s^{-1}Y_q)\circ dM_s^j\}.$$

Set

$$F_t(x) = K_x(\pi_t^*\theta^1,\dots,\pi_t^*\theta^p, \pi_t^{-1}Y_1,\dots,\pi_t^{-1}Y_q)$$

and apply Theorem 1.2 to $F_t(\xi_t(x))$. Then

$$(2.19)\qquad F_t(\xi_t(x)) - F_0(x)$$

$$= \sum_{j=1}^{r} \{\int_0^t X_j F_s(\xi_s(x))\circ dM_s^j$$

$$- \sum_k \int_0^t K_{\xi_s(x)}(\pi_s^*\theta^1,\dots,\pi_s^*\nabla_{X_j}\theta^k,\dots,\pi_s^{-1}Y_1,\dots,\pi_s^{-1}Y_q)\circ dM_s^j$$

$$- \sum_\ell \int_0^t K_{\xi_s(x)}(\pi_s^*\theta^1,\dots,\pi_s^{-1}Y_1,\dots,\nabla_{X_j}\pi_s^{-1}Y_\ell,\dots,\pi_s^{-1}Y_q)\circ dM_s^j\}.$$

Noting the relation (2.6), we see that the right hand side of (2.19) is

$$\sum_{j=1}^{r}\int_0^t \pi_s\nabla_{X_j}K(\theta^1,\dots,\theta^p,\ Y_1,\dots,Y_q)\circ dM_s^j\ .$$

The proof is complete.

Remark. The inverse π_t^{-1} is defined by

$$(\pi_t^{-1}K)_x(\theta^1,\dots,\theta^p,\ Y_1,\dots,Y_q) = K_{\xi_t^{-1}(x)}(\pi_t\theta^1,\dots,\pi_t\theta^p,\ \pi_tY_1,\dots,\pi_tY_q).$$

Then similarly as Theorem 2.4, we have

$$\pi_t^{-1}K = K - \sum_{j=1}^{r}\int_0^t \nabla_{X_j}\pi_s^{-1}K\circ dM_s^j$$

$$= K - \sum_{j=1}^{r}\int_0^t \nabla_{X_j}\pi_s^{-1}K dM_s^j - \frac{1}{2}\sum_{j,k}\int_0^t \nabla_{X_j}\nabla_{X_k}\pi_s^{-1}K d\langle M^j,M^k\rangle_s.$$

The Ito formula (2.18) can be applied to getting a heat equation for tensor fields. Suppose that ξ_t is determined by

$$(2.20)\qquad d\xi_t = \sum_{j=1}^{r} X_j(\xi_t)\circ dB_t^j + X_0(\xi_t)dt,$$

where $(B_t^1,\dots,B_t^r)$ is a Brownian motion. Then,

$$(2.21)\qquad \pi_t K - K = \sum_{j=1}^{r}\int_0^t \pi_s \nabla_{X_j} K dB_s^j + \int_0^t \pi_s(\frac{1}{2}\sum_{j=1}^{r}\nabla_{X_j}^2 + \nabla_{X_0})Kds$$

Theorem 2.5. Define for each t a tensor field K_t by

$$(K_t)_x(\theta^1,\dots,\theta^p, Y_1,\dots,Y_q) = E[(\pi_t K)_x(\theta^1,\dots,\theta^p, Y_1,\dots,Y_q)].$$

Then it satisfies the heat equation

$$\frac{\partial K_t}{\partial t} = (\frac{1}{2}\sum_{j=1}^{r}\nabla_{X_j}^2 + \nabla_{X_0})K_t, \quad K_0 = K.$$

Proof. We shall omit $\theta^1,\dots,\theta^p, Y_1,\dots,Y_q$ for simplicity. Set $K_s = E[\pi_s K]$. Taking expectation to both sides of (2.21), we have

$$K_s - K = \int_0^s E[\pi_u(\frac{1}{2}\sum_j \nabla_{X_j}^2 + \nabla_{X_0})K]du.$$

Since K_x is smooth relative to x, so is $(K_s)_x$.

Let us substitute K_t to the above formula. Then

$$E[\pi_s K_t] - K_t = \int_0^s E[\pi_u(\frac{1}{2}\sum_{j=1}^{r}\nabla_{X_j}^2 + \nabla_{X_0})K_t]du.$$

Now it holds $\pi_{t+s} = \pi_t\hat{\pi}_s$, where $\hat{\pi}_s$ is the parallel displacement along ξ_u, $t \le u \le t+s$ from $\xi_{t+s}(x)$ to $\xi_t(x)$. Then by Markov property, we have

$$E[\pi_{s+t}K] = E[\pi_t \hat{\pi}_s K] = E[\pi_t K_s].$$

Consequently,

$$K_{t+s} - K_t = \int_0^s E[\pi_u(\frac{1}{2}\sum_j \nabla_{X_j}^2 + \nabla_{X_0})K_t]du,$$

so that we have

$$\frac{\partial}{\partial t} K_t = (\frac{1}{2}\sum_{j=1}^{r} \nabla_{X_j}^2 + \nabla_{X_0})K_t.$$

The proof is complete.

3. Ito's formula for ξ_t^* acting on tensor fields.

In this section, we shall obtain an Ito's formula for stochastic maps ξ_t^* acting on tensor fields, which is induced by the solution $\xi_t(x)$ of (1.7). The formula looks similar to the one for parallel displacement. The only difference is that Lie derivative is involved in place of covariant derivative. The formula has been obtained by S. Watanabe [9]. His approach is based on the lift of the process to a frame bundle in a suitable way and the use of scalarization of tensor field on the bundle. On the other hand, our proof is very close to the method in previous section.

Given a diffeomorphism ϕ of M, the differential ϕ_{*x} is a linear map of $T_x(M)$ onto $T_{\phi(x)}(M)$. The dual map ϕ_x^* of the differential ϕ_{*x} is a linear map of $T_{\phi(x)}(M)^*$ onto $T_x(M)^*$. Let Y be a vector field. The ϕ-related vector field $\phi_*(Y)$ is defined by the relation $\phi_*(Y)_x = \phi_{*\phi^{-1}(x)} Y_{\phi^{-1}(x)}$. For 1-form θ, $\phi^*(\theta)$ is defined by $\phi^*(\theta)_x = \phi^*_x \theta_{\phi(x)}$. The inverse $\phi^{*-1}(\theta)$ is defined in the same way.

Let K be a tensor field of type (p,q). We define a tensor field $\phi^* K$ by the relation

$$(3.1) \qquad (\phi^* K)_x(\theta^1, \ldots, \theta^p, Y_1, \ldots, Y_q)$$

$$= K_{\phi(x)}(\phi^{*-1}(\theta^1), \ldots, \phi^{*-1}(\theta^p), \phi_*(Y_1), \ldots, \phi_*(Y_q)).$$

If K is a vector field, it holds $\phi^* K = \phi_*^{-1}(K)$ and if K is a 1-form, it holds $\phi^* K = \phi^*(K)$.

Remark. The definition of the above ϕ^* is not equal to that of $\tilde{\phi}$ in Kobayashi-Nomizu [5], p. 28. The relation of these is $\tilde{\phi}^{-1} = \phi^*$ or $\tilde{\phi} = (\phi^{-1})^*$.

Let X be a complete vector field and ϕ_t, $t \in (-\infty, \infty)$ be the one parameter group of transformations generated by X. The Lie derivative of tensor field K with respect to X is defined by

$$(3.2) \qquad L_X K = \lim_{t \downarrow 0} \frac{1}{t}\{\phi_t^* K - K\}.$$

The following properties are well known. (i) If K is a scalar function, then $L_X K = X(K)$. (ii) If K is a vector field, then $L_X K = [X,K]$, where [,] is the Lie bracket. (iii) If Y is a vector field and θ is a 1-form, then

$$(3.3) \qquad \langle L_X\theta , Y\rangle + \langle\theta , L_X Y\rangle = X\langle\theta , Y\rangle.$$

(iv) If K is a tensor field of type (p,q), then

$$(3.4)\qquad (L_X K)_x(\theta^1,\dots,\theta^p, Y_1,\dots,Y_q) = X(K_x(\theta^1,\dots,\theta^p, Y_1,\dots,Y_q))$$
$$- \sum_{k=1}^{p} K_x(\theta^1,\dots,L_X\theta^k,\dots,\theta^p, Y_1,\dots,Y_q)$$
$$- \sum_{\ell=1}^{q} K_x(\theta^1,\dots,\theta^p, Y_1,\dots,L_X Y_\ell,\dots,Y_q).$$

Now let $\xi_t(x)$ be a solution of stochastic differential equation (1.7). Then $\xi_t^* K$ is a stochastic tensor field. We shall obtain Ito's formula for $\xi_t^* K$ and $(\xi_t^*)^{-1}K$. We first consider the case that K is a vector field and then the case that K is a 1-form

Lemma 3.1. (c.f. [7], Proposition 5.2 and 5.3). Let Y be a vector field. Then it holds

$$(3.5)\qquad \xi_t^* Y = Y + \sum_{j=1}^{r}\int_0^t \xi_s^* L_{X_j} Y\circ dM_s^j$$

$$(3.6)\qquad \xi_{t*}(Y) = Y - \sum_{j=1}^{r}\int_0^t L_{X_j}\xi_{s*}(Y)\circ dM_s^j$$

Lemma 3.2. Let θ be a 1-form. Then it holds

$$(3.7)\qquad \xi_t^*\theta = \theta + \sum_j \int_0^t \xi_s^* L_{X_j}\theta\circ dM_s^j$$

$$(3.8)\qquad (\xi_t^*)^{-1}\theta = \theta - \sum_j \int_0^t L_{X_j}(\xi_s^*)^{-1}\theta\circ dM_s^j.$$

Proof. We shall prove (3.8) only since (3.7) is a special case of the next theorem. It holds

$$<(\xi_t^*)^{-1}\theta\ ,\ Y>_x\ =\ <\xi_t^{*-1}\theta\ ,\ Y>_x\ =\ <\theta\ ,\ \xi_t^* Y>_{\xi_t^{-1}(x)}\ .$$

Then similarly as the proof of Proposition 2.3, we have

$$<\theta\ ,\ \xi_t^* Y>_{\xi_t^{-1}(x)}\ -\ <\theta,Y>_x = -\sum_j \int_0^t <L_{X_j}(\xi_s^*)^{-1}\theta,Y>_x \circ dM_s^j\ .$$

This proves (3.8).

Formulas (3.5), (3.6), (3.7) and (3.8) correspond formulas (2.7), (2.12), (2.15) and (2.16), respectively. Then the next Ito's formula for tensor field $\xi_t^* K$ is proved in the same way as the case of parallel displacement.

Theorem 3.3. (c.f. S. Watanabe [9]). Let K be a smooth tensor field of type (p,q). Then it holds

$$(3.9)\qquad \xi_t^* K = K + \sum_{j=1}^{r} \int_0^t \xi_s^* L_{X_j} K \circ dM_s^j$$

$$= K + \sum_{j=1}^{r} \int_0^t \xi_s^* L_{X_j} K\, dM_s^j + \frac{1}{2} \sum_{j,k} \int_0^t \xi_s^* L_{X_j} L_{X_k} K d<M^j, M^k>_s .$$

Similarly as Theorem 2.4, we have

Theorem 3.4. Let ξ_t be a solution of (2.20). Set

$$K_t = E[\xi_t^* K].$$

Then it satisfies

$$\frac{\partial}{\partial t}K_t = \frac{1}{2}\left(\sum_{j=1}^{r} L_{X_j}^2 + L_{X_0}\right)K_t,$$

$$K_0 = K.$$

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