MICHAEL I. TAKSAR

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REGENERATIVE SETS ON REAL LINE

M. I. Taksar
Cornell University

A number of papers are devoted to studying regenerative sets on a positive half-line, i.e. random sets M which form a replica of themselves after each stopping time $\tau \in M$. Our objective is to construct translation invariant sets of this type on the entire real line. Besides we start from a weaker definition of regenerativity, involving only special times τ-infima of intersections M with half lines $[t,\infty[$.

1. INTRODUCTION

Let (Ω,F,P) be a probability space and let T be the real line $]-\infty,\infty[$. A subset M of $T \times \Omega$ is called a random set if M is $\mathcal{B} \times F$-measurable and $M(\omega)$ is nonempty for a.e. ω, where $\mathcal{B}$ is the Borel σ-field of T and $M(\omega)$ is the ω-section of M.

We say that M is a closed (discrete, perfect, etc.) random set if $M(\omega)$ is closed (discrete, perfect, etc.) for almost all ω. We consider only closed random sets, so we shall not mention this explicitly each time. The complement of $M(\omega)$ is a countable union of disjoint open intervals $]\gamma,\delta[$. Let $I(t)$ stand for the interval $]\gamma,\delta[$ which covers t and z_t^-, z_t^+ stand for the ends of $I(t)$. (We put $z_t^- = z_t^+ = t$ if $t \in M$.) We associate with M a $(T)^2$-valued stochastic process $z_t = (z_t^-, z_t^+)$, $t \in T$. The σ-algebra in Ω generated by this process is denoted by $\sigma(M)$ and independence (or conditional independence) of random sets means independence (or conditional independence) of corresponding associated processes.

We denote by MI the intersection of M with an interval I and by $\sigma(MI)$ the corresponding σ-algebra in Ω. We write M_t and M^t for intersections of M with $I =]-\infty,t]$ and $[t,+\infty[$.

A random set M is called Markov if

1. A. For each t

   a) M_t and M^t are conditionally independent, given z_t^+.

   b) M_t and M^t are conditionally independent, given z_t^-.

A random set M is called right regenerative (r.r.) if

1. B. For any t, $M_{z_t^+}$ and $\tilde{M}^t = M^{z_t^+} - z_t^+$ are independent.

A random set is called left regenerative (l.r.) if

1. B'. For any t, $M^{z_t^-}$ and $\tilde{M}_t = M_{z_t^-} - z_t^-$ are independent.

The set satisfying both 1.B and 1.B' is called double regenerative (d.r.). It is obvious that 1.B implies 1.A.a and 1.B' implies 1.A.b and thus any d.r. set is Markov.

A random set is called translation invariant (t.i.) if

1. C. The distribution of $M + t$ does not depend on t.

This is equivalent to an assumption that $(z_t - t, P)$ is a stationary process, where $z_t - t = (z_t^- - t,\ z_t^+ - t)$.

Processes with independent increments can be used for constructing r.r. random sets. The following facts on these processes can be found, for example, in [1].

Let α be a nonnegative constant and Π be a measure on $]0,\infty[$ such that

$$\int_0^\infty x \wedge 1\ \Pi(dx) < \infty \tag{1.1}$$

Then there exists a right-continuous increasing process y_t with independent increments, $t \in T_+ = [0,\infty[$, with transition probabilities P_ℓ and the set of discontinuities J such that

1. D. For any function f on T

$$P_\ell \sum_{t \in J,\, c < t \leq d} f(y_t - y_{t-}) = (d - c) \int_0^\infty f(x)\ \Pi(dx) \qquad {}^*)$$

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*We denote by one letter a measure and the integral with respect to this measure. Thus for a random variable ξ $P\xi$ means its mathematical expectation with respect to P.

1. E. $y_d - y_c = \alpha(d - c) + \sum_{t \in J, c < t \leq d} (y_t - y_{t-})$.

We call y_t an (α,Π)-process. The constant α is called translation constant and Π is called the Levy measure of the process. An (α,Π)-process is uniquely determined by its initial distribution at time 0. Put

$$e(\Pi) = \int_0^\infty x\Pi(dx) = \int_0^\infty \Pi(]x,\infty[)dx$$

The condition

$$e(\Pi) < \infty \tag{1.2}$$

is necessary and sufficient for $P_\ell(y_t - y_s) < \infty$ for all ℓ, s and t.

It follows from the results of Section 6 that the range of y_t (i.e. the closure of the set of values of y_t) is r.r. and Markov.

A set M is called (α,Π)-generated if for every $s > -\infty$ there exist an (α,Π)-process whose range restricted to $[s,\infty[$ has the same distribution as M^s. Our main result is the following.

THEOREM 1. Each right regenerative translation invariant closed random set M is left regenerative. There exists $\alpha \geq 0$ and Π subject to (1.2) such that M is (α,Π)-generated. The vector (α,Π) is unique up to proportionality and satisfies the following relations:

$$P\{t \in M\} = \alpha/(\alpha + e(\Pi)) \; ; \tag{1.3}$$

for any function f on $T \times T$

$$P\{\sum_\gamma f(\gamma,\delta)\} = c \int_{-\infty}^\infty \{\int_0^\infty f(s, s + y)\, \Pi(dy)\}ds \tag{1.4}$$

where

$$c = (\alpha + e(\Pi))^{-1} \tag{1.5}$$

Let $\alpha \geq 0$ and Π satisfy (1.2) and $\alpha + e(\Pi) > 0$. Then it is possible to construct one and only one double regenerative translation invariant closed set M which is (α,Π)-generated.

A random set M is called thin if for any t

$$P\{t \in M\} = 0$$

THEOREM 2. Each thin t.i set M subject to 1.A.a is d.r. and thus is $(0,\Pi)$-generated for some Π subject to (1.2).

Discrete t.i.r.r. sets can be considered in the domain of Renewal Theory. To each set M of such type there corresponds a random flow whose times of arrivals coincide with the points of M. In this case the property 1.C is equivalent to the stationarity of the flow, that is the distribution of the number of arrivals which occur in the intervals $I_1, I_2, \ldots, I_k$ is the same as that of $I_1 + t$, $I_2 + t, \ldots, I_k + t$ (see [2], p. 339). The property 1.B is equivalent to the independence of the lengths of the intervals between successive arrivals (waiting times). By virtue of Fubini's theorem any discrete M is thin, consequently it is $(0,\Pi)$-generated. The range of a $(0,\Pi)$-process is discrete iff Π is a finite measure (see [3], Ch. XI, TXI.1). Hence Theorem 1 implies the following known result.

THEOREM 3. All the stationary flows on the real line with independent waiting times between successive arrivals are in one-to-one correspondence with probability measures Π on $]0,\infty[$, subject to (1.2). The measure Π is the distribution of the waiting time between successive arrivals.

In the theory of regenerative sets on T_+ (see, for instance, [3], where further references may be found) it is supposed that M contains 0 and that M_t is adapted to an increasing family of σ-fields A_t in Ω. The definition of (right) regenerativity in the case of unbounded sets M is equivalent to the following:

1. F. For every stopping time τ with respect to A_t such that $\tau \in M$

   a) $M^\tau - \tau$ and M_τ are independent

   b) the distribution of $M^\tau - \tau$ does not depend on τ.

In our case 1.F holds for $\tau = z_t^+$: the relation 1.F.a is just the same as 1.B; and 1.F.b follows from translation invariance. Since we use only a very limited initial class of random times τ, our construction of the (α,Π)-process starting from a r.r. set is much more complicated than the analogous one in [3].

In Section 2 we prove some properties of the (α,Π)-processes. In the next section we introduce the families of the σ-fields generated by a random set M and prove that every t.i.r.r. set is either perfect or discrete. Then we construct, for each t, an (α,Π)-process $y_{\cdot}$ whose range coincides with M^t a.s. In the next two sections we prove that there exists no more than one t.i. (α,Π)-generated set and we construct such a set, given α and Π.

The main idea of the construction is rather simple. We take a sequence of (α,Π)-processes whose initial distributions are uniform on $[-n,0]$ and take the weak limit of their ranges, when $n \to \infty$. This simple idea however, causes a lot of technical problems; the most difficult is to show that all the properties of (α,Π)-generated sets are stable under a weak limit.

Section 7 is devoted to the proof of Theorem 2. We give an example of a t.i. Markov set which is not r.r.

The word "function" will always stand for a nonnegative bounded measurable function. All subsets Γ, Δ of T and $(T)^n$ are supposed to be Borel. We denote $[t,\infty[$ and $]-\infty,t]$ by T^t and T_t respectively. If Γ is a subset of T then the writing $\Gamma \geq t$ (or $\Gamma \leq t$) means that $\Gamma \subset T^t$ (or $\Gamma \subset T_t$).

If ξ is a random variable, then writing $\xi \in M$ means that $\xi(\omega) \in M(\omega)$ for a.e. ω.

If we have a Euclidean space E and we define a measure ν only on the subset Δ of the space E then it is always assumed that $\nu(E\setminus\Delta) = 0$.

2. PROPERTIES OF (α,Π)-PROCESSES

Fix α and Π subject to (1.1) and consider an (α,Π)-process y_t. Put

$$\sigma_\ell = \inf\{t : y_t > \ell\} = \inf\{t : y_{t-} > \ell\} \; ; \tag{2.1}$$
$$U_\ell = y_{\sigma_\ell -}, \quad V_\ell = y_{\sigma_\ell} \quad , \quad Y_\ell = (U_\ell, V_\ell) \; .$$

We call $Y_\ell = (U_\ell,V_\ell)$ the jump over ℓ.

Denote

$$\Pi(s;\Gamma) = \Pi(\Gamma - s) \; , \qquad \Gamma \subset T$$
$$\Pi_s(\Delta \times T) = 1_\Delta(s)\, \Pi(s;\Gamma) \; , \qquad \Gamma,\, \Delta \subset T$$

For f being a function on $(T)^2$ set

$$A_f(s,u) = \sum_{t \in J,\, s < t \le u} f(y_{t-}, y_t) \tag{2.2}$$

If $\Delta \subset T \times T$ we write $A_\Delta(s,u)$ instead of $A_{1_\Delta}(s,u)$. Writing A_f without any arguments stands for $A_f(0,\infty)$.

LEMMA 2.1. For any function f on $T \times T$,

$$P_b\{A_f\} = \int \lambda_b(dx)\, \Pi_x(f)$$

where

$$\lambda_b(\Gamma) = P_b \int_0^\infty 1_\Gamma(y_t)\,dt \; , \qquad \Gamma \subset T \tag{2.3}$$

The proof of this Lemma is well-known. •

LEMMA 2.2. For any $\Gamma \subset T^\ell$

$$P_b\{V_\ell \in \Gamma \mid U_\ell\} = \Pi(U_\ell;\Gamma)/\Pi(U_\ell;T^\ell) \quad \text{on the set } \{U_\ell < \ell\} \tag{2.4}$$

<u>Proof</u>. Let $\Gamma_1, \Gamma_2 \subset T$, $\Gamma_1 < \ell \leq \Gamma_2$. Put $\Delta = \Gamma_1 \times \Gamma_2 \subset T \times T$. By virtue of Lemma 2.1

$$P_b\{U_\ell \in \Gamma_1, V_\ell \in \Gamma_2\} = P_b\{Y_\ell \in \Delta\} = P_b\{A_\Delta\} = \int \lambda_b(dx)\Pi_x(\Delta)$$

$$= \int_{\Gamma_1} \lambda_b(dx)\ \Pi(x;\Gamma_2) \tag{2.5}$$

Similarly

$$P_b\{U_\ell \in \Gamma_1\} = \int_{\Gamma_1} \lambda_b(dx)\ \Pi(x;T^\ell) \tag{2.6}$$

Comparing (2.5) and (2.6) we obtain (2.4).

LEMMA 2.3. Let (y_t,P) be an (α,Π)-process. If $e(\Pi) = \infty$, then for each $a > 0$

$$P\{V_N < N+a\} \to 0 \qquad \text{as} \quad N \to \infty .$$

<u>Proof</u>. Suppose $P = P_0$. Put $\eta(N) = \inf\{k:k\text{-integer},\ y_k \geq N\}$. The conditions of the Lemma imply that $P_0\{y_1\} = \infty$. It is known (see [4], for example) that for each $a > 0$

$$P_0\{y_{\eta(N)} < N+a\} \to 0 \qquad \text{as} \quad N \to \infty .$$

Choose m such that $P_0\{y_1 > m\} < \varepsilon$. Let N be such that for any $L > N$, $P_0\{y_{\eta(L)} < L+m+a\} < \varepsilon$. Since $\eta(L) \leq \sigma_L + 1$, we have $y_{\eta(L)} \leq y_{\sigma_L+1}$. Therefore, for any $L > N$,

$$P_0\{V_L < L+a\} \leq P_0\{y_{\eta(L)} \leq L+m+a\} + P_0\{y_{\eta(L)} > L+m+a\ ,\ y_{\eta(L)} - y_{\sigma_L} > m\}$$

$$\leq \varepsilon + P_0\{y_1 > m\} \leq 2\varepsilon\ .$$

The passage from P_0 to an arbitrary P is trivial.

LEMMA 2.4. Let (y_t,P_b) be an (α,Π)-process and σ_N defined by (2.1). Then

$$\lim_{N\to\infty} P_b\{\sigma_N\}/N = (\alpha + e(\Pi))^{-1} \tag{2.7}$$

Moreover, the convergence is uniform for all $b \in [c,a]$, $c < a$.

Proof. The fundamental theorem of Renewal Theory implies that

$$\lim_{N\to\infty} P_b\{\phi_N\}/N = (P_b\{y_1\})^{-1} = (\alpha + e(\Pi))^{-1}$$

where $\phi_N = \sup\{k:k\text{-integer},\ y_k < N\}$. (See [5], Ch. 9.)

Since $\phi_N \le \sigma_N \le \phi_N+1$, we have (2.7)

Inasmuch as for any $\ell > 0$

$$N^{-1}P_{a-\ell}\{\sigma_N\} = (N+\ell)^{-1}\ P_a\{\sigma_{N+\ell}\}(N+\ell)/N$$

the convergence in the left side of (2.7) for any fixed a implies the uniform convergence of (2.7) for all $b \in [c,a]$.

LEMMA 2.5. Let $f(x)$ be a bounded function on T^t. Suppose f has at most a countable number of discontinuities. Then so do the functions

$$\hat{f}(x) = \Pi(x;f)\ ,$$

$$\bar{f}(x) = \Pi(x;f)/\Pi(x;T^t), \qquad x \in\ ]-\infty,t[\ .$$

Proof. Let Λ_1 be the set of discontinuities f and Λ_2 be the countable set of atoms of Π. Put

$$\Lambda = \{y:y = x_1 - x_2,\ x_i \in \Lambda_i\}$$

The family of measures $\Pi(x;-)$, $x \in T_{t-\varepsilon}$ is uniformly bounded by $\Pi(T^\varepsilon)$ and is weakly continuous with respect to x (being the shift on x of a single measure Π). Therefore (see [6], Th. 5.1) $\hat{f}(x)$ is continuous for all x such that $\Pi(x;-)$ does not charge Λ_1, that is for all $x \bar{\in} \Lambda$.

LEMMA 2.6. Let f be a continuous function on $(T \times T)^k$ and let $t_1 < t_2 < \cdots < t_k$. Then

$$P_b\{f(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_k})\}$$

is a left-continuous function of b on the set $\{b < t_1\}$.

Proof. Consider $k = 1$. (The case $k > 1$ is similar.) Let $t > a$, and $b_n \uparrow a$. Put $t_n = t - b_n$. Since $\sigma_{t_n} = \sigma_t$ on the set $\{V_t > t_n\}$, we have

$$P_{b_n}\{f(Y_t)\} = P_a\{f(Y_{t_n})\} = P_a\{f(Y_{t_n});V_t > t_n\} + P_a\{f(Y_{t_n});V_t \leq t_n\} \quad (2.8)$$

Since $t_n \downarrow t$, then

$$\sigma_{t_n} \downarrow \sigma_t; \quad \{V_t > t_n\} \uparrow \{V_t > t\}; \quad \{V_t \leq t_n\} \downarrow \{V_t = t\} \quad .$$

Using the bounded convergence theorem we get that the limit of (2.8) is equal to

$$P_a\{f(Y_t);V_t > t\} + P_a\{f(t,t);V_t = t\} = P_a\{f(Y_t)\} \; .$$

3. THE STRUCTURE OF A T.I.R.R. SET

In this section we prove that each t.i.r.r. set M is either discrete or perfect.

We put for convenience $u_t = z_t^-$, $v_t = z_t^+$. Set

$D_t^\varepsilon = \{v_t \in M, \;\;]v_t, v_t + \varepsilon[\cap M = \emptyset\}$, $\quad D_t^0 = \bigcup_{\varepsilon > 0} D_t^\varepsilon$

$C_t = \{v_t \in M$ and for each $\varepsilon > 0$ $]v_t, v_t + \varepsilon[\cap M \neq \emptyset\}$

LEMMA 3.1. Either

$$P\{D_t^0\} = 1 \qquad \text{for all } t \quad (3.1)$$

or

$$P\{C_t\} = 1 \qquad \text{for all } t. \quad (3.2)$$

Proof. 1^0. We have $P\{M \neq \emptyset\} = 1$, therefore

$$\lim_{t \to -\infty} P\{M^t \neq \emptyset\} = 1$$

Since M is t.i. then $P\{M^t \neq \emptyset\}$ does not depend on t and therefore is equal to 1. This implies $v_t < \infty$ a.s., so

$$v_t \in M \;\; \text{a.s.} \quad (3.3)$$

2^0. Denote $\alpha^\varepsilon = P\{D_t^\varepsilon\}$, $\beta = P\{C_t\}$. Relation (3.3) implies

$$\alpha^0 + \beta = P\{v_t \in M\} = 1 \tag{3.4}$$

Put

$$t(k,n) = k2^{-n} \tag{3.5}$$

$$L(k,n) = [t(k-1,\ n),\ t(k,n)[\tag{3.6}$$

$$\ell_n(s) = t(k,n) \quad \text{if } s \in L(k,n) \tag{3.7}$$

Let $\phi = \ell_n(v_t)$. Calculate

$$P\{C_t\ D_\phi^\varepsilon\} = \sum_k P\{C_t, \phi = t(k,n),\ D_{t(k,n)}^\varepsilon\} \tag{3.8}$$

Since $\{C_t, \phi = t(k,n)\}$ is $M_{t(k,n)}$-measurable and $D_{t(k,n)}^\varepsilon$ is $\widetilde{M}^{t(k,n)}$-measurable, we may apply 1.B to (3.8) and obtain

$$P\{C_t D_\phi^\varepsilon\} = P\{D_t^\varepsilon\}\ P\{\sum_k (C_t, \phi = t(n,k))\} = \alpha^\varepsilon P\{C_t\} = \alpha^\varepsilon \beta \tag{3.9}$$

Now let $n \to \infty$. On the set $C_t\ v_\phi \downarrow v_t$ and $1_{D_\phi^\varepsilon} \to 0$. Therefore the left side of (3.9) tends to 0 when $n \to \infty$. We get $\alpha^\varepsilon \beta = 0$ for each $\varepsilon > 0$. Since $\alpha^0 = \sup \alpha^\varepsilon$ then $\alpha^0 \beta = 0$, Comparing the last equality with (3.4) we get the statement of the lemma.

LEMMA 3.2. If M satisfies (3.1) then M is discrete, if M satisfies (3.2) then M is perfect.

Proof. 1^0. Put

$$\tau(0,t) = \tau_t = v_{t+} = \inf\{s > t,\ s \in M\} \tag{3.10}$$

$$\tau(k,t) = \tau(0, \tau(k-1,t))$$

If (3.1) holds then expressions similar to those of (3.8) and (3.9) show that for each k

$$P\{]\tau(k,t),\ \tau(k,t) + \varepsilon[\ \cap M = \emptyset \ \text{ for some } \ \varepsilon > 0\} = 1$$

and all $\eta_k = \tau(k,t) - \tau(k-1,t)$ are independent and identically distributed. Consequently $\tau(k,t) \to \infty$ as $k \to \infty$; M is equal to the union of the graphs of $\tau(k;t)$; as a result, M is discrete.

2^0. Suppose (3.2) holds and ϕ is an isolated point of $M(\omega)$. Then there exists $\varepsilon > 0$ such that $I =]\phi-\varepsilon, \phi[\cap M(\omega) = \emptyset$. Hence $\omega \in D_t^0$ for all $t \in I$. Applying the Fubini theorem, we see that this can happen only for ω with P-measure zero.

4. CONSTRUCTION OF THE GENERATING (α,Π)-PROCESS

In this section we construct an (α,Π)-process whose range is indistinguishable from M^t.

The case in which M is discrete has been already treated. In Section 1^0 of Lemma 3.2 we showed that for each t M^t is indistinguishable from the union of the graphs of the sums of i.i.d. positive random variables η_k. Thus M is $(0,\Pi)$-generated for $\Pi(\Gamma) = P\{\eta_1 \in \Gamma\}$

In the case when M is perfect the natural candidate for a generating process is the inverse of the local time of M. Since we can use regenerativity of M only for a very restricted class of stopping times we must construct a local time μ_t in such a way that μ_t has no discontinuity when t is the left endpoint of an interval contiguous to M. For this purpose we introduce the notion of regular and irregular points of a set and prove that the structure of the set of regular points on the interval $[a,\infty[$ depends only on the structure of the original set on the same interval (Lemma 4.1).

Put $N = M^0$. Denote

$$F_t = \bar{\sigma}(N[0,t]),\ A_s = F_{s+} = \bigwedge_{t > s} F_t\ ,$$

$\bar{\sigma}(M)$ being the minimal σ-field generated by N and all sets of P-measure 0. Let τ_s be defined by (3.10) and $\hat{\xi}_t = \exp(t - \tau_t)$. Let ξ_t stand for the well-measurable projection of $\hat{\xi}_t$ with respect to A_t. Denote by $\overleftarrow{N}$ the set of left endpoints of the intervals contiguous to N and put

$$\overleftarrow{N}_{reg} = \{t:\xi_t < 1\} \cap \overleftarrow{N} = \{t > 0: t = \gamma,\ \xi_t < 1\}$$

$$\overleftarrow{N}_{ir} = \overleftarrow{N}\setminus\overleftarrow{N}_{reg} = \{t > 0: t = \gamma,\ \xi_t \geq 1\}$$

The definition of $\overleftarrow{L}_{reg}$ and $\overleftarrow{L}_{ir}$ for an arbitrary set L is similar to the one given above. First we consider σ-fields $\mathcal{B}_t$ generated by the set L. Then we consider the family of stopping times (with respect to $\mathcal{B}_t$) τ_t which are the first hitting times of L after t. We consider the well-measurable projection ξ_t of $\exp(t - \tau_t)$, with respect to the filtration $\mathcal{B}_t$. The set of the left endpoints s of intervals contiguous to L such that $\xi_s < 1$ (such that $\xi_s = 1$) is denoted $\overleftarrow{L}_{reg}$ (is denoted $\overleftarrow{L}_{ir}$).

LEMMA 4.1. For any $u \geq 0$

$$(\widetilde{M}^u)^{\leftarrow}_{reg} = \{\overleftarrow{N}_{reg} - v_u\} \cap [0,\infty[\tag{4.1}$$

Proof. Denote $\mathcal{B}_s = \bigcap_{t>s} \bar{\sigma}(\widetilde{M}^u[0,t])$, $\widetilde{\tau}_t = \inf[s > t : s \in \widetilde{M}^u\} = \tau_{v_u+t} - t$, $\hat{\eta}_t = \exp(t - \widetilde{\tau}_t)$. Let η_t stand for the well-measurable projections of $\hat{\eta}_t$ with respect to $\mathcal{B}_t$. (See [7], Ch. V for the definition and details.) The statement of the Lemma follows from the following equality

$$\eta_t = \xi_{t+v_u} \quad \text{for all } t \quad \text{a.s.}, \tag{4.2}$$

which we are going to prove. By [7], Ch. IV, T28 ξ_t and η_t are a.s. right-continuous, hence it is enough to prove (4.2) for any fixed t. Put $\sigma = t + v_u$. Since σ is a stopping time with respect to $\mathcal{A}_t$, then $\mathcal{B}_t \subset \mathcal{A}_\sigma$. For any $A \in \mathcal{B}_t$

$$P\{1_A\xi_\sigma\} = P\{1_A\hat{\xi}_\sigma\} = P\{1_A\hat{\xi}_{t+v_u}\} = P\{1_A\hat{\eta}_t\} = P\{1_A\eta_t\}$$

Therefore

$$P\{\xi_\sigma/\mathcal{B}_t\} = P\{\eta_t/\mathcal{B}_t\} = \eta_t \quad \text{a.s.} \tag{4.3}$$

Prove that ξ_σ is $\mathcal{B}_t$-measurable. Define $L(k,n)$ by (3.6) and put $A(k,n) = \{v_u \in L(k,n)\}$. Put $\varepsilon = 2^{-n}$ and $a = v_u + \varepsilon$. We have $\xi_\sigma = \sum_k 1_{A(k,n)}\xi_\sigma$ is $\bar{\sigma}(N_{a+t})$-measurable. Since $\hat{\eta}_t$ is $\sigma(\widetilde{M}^u)$-measurable and

$$\bar{\sigma}(N_{a+t}) = \bar{\sigma}(N_{v_u}) \vee \bar{\sigma}\{\widetilde{M}^u[0, t+\varepsilon]\} \tag{4.4}$$

we obtain

$$P\{\xi_\sigma | N_{v_u}\} = P\{P\{\xi_\sigma | A_\sigma\} | N_{v_u}\} = P\{\hat{\xi}_\sigma | N_{v_u}\} = P\{\hat{\eta}_t | N_{v_u}\} = P\{\hat{\eta}_t\}$$

$$= P\{\hat{\xi}_\sigma\} = P\{\xi_\sigma\} \tag{4.5}$$

The expression (4.5) shows that ξ_σ and N_{v_u} are independent. Comparing this with (4.4) and 1.B we see that ξ_σ is $\bar{\sigma}(\tilde{M}^u[0,t+\varepsilon])$-measurable. In view of arbitrariness of ε we get that ξ_σ is $\mathcal{B}_t$-measurable.

For a random set M and real numbers a and b put

$$\zeta(M,a,b) = m(M]a,b]) + \sum 1 - \exp(\gamma-\delta) ,$$

where the sum is taken over all (γ,δ) such that $a < \gamma \leq b$; $\gamma \in \overleftarrow{M}_{ir}$; and m is the Lebesgue measure. The functional ζ is used for the construction of a local time μ_t. We want μ_t to be "homogeneous," that is $\mu_{s+b} - \mu_s$ to depend only on the "shape" of $N[s,s+b]$ but not on s. For this reason we need the following

LEMMA 4.2. For any s, $b > 0$

$$\zeta(N,v_s,v_s+b) = \zeta(\tilde{M}^s,0,b)$$

This follows immediately from Lemma 4.1.

LEMMA 4.3. If τ is a stopping time with respect to A_t then

$$P\{\tau \in \overleftarrow{M}_{ir}\} = 0$$

<u>Proof</u>. Put $A = \{\omega:\tau(\omega) \in \overleftarrow{M}_{ir}\}$. Let $\sigma(\omega) = \tau(\omega)$ if $\omega \in A$ and $\sigma(\omega) = \infty$, if $\omega \notin A$. Since σ is a stopping time and $A \in A_\sigma$ we have (see [7], Ch. V, T37)

$$P\{(\xi_\sigma - \hat{\xi}_\sigma)1_A\} = 0 \tag{4.6}$$

But $\xi_\sigma = \xi_\tau \geq 1$ on A and $\hat{\xi}_\sigma = \exp(\sigma-\tau_\sigma) < 1$. Therefore (4.6) implies $P\{A\} = 0$.

Put $\hat{\mu}_t = \zeta(N,0,t)$ and let μ_t stand for the dual well-measurable projection of $\hat{\mu}_t$ with respect to A_t. (See [7], Ch. V for the definition and details.) Put

$$y_s = \inf\{u:\mu_u > s\} \tag{4.7}$$

We prove that y_s generates N (Lemma 4.5). This proof uses a common technique of the general theory of processes (see [7], Ch. IV, V). Lemma 4.6 proves that y_s is a homogeneous process with independent increments. In order to apply 1.B we have to approximate the stopping time y_s by the stopping times η^n such that $\eta^n \downarrow y_s$ and η^n belongs to the set of the right endpoints of the intervals contiguous to M. Such an approximation is possible if y_s differs from all left endpoints of this type intervals. This fact follows from Lemma 4.4.

LEMMA 4.4. For any s

$$P\{y_s \in \overleftarrow{N}\} = 0 . \tag{4.8}$$

<u>Proof</u>. 1^0. Since $\hat{\mu}_t$ has discontinuities only when $t \in \overleftarrow{M}_{ir}$, then by Lemma 4.3 for any stopping time σ, $\hat{\mu}_\sigma = \hat{\mu}_{\sigma-}$ a.s. By [7], Ch. V, T30 for any well-measurable with respect to A_t process ϕ_t

$$P\{\int_0^\infty \phi_t \, d\hat{\mu}_t\} = P\{\int_0^\infty \phi_t d\mu_t\} \tag{4.9}$$

Taking $\phi_t = 1_{t=\sigma}$, we find out that $\mu_\sigma = \mu_{\sigma-}$ a.s. for any stopping time σ. By [7], Ch. IV, T30 μ_t is a continuous process.

2^0. Fix s and put $y_s = z$. Since $\mu.$ is continuous, $\mu_z = s$; and for any $\varepsilon > 0$, $\mu_{z+\varepsilon} > s$. Put $A = \{z \in \overleftarrow{N}\}$, $I = [z,\tau_z[$, τ_t being defined by (3.10) and set $\phi_t = 1_A 1_I(t)$. Applying (4.9) to ϕ_t, we get

$$P\{1_A(\mu_{\tau_z} - \mu_z)\} = P\{1_A \, m(M[z,\tau_z[)\} + P\{1_A 1_{\overleftarrow{M}_{ir}}(z)(1 - \exp(z-\tau_z)\} \tag{4.10}$$

The first summand in the right side of (4.10) vanishes, because $m(M[z,\tau_z[) = 0$ for any z. The second summand is also equal to zero, because z is a stopping time and by Lemma 4.3 $P\{z \in \overleftarrow{M}_{ir}\} = 0$. Since $\mu_{\tau_z} > \mu_z$ a.s. on A, we get $P\{A\} = 0$.

LEMMA 4.5. The range $\tilde{N}$ of the process y_s is indistinguishable from N.

Proof. 1^0. Since y_s is an increasing right continuous process, then $\tilde{N}$ is a closure of the set $\{t: t = y_r,\ r\text{-rational}\}$. By [7], Ch. VI, T4, $\tilde{N}$ is a well-measurable set. The set N is also well-measurable, because N is closed and N_t is adapted to $\mathcal{A}_t$. Put $A_k = \overset{\leftarrow}{N} \cap \{t: \xi_t < 1 - k^{-1}\}$. By [7], Ch. VI, T2, A_k is progressive measurable. Usual arguments show that A_k is discrete. By [7], Ch. VI, T4 it is well-measurable. Inasmuch as $\overset{\leftarrow}{N}_{reg} = \bigcup_k A_k$ we get that $\overset{\leftarrow}{N}_{reg}$ is well-measurable the same as $N \setminus \overset{\leftarrow}{N}_{reg}$.

2^0. Since N and $N \setminus \overset{\leftarrow}{N}_{reg}$ have the same closure it is enough to show that

$$N \supseteq \tilde{N} \qquad \text{a.s.} \tag{4.11}$$

$$\tilde{N} \supseteq N \setminus \overset{\leftarrow}{N}_{reg} \quad \text{a.s.} \tag{4.12}$$

Let σ be a stopping time such that $\sigma \in N \setminus \overset{\leftarrow}{N}_{reg}$ a.s. on $\{\sigma < \infty\}$. By Lemma 4.3 $P\{\sigma \in \overset{\leftarrow}{N}\} = 0$, hence $\hat{\mu}_{\sigma+\varepsilon} - \hat{\mu}_\sigma > 0$ for all $\varepsilon > 0$. The same reasoning as in Section 2^0 of Lemna 4.4 shows that $\mu_{\sigma+\varepsilon} - \mu_\sigma > 0$ a.s. on $\{\sigma < \infty\}$. Therefore $y_{\mu_\sigma} = \sigma$ a.s. on $\{\sigma < \infty\}$ and we have $\sigma \in \tilde{N}$ a.s. on $\{\sigma < \infty\}$. By [7], Ch. IV, T13, this implies (4.12).

By (4.9)

$$P\{\int_0^\infty 1_{T\setminus N}(t)\,d\mu_t\} = P\{\int_0^\infty 1_{T\setminus N}(t)\,d\hat{\mu}_t\} = 0 .$$

Hence μ_t does not increase on $T \setminus N$ a.s.; and $P\{y_r \in T \setminus N$ for any $r > 0\} = 0$. This implies (4.11).

LEMMA 4.6. The process (y_t, P) is a homogeneous process with independent increments.

Proof. 1^0. Let us show that for each $r \geq 0$

$$P\{r \in \overset{\leftarrow}{N}\} = 0 \tag{4.13}$$

In view of 1.C the left side of (4.13) does not depend on r; therefore

$$P\{r \in \overset{\leftarrow}{N}\} = \int_0^\infty e^{-u}\, P\{u \in \overset{\leftarrow}{N}\}\,du = P\{\int_0^\infty e^{-u}\, 1_{\overset{\leftarrow}{N}}(u)\,du\} = 0$$

2^0. Let $t(k,n)$, $L(k,n)$ and ℓ_n be defined by formulae (3.5), (3.6) and

(3.7) respectively. Fix $0 \le s < t$ and put $z = y_s$; $z^n = \tau_{\ell_n(y_s)}$. Fix $a \ge 0$. Let $\eta = \tau_{t(k,n)}$; put

$$B = \{\mu_{z+a} - \mu_z < t-s\}, \qquad B_n = \{\mu_{z^n+a} - \mu_{z^n} < t-s\},$$

$$B_n^k = \{\mu_{\eta+a} - \mu_\eta < t-s\}, \qquad C_n^k = \{z \in L(n,k)\}$$

In view of 1.C $P\{\zeta(\widetilde{M}^u,0,c) < b\}$ does not depend on u; we denote this number by $r(c,b)$.

Let $A \in A_{y_s}$. Consider

$$P\{A, y_t - y_s > a\} = P\{A, \mu_{z+a} - \mu_z < t-s\} = P\{AB\} \tag{4.14}$$

By Lemma 4.4. $P\{y_s \in \overleftarrow{M}\} = 0$; therefore, $z^n \downarrow z$ a.s. In view of continuity of μ_t, $B^n \to B$; hence

$$P\{AB\} = \lim_{n\to\infty} P\{AB^n\} = \lim_{n\to\infty} P\{\sum_{k=1}^{\infty} AC_n^k B_n\} \tag{4.15}$$

Note that $C_n^k B_n = C_n^k B_n^k$. Put $D = AC_n^k$, and $\phi_t = 1_D 1_{\eta < t < \eta+a}$. Applying (4.9) to ϕ, we get

$$P\{DB_n\} = P\{DB_n^k\} = P\{D, \hat{\mu}_{\eta+a} - \hat{\mu}_\eta < t-s\} = P\{D, \zeta(N,\eta,\ \eta+a) < t-s\} \tag{4.16}$$

Formula (4.13) implies $\tau_{t(k,n)} = v_{t(k,n)}$ a.s., hence we can replace in (4.16) $\tau_{t(k,n)}$ by $v_{t(k,n)}$ and apply Lemma 4.2. Doing so, we get

$$P\{DB_n^k\} = P\{D, \zeta(\widetilde{M}^{t(k,n)},0,a) < t-s\} \tag{4.17}$$

Since D is $\bar{\sigma}(M_{t(k,n)})$-measurable, we can apply 1.B to (4.17)

$$P\{DB_n^k\} = P\{D\}\ r(a,t-s) \tag{4.18}$$

Comparing (4.18), (4.15) and (4.14) we obtain

$$P\{y_t - y_s > a | A_{y_s}\} = r(a,t-s)$$

and that proves the lemma.

LEMMA 4.7. The process y_s is an increasing (α,Π)-process with Π subject to (1.2).

Proof. We have already proved that y_s is a homogeneous process with independent increments; that is an (α,Π)-process.

Since y_s generates M^0, the distribution of $V_t - t$ is equal to that of $\tau_t - t$ for each $t > 0$. By 1.C the distribution of $\tau_t - t$ does not depend on t. Choose a such that $P\{\tau_t - t < a\} > 0.5$. Suppose Π does not satisfy (1.2). By virtue of Lemma 2.3 $P\{V_t - t < a\}$ tends to 0 when $t \to \infty$. Therefore, we come to a contradiction.

LEMMA 4.8. If M is a t.i. (α,Π)-generated set then the vector (α,Π) satisfies the equations (1.3) and (1.4), which determine it up to proportionality.

Proof. 1^0. Let μ_t be the distribution of v_t and let λ_b be defined by (2.3). Consider

$$\Lambda_t(\Gamma) = \int_0^\infty \lambda_b(\Gamma)\, \mu_t(db) , \qquad \Gamma \subset T$$

It is easy to see that for each $a \in T$ and $\Delta \subset T$, $\lambda_b(\Delta) = \lambda_{b+a}(\Delta + a)$. In view of t.i. the same is true for the family of measures μ_t. Therefore,

$$\Lambda_t(\Gamma) = \Lambda_{t+a}(\Gamma + a) \tag{4.19}$$

Let π be a measure on $T \times T$ defined

$$\pi(\Gamma) = P\{\sum_\gamma 1_\Gamma(\gamma,\delta)\} , \qquad \Gamma \subset T \times T$$

Let A_f be defined by (2.2). If $f(x,y)$ is a function on $T \times T$ such that $f(x,y) = 0$ for $x \leq t$, then

$$\pi(f) = P\{\sum_\gamma f(\gamma,\delta)\} = \int_t^\infty \mu_t(db)\, P_b\{A_f\} = \int_t^\infty \Lambda_t(dx)\, \Pi_x(f) \tag{4.20}$$

Taking $s < t$, and applying the same computations we get

$$\pi(f) = \Lambda_s(\Pi.(f)) \tag{4.21}$$

Put $I = [a,b]$, $t < a < b$ and put $f(x,y) = (e(\Pi))^{-1} 1_I(x)(y-x)$. Applying to f (4.20) and (4.21) we obtain

$$\pi(f) = \Lambda_s(I) = \Lambda_t(I) \tag{4.22}$$

In view of (4.19) the relation (4.22) is equivalent to $\Lambda_t(I) = \Lambda_t(I + t - s)$. Therefore, $\Lambda_t(dx) = cm(dx)$ (on $[t,\infty[$). Substituting the expression for Λ_t into (4.20), we get (1.4) for f with support in $T^t \times T$. Standard arguments show that (1.4) holds for all f.

2^0. Let $g(x,y) = y - x$ and $\widetilde{P} = \int \mu_0(db)P_b$. Since M is t.i. we have

$$P\{u \in M\} = L^{-1} \int_0^L P\{t \in M\}dt = \lim_{L\to\infty} L^{-1} P\{\int_0^L 1_M(t)dt\}$$

$$= \lim_{L\to\infty} L^{-1} \widetilde{P}\{V_L - y_0 - A_g(0,\sigma_L)\}$$

$$= \lim_{u\to\infty} \lim_{L\to\infty} P^{(u)}\{V_L - y_0 - A_g(0,\sigma_L)\}, \tag{4.23}$$

where $P^{(u)} = \int_0^u P_b\, \mu_0(db) + \mu_0[u,\infty[P_u$. (The last equality in (4.23) is due to the fact that $|\widetilde{P}(C) - P^{(u)}(C)| \to 0$ for each event C.) By virtue of 1.E the expression under $P^{(u)}$ is equal to $\alpha\sigma_L$. Applying Lemma 2.4 we see that (4.23) is equal to $\alpha/(\alpha + e(\Pi))$.

3^0. Formulae (1.3) and (1.4) imply

$$\Pi(\Gamma) = c^{-1} P\{\sum_{0<\gamma<1} 1_\Gamma(\delta - \gamma)\}, \qquad \Gamma \subset T \tag{4.24}$$

$$\alpha = P\{t \in M\}\, e(\Pi)/(1 - P\{t \in M\}) \tag{4.25}$$

The expressions (4.25) and (4.24) determine (α,Π) up to a constant c.

COROLLARY. The constant c in (1.4) is given by (1.5).

<u>Proof</u>. Since $P\{t \in M\} + P\{u_t < t < v_t\} = 1$ we get

$$\alpha/(\alpha + e(\Pi)) + c \int_{-\infty}^t \Pi(x;T^t) = \alpha/(\alpha + e(\Pi)) + ce(\Pi) = 1$$

and this is equivalent to (1.5).

5. CONSTRUCTION OF A T.I. SET, GIVEN α AND Π.

In this section first we prove that (z_t,P) is a Markov process whose one-dimensional distributions and transition function are uniquely determined by α and Π. This implies the uniqueness of a t.i. (α,Π)-generated set.

The main part of this section is devoted to constructing a t.i. set with given α and Π. First we consider the (α,Π)-process with initial distribution uniform on $[-n,0]$. Let (Y_t,P^n) be the corresponding jump process. We get the process (z_t,P) by passing to a weak limit as $n \to \infty$. To justify this we make use of the following Lemma, proved in [6] (see Th.5.1).

LEMMA 5.1. If ρ^n is a sequence of measures on a topological space X and ρ^n converges weakly to ρ then

$$\rho^n(f) \to \rho(f)$$

for each f whose set of discontinuities has ρ-measure zero.

We apply this lemma to the case of an open half-line X, an absolutely continuous (with respect to Lebesgue's measure m) measure ρ, and a function f with at most a countable number of discontinuities. We use also the following fact, the proof of which is trivial.

LEMMA 5.2. If ρ^n is a sequence of measures on X and any subsequence of ρ^n has a sub-subsequence which converges weakly to a measure ρ, then ρ^n converges weakly to ρ.

The plan of the construction of (z_t,P), given α and Π is the following. First we show that the sequence of distributions of U_t under P_n is tight (Lemma 5.4). Then we show that this sequence is weakly convergent and we find the limit measure (Lemma 5.6). After that we find the conditional distribution of $(V_t, Y_{t_1}, Y_{t_2}, \ldots, Y_{t_k})$, $t \leq t_1 < \cdots < t_k$, given U_t and show that this distribution does not depend on n (Lemma 5.7). Applying Lemma 5.1 we find that the finite dimensional distributions of P^n weakly converge to those of measure P (Lemma 5.9).

Further improvement of the trajectories of (Y_t,P) and the construction of the set is done in Section 6.

LEMMA 5.3. If M is a t.i. (α,Π)-generated set then the associated process (z_t,P) is Markov with one-dimensional distributions

$$\nu_t(\Gamma) = c(\alpha 1_{(t,t)}(\Gamma) + \int_{-\infty}^{t} \Pi_x(\Gamma)dx) \tag{5.1}$$

c given by (1.5), $\Gamma \subset T_t \times T^t$, and transition function

$$p(s,z;t,\Gamma) = \begin{cases} 1_\Gamma(z) \ , & \text{if } y \geq t \\ P_y\{Y_t \in \Gamma\} & \end{cases} \tag{5.2}$$

Here $z = (x,y) \in T \times T$, $\Gamma \subset T_t \times T^t$, P_y the transition probabilities of (α,Π)-process, $Y_t = (U_t,V_t)$ is a jump over t, defined in Section 2.

<u>Proof</u>. If M is t.i. then $P\{t = \gamma\}$ does not depend on t and

$$P\{t=\gamma\} = \int_0^\infty e^{-s} P\{s=\gamma\}ds = P\{\int_0^\infty e^{-s}1_\gamma(s)ds\} = 0 \ .$$

Similarly $P\{t=\delta\} = 0$. Therefore for each t a.s. $v_t = v_{t+}$, $u_t = u_{t-}$. If the range of (α,Π)-process (y_s,P) coincides with M^0, then $v_{t+} = V_t$ and $u_{t-} = U_t$.

The strong Markov property of y_t implies that $z_t = (u_t,v_t)$ is a Markov process with the transition function (5.2). It is obvious that the distribution of z_t is concentrated on $T_t \times T^t$. Let Γ be a subset of $T_t \times T^t$. By virtue of Lemma 4.8 α and Π satisfy (1.3) and (1.4); hence

$$\begin{aligned} P\{z_t \in \Gamma\} &= P\{z_t = (t,t),\ (t,t) \in \Gamma\} + P\{u_t < t,\ z_t \in \Gamma\} \\ &= P\{t \in M\}1_{(t,t)}(\Gamma) + P\{\sum_\gamma 1_{\gamma < t < \delta}\, 1_\Gamma(\gamma,\delta)\} \\ &= 1_\Gamma(t,t)\alpha c + c\int_{-\infty}^{t} \Pi_x(\Gamma)dx = \nu_t(\Gamma) \ . \end{aligned}$$

Lemma 5.3 points to a natural way of constructing a t.i. (α,Π)-generated set. First we have to construct a Markov process (z_t,P) with transition function (5.2) and one-dimensional distributions (5.1) and define M as the range of z_t^+.

Unfortunately, we cannot prove directly that ν_t is an entrance law with respect to p; that is why we use a long and cumbersome procedure to construct (z_t,P). Let λ_b be defined by (2.3) and λ stand for λ_0.

LEMMA 5.4. For any $s \geq 0$ and $a > 0$

$$\lambda[0,a] \geq \lambda[s,s+a] \ . \tag{5.3}$$

There exist $N \geq 0$ and $d > 0$ such that

$$\lambda[s,s+N] \geq d\lambda[0,N] \ . \tag{5.4}$$

<u>Proof</u>. 1^0. Applying strong Markov property, we have

$$\lambda[s,s+a] = P_0\{\int_0^\infty 1_{[s,s+a]}(y_t)dt\} = P_0\{\lambda_{V_s}[V_s,s+a]\}$$

$$\leq P_0\{\lambda_{V_s}[V_s,V_s+a]\} = \lambda[0,a] \ .$$

2^0. Since Π is subject to (1.2), y_t has a finite mean; therefore, we can apply to the sequence y_k, $k = 0,1,2,\ldots$, Renewal Theorem (see [5], p. 363). By virtue of this theorem there exist N_1 and N_2 such that for any $s > N_1$

$$P_0\{y_i \in [s,s+N_2] \text{ for some integer } i\} > 0.5 \ .$$

Therefore for any $s > N_1$, $P_0\{V_s-s < N_2\} > 0.5$. In view of right continuity of y_t, $\sigma_1 > 0$ a.s. P_0; therefore, $d_1 = \lambda[0,1] = P_0\{\sigma_1\} > 0$. Take $N = N_1 + N_2 + 1$. Let $s > 0$ and let $u = s \vee N_1$. We have

$$\lambda[s,s+N] \geq \lambda[u,u+N_2+1] \geq P_0\{1_{V_u-u<N_2} P_{V_u}\{\sigma_{u+1} - \sigma_u\}\}$$

$$\geq 0.5 \ P_0\{\sigma_1\} = d_1/2 \ . \tag{5.5}$$

The inequality (5.5) implies (5.4) with $d = d_1/2\lambda[0,N]$.

COROLLARY. For any t and any $\varepsilon > 0$ there exists m such that for any $b < t$

$$P_b\{U_t < m\} < \varepsilon \ . \tag{5.6}$$

Proof. Let $t = 0$. Put $n = \inf\{i : i \text{ is integer, } i \geq b\}$. For m being negative integer, we have

$$P_b\{U_0 < m\} = P_b\{\int_0^\infty 1_{y_s < m}\, \Pi(y_s;T^0)ds\} = \int_b^m \lambda_b(dx)\, \Pi(x;T^0)$$

$$= \sum_{k=n}^{m} \int_{k-1}^{k} \lambda_b(dx)\, \Pi(x,T^0) \leq \lambda[0,1] \sum_{k=n}^{m} \Pi(k;T^0)$$

$$\leq \lambda[0,1] \sum_{k=-\infty}^{m} \Pi(k;T^0) \leq \lambda[0,1] \int_{-\infty}^{m+1} \Pi(x;T^0)dx \tag{5.7}$$

(The first inequality in (5.7) is due to Lemma 5.2.) In view of (1.2) the right side of (5.7) tends to zero, when $m \to -\infty$.

Consider the sequence of measures

$$P^n = n^{-1} \int_{-n}^{0} P_b\, db\ .$$

As it was mentioned, we are interested in the limit behavior of the finite dimensional distributions of the processes (Y_t, P^n). We want to study separately the singular and the regular parts (with respect to the Lebesgue measure) of the one-dimensional distributions of the above processes. For this purpose we need the following

LEMMA 5.5. For any t and any $n \geq 1$

$$P^n\{U_t = t,\ V_t > t\} = P^n\{V_t = t,\ U_t < t\} = 0\ ; \tag{5.8}$$

and

$$\lim_{\varepsilon \to 0} \sup_n P^n\{t-\varepsilon < U_t < t\} = \lim_{\varepsilon \to 0} \sup_n P^n\{t < V_t < t+\varepsilon\} = 0\ . \tag{5.9}$$

<u>Proof</u>. 1^0. We suppose $t = 0$ (the case in which $t \neq 0$ is similar)

$$P^n\{U_0=0,\ V_0 > 0\} = n^{-1} \int_{-n}^{0} P_b\{V_0 > 0,\ U_0 = 0\}db = n^{-1} \int_0^n P_0\{V_s > s,\ U_s = s\}ds$$

$$= n^{-1}\, P_0\{\int_0^\infty 1_{s \in J}\, 1_{U_s \leq n}\ ds\} = 0\ . \tag{5.10}$$

2^0. Put $f_\varepsilon(x) = x \wedge \varepsilon$, $g_\varepsilon(x,y) = f_\varepsilon(y-x)$. The computations similar to those of (5.10) show

$$P^n\{0 < V_0 < \varepsilon\} = n^{-1} P_0\{\int_0^n 1_{s < V_s < s+\varepsilon}\, ds\}$$

$$\leq n^{-1} P_0\{A_{g_\varepsilon}(0,\sigma_n)\} = \Pi(f_\varepsilon)\, P_0\{\sigma_n\}/n.$$

By Lemma 2.4, $P_0\{\sigma_n\}/n$ has a limit when $n \to \infty$; therefore, this quantity is uniformly bounded for all n. By monotone convergence theorem $\Pi(f_\varepsilon) \to 0$ as $\varepsilon \to 0$.

Let δ_t be a unit mass concentrated at the point t. Denote

$$\kappa_t^n(\Delta) = P^n\{U_t \in \Delta\}, \qquad \Delta \subset T \tag{5.11}$$

$$\beta_t^n = \kappa_t^n\{t\} = P^n\{U_t = t\}, \tag{5.12}$$

$$\rho_t^n = \kappa_t^n - \beta_t^n \delta_t . \tag{5.13}$$

Put

$$\rho_t(\Gamma) = c \int_\Gamma \Pi(x;T^t)\,dx, \qquad \Gamma \subset T_t \tag{5.14}$$

where c is given by (1.5).

LEMMA 5.6. Let $\beta = \alpha c$. Then for each t

$$\lim_{n\to\infty} \beta_t^n = \beta; \tag{5.15}$$

$$w - \lim_{n\to\infty} \rho_t^n = \rho_t ; \tag{5.16}$$

$$w - \lim_{n\to\infty} \kappa_t^n = \kappa_t ; \tag{5.17}$$

where $\kappa_t = \rho_t + \beta\delta_t$; the sign "w-lim" means the weak limit of measures.

<u>Proof</u>. 1^0. Fix t. The Corollary to Lemma 5.4 implies that the sequence of measures κ_t^n on T_t is tight. By virtue of Lemma 5.5 so is the sequence of measures ρ_t^n on $]-\infty,t[$. Therefore there exist measures $\tilde{\kappa}_t$ on T_t and $\tilde{\rho}_t$ on $]-\infty,t[$ such that w-lim $\kappa_t^{n(k)} = \tilde{\kappa}_t$ and w-lim $\rho_t^{n(k)} = \tilde{\rho}_t$ for some sequence $n(k)$

of positive integers. In view of (5.13) there exists a constant β_t such that $\beta_t^{n(k)} \to \beta_t$.

2^0. Let $s \neq t$. Let f be a positive continuous function bounded by 1. Put $g(x) = f(x + s - t)$. It is obvious that

$$P^n\{g(U_s)\} = n^{-1} \int_{-n+t-s}^{t-s} P_b\{f(U_t)\}db .$$

Therefore

$$|\kappa_s^n(g) - \kappa_t^n(f)| = |n^{-1} \int_{-n+t-s}^{-n} P_b\{f(U_t)\}db - n^{-1} \int_{t-s}^{0} P_b\{f(U_t)\}db| \leq 2|t-s|/n. \tag{5.18}$$

The expression (5.18) tends to zero uniformly for all s belonging to a compact set as $n \to \infty$. Consequently, $\kappa_s^{n(k)}$ converges weakly, the convergence is uniform on any bounded set of s and

$$\tilde{\kappa}_s(\Gamma) = \tilde{\kappa}_t(\Gamma + t - s) . \tag{5.19}$$

Formula (5.19) implies that β_t does not depend on t and is equal to a constant $\tilde{\beta}$.

The computations similar to those of Section 2^0 of Lemma 4.8 show

$$\begin{aligned}
\tilde{\beta} &= \lim_{k \to \infty} P^{n(k)}\{U_0 = 0\} \\
&= \lim_{k \to \infty} n(k)^{-1} \int_{-n(k)}^{0} P_b\{U_0 = 0\}db \\
&= \lim_{k \to \infty} n(k)^{-1} \int_{0}^{n(k)} P_0\{U_t = t\}dt \\
&= \lim_{k \to \infty} P_0\{\alpha\sigma_{n(k)} | n(k)\} = \alpha c .
\end{aligned}$$

(The last equality is due to Lemma 2.4.) Therefore $\tilde{\beta} = \beta$.

3^0. Set

$$\lambda^k(\Gamma) = n(k)^{-1} \int_{-n(k)}^{0} \lambda_b(\Gamma)db = P^{n(k)} \{\int_0^\infty 1_\Gamma(y_t)dt\} , \qquad \Gamma \subset T .$$

Let N be the same as in Lemma 5.4. By virtue of this lemma λ^k restricted to the interval $[-\ell N, \ell N]$, ℓ being an integer, is a sequence of measures uniformly bounded above and away from zero. Hence, there exists a subsequence $k(q)$ (which depends on ℓ) such that $\lambda^{k(q)}$ is weakly convergent on $[-\ell N, \ell N]$. Using the diagonal method, we can choose a subsequence $k(m)$ and a measure $\tilde{\lambda}$ on $]-\infty,\infty[$ such that $\lambda^{k(m)}(f) \to \tilde{\lambda}(f)$ for any continuous f with compact support, as $m \to \infty$. Let $g(x) = f(x-r)$, $r \in T$. Let $\phi = n(k(m))$

$$\begin{aligned}
\tilde{\lambda}(f) - \tilde{\lambda}(g) &= \lim_{m\to\infty} \lambda^{k(m)}(f-g) = \lim_{m\to\infty} \phi^{-1} \int_{-\phi}^{0} \lambda_b(f-g)\,db \\
&= \lim_{m\to\infty} \phi^{-1}\{\int_{-\phi}^{0} \lambda_b(f)\,db - \int_{-\phi+r}^{r} \lambda_b(f)\,db\} \\
&= \lim_{m\to\infty} \phi^{-1}\{\int_{-\phi}^{r-\phi} \lambda_b(f)\,db - \int_{0}^{r} \lambda_b(f)\,db\} = 0 \,.
\end{aligned} \tag{5.20}$$

The relation (5.20) shows that $\lambda(\Gamma) = \lambda(\Gamma + r)$. Therefore $\tilde{\lambda}(dx) = d\cdot m(dx)$ where d is a constant.

4^0. Let f be a continuous function on T_t with compact support. Put $h(x,y) = f(x)\, 1_{T^t}(y)$; $\hat{f}(x) = f(x)\, \Pi(x;T^t) 1_{x<t}$. By Lemma 2.5 $\hat{f}(x)$ has at most a countable number of discontinuities. Applying successively Lemma 2.1 and Lemma 2.5 we get

$$\begin{aligned}
\tilde{\rho}_t(f) &= \lim_{k\to\infty} \rho_t^{n(k)}(f) = \lim_{m\to\infty} \rho_t^{\phi}(f) = \lim_{m\to\infty} P^{\phi}\{f(U_t)\} \\
&= \lim_{m\to\infty} P^{\phi}\{A_h\} = \lim_{m\to\infty} \int_{-\infty}^{t} \hat{f}(x)\, \tilde{\lambda}^{\phi}(dx) = \tilde{\lambda}(\hat{f}) \\
&= d \int_{-\infty}^{\infty} f(x)\, 1_{x<t}\, \Pi(x;T^t)\,dx = (d/c)\, \rho_t(f) \,.
\end{aligned}$$

Thus we see that $\tilde{\rho}_t(f) = (d/c)\, \rho_t(f)$ for any continuous f with compact support; therefore for all f. Taking $f(x) = 1_{x<t}$ and noticing that $\tilde{\rho}_t(f) + \beta = 1$, we get $d = c$.

5^0. The same reasoning shows that ρ_t is a weak limit point of each subsequence of ρ_t^n. By Lemma 5.2 ρ_t^n converges weakly to ρ_t. Similarly $\beta_t^n \to \beta$ and $\kappa_t^n \to \kappa_t$.

For $x \leq t \in T$, $\Gamma \subset T^t$ put

$$K(t,x;\Gamma) = \begin{cases} 1_\Gamma(x)\ , & \text{if } x = t \\ \\ \Pi(x;\Gamma)/\Pi(x;T^t)\ , & \text{if } x < t\ . \end{cases} \tag{5.21}$$

For $z = (x,y) \in T \times T$, $\Gamma \subset (T \times T)^n$ and $s < t_1 < t_2 < \cdots < t_n \in T$ put

$$p^n(s,z;t_1,t_2,\ldots,t_n;\Gamma)$$

$$= \int_\Gamma p(x,z;t_1,dz_1)\ p(t_1,z_1;t_2,dz_2) \cdots p(t_{n-1},z_{n-1};t_n,dz_n)\ . \tag{5.22}$$

Let $\Delta \subset T \times (T \times T)^n$. Let Δ^1 be the projection of Δ on the first axis and Δ_x be the section of Δ_x when the first coordinate is equal to x. For $t < t_1 < t_2 < \cdots < t_n$ put

$$Q(x,t,t_1,\ldots,t_n;\Delta) = \int_{\Delta^1} K(t,x;dy)\ p^n(t,(x,y);t_1,\ldots,\ t_n;\Delta_y)\ . \tag{5.23}$$

Now we prove that the conditional distribution of V_t, $Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n}$ given U_t is defined by the kernel Q.

LEMMA 5.7. Let G be a function on $(T)^{2n+1}$ and h be a function on T. Let $t < t_1 < t_2 < \cdots < t_n$. If (y_t,R) is an (α,Π)-process such that

$$R\{U_y = t,\ V_t > t\} = 0 \tag{5.24}$$

then

$$R\{h(U_t)\ G(V_t,Y_{t_1},Y_{t_2},\ldots\ ,\ Y_{t_n})\} = R\{h(U_t)\ Q(U_t,t,t_1,\ldots,t_n;G)\}\ . \tag{5.25}$$

<u>Proof</u>. By the strong Markov property of (y_t,R)

$$R\{h(U_t)\ G(V_t,Y_{t_1},\ldots,\ Y_{t_n})\} = R\{h(U_t)\ \pi(U_t,V_t)\} \tag{5.26}$$

where $\pi(x,y) = \int p^n(t,(x,y);t_1,\ldots,t_n,dz_1,\ldots,dz_n)\ G(y,z_1,\ldots,z_n)$. By (5.24) and Lemma 2.2,

$$\begin{aligned}
&R\{h(U_t)\ \pi(U_t,V_t)\} \\
&\quad = R\{h(t)\ \pi(t,t)1_{U_t=t}\} + R\{h(U_t)\ \pi(U_t,V_t)1_{U_t<t}\} \\
&\quad = R\{h(U_t) \int K(t,U_t;dx)\ \pi(U_s,x)\}\ . \qquad (5.27)
\end{aligned}$$

Comparing (5.27) and (5.26) we get (5.25)

LEMMA 5.8. Fix $t_1 < t_2 < \cdots < t_k$. For G being a function on $(T)^{2k}$, put

$$Q_G(x) = \int Q(x;t_1,t_2,\ldots,t_k,dy,dz_1,dz_2,\ldots,dz_{k-1})\cdot G(x,y,z_1,\ldots,z_{k-1})\ . \qquad (5.28)$$

Here $x, y \in T$, $z_1, z_2, \ldots, z_k \in T \times T$. If G is a continuous function with compact support then $Q_G(x)$, $x \in]-\infty,t_1]$ has at most a countable number of discontinuities.

Proof. The family of functions for which the statement of the Lemma holds is invariant under linear operations and uniform convergence. Hence, it is enough to prove the Lemma for functions G of the form

$$G(z_1,z_2,\ldots,z_k) = h_1(z_1)\ \cdots\ h_k(z_k) \qquad (5.29)$$

where $z_i = (x_i,y_i) \in T \times T$, $h_i(z) = f_i(x)\ g_i(y)$, f_i and g_i are continuous functions. Put $t_0 = -\infty$,

$$\begin{aligned}
&q_{k+1}(y) \equiv 1,\ q_i(y) = P_y\{h_i(Y_{t_i})\ h_{i+1}(Y_{t_{i+1}})\cdots h_k(Y_{t_k})\},\quad y < t_i\ ; \\
&F_i(y) = 1_{[t_{i-1},t_i[}(y)\ q_i(y)\ g_1(y)\ g_2(y)\ \cdots\ g_{i-1}(y)\ , \\
&H_i(x) = f_1(x)\ f_2(x)\ \cdots\ f_{i-1}(x)\ K(t_1,x;F_i),\quad i = 1,2,\ldots,k+1;
\end{aligned}$$

Direct computations show that $Q_G(x) = \sum_{i=1}^{k+1} H_i(x)$. By Lemma 2.6 all the functions q_i have at most a countable number of discontinuities. It is obvious that so does F_i and by Lemma 2.5 so does H_i.

LEMMA 5.9. Let $\mu_{t_1 t_2 \cdots t_k}$ be a measure on $(T \times T)^k$, whose projection on the first axis is κ_t and the conditional distribution of $(y_1, x_2, y_2, x_3, y_3, \ldots, x_k, y_k)$ given x_1 is $Q(x_1, t_1, t_2, \ldots, t_k; -)$. Put $\mu^n_{t_1 t_2 \cdots t_k}(\Gamma) = P^n\{Y_{t_1}, \ldots, Y_{t_k}) \in \Gamma\}$. Then $\mu^n_{t_1 \cdots t_k}$ converges weakly to $\mu_{t_1 \cdots t_k}$.

<u>Proof</u>. 1^0. Since $U_{t_1} \leq V_{t_1} \leq U_{t_2} \leq \cdots \leq U_{t_k} \leq V_{t_k}$, then $\max(|U_{t_1}|, |V_{t_1}|, \ldots, |U_{t_k}|, |V_{t_k}|) = \max(|U_{t_1}|, |V_{t_k}|)$. The computations similar to those of Lemma 5.4 show that $P^n\{\max(|U_{t_1}|, |V_{t_k}|) > N\}$ tends to 0 when $N \to \infty$ uniformly for all n. Therefore the sequence of measures $\mu^n_{t_1 t_2 \cdots t_k}$ is tight and has a weak limit point $\tilde{\mu}$.

2^0. Let G be of the form (5.29) and $Q_G(x)$ be given by (5.28). By Lemma 5.7 and (5.8)

$$P^n\{G(Y_{t_1}, Y_{t_2}, \ldots, Y_{t_n})\} = P^n\{Q_G(U_{t_1})\} = \kappa^n_{t_1}(Q_G) = \rho^n_{t_1}(Q_G) + \beta^n_{t_1} Q_G(t) \quad . \quad (5.30)$$

By Lemma 5.6 the sequence of real numbers $\beta^n_{t_1}$ converges to β. The sequence of measures $\rho^n_{t_1}$ converges weakly to ρ_t which is a continuous measure. By Lemma 5.8 and Lemma 5.1 $\rho^n_{t_1}(Q_G) \to \rho_{t_1}(Q_G)$ when $n \to \infty$. Therefore $\tilde{\mu}(G) = \rho_t(Q_G) + \beta Q_G(t)$ $= \kappa_t(Q_G) = \mu_{t_1 \cdots t_k}(G)$ for each G of the form (5.29), hence for any function G on $(T \times T)^k$.

3^0. We can apply the same arguments to any subsequence of $\mu^n_{t_1 \cdots t_k}$. By Lemma 5.2 $\mu^n_{t_1 \cdots t_k}$ converges weakly when $n \to \infty$.

The family of measures $\mu_{t_1 \cdots t_k}$ is obtained as a weak limit of the families of finite dimensional distributions of (Y_t, P^n). Therefore $\mu_{t_1 \cdots t_k}$ is a consistent family of distributions. By the Kolmogorov theorem there exists a process (Y_t, P), $t \in T$, $Y_t = (U_t, V_t) \in T \times T$ with finite dimensional distributions $\mu_\bullet$.

6. CONSTRUCTION OF A T.I. SET GIVEN α AND Π (CONTINUATION)

If $z_t = (u_t, v_t)$ is the process associated with a random set M, then M is equal to the complement of the union of the intervals $]u_t, v_t[$. We start from the process Y_t, defined in Section 5 and we construct $z_t = (u_t, v_t)$ such that

(a) $z_t = Y_t$ a.s. P for each t

(b) u_t is right-continuous, v_t is left-continuous a.s.P. To construct z_t we establish first certain properties for Y_t (Lemmas 6.1 and 6.2).

LEMMA 6.1. Fix $u < s < t$. Then a.s.P

$$U_s \neq u, \quad V_s \neq t \tag{6.1}$$

$$V_s < U_t \quad \text{on the set} \quad \{V_s < t\} \cup \{U_t > s\} \tag{6.2}$$

$$Y_s = Y_t \quad \text{on the set} \quad \{V_s > t\} \cup \{U_t < s\} \tag{6.3}$$

$$U_s \leq V_s \tag{6.4}$$

To prove (6.1), (6.2), (6.3) and (6.4) it suffices to consider the one-dimensional and two-dimensional distributions of (Y_t, P), the formula for which was obtained in Lemma 5.9.

LEMMA 6.2. The process (Y_t, P) is a stochastically continuous Markov process with one-dimensional distributions (5.1) and transition function (5.2).

Proof. 1^0. The finite dimensional distributions of P have been already calculated. Easy computations show that μ_t is equal to ν_t, where ν_t is given by (5.1).

Let f be a continuous function on $(T \times T)^m$ and g be a continuous function on $T \times T$. To get the Markov property for (Y_t, P) it is sufficient to pass to a limit in the relation

$$P^n\{f(Y_{t_1}, \ldots, Y_{t_m})\, g(Y_t)\} = P^n\{f(Y_{t_1}, \ldots, Y_{t_m})\, p(t_m, Y_{t_m}; t, g\}, \quad t_1 < \cdots < t_m < t$$

(which is justified by Lemmas 5.8 and 5.1).

2^0. Fix $\varepsilon > 0$. The right-continuity of the (α,Π)-process y_t implies that there exists $\psi > 0$ such that

$$P_0\{|V_\psi| > \varepsilon\} < \varepsilon/2 \tag{6.5}$$

Let $\phi > 0$ be such that

$$c\int_{-\infty}^{0} \Pi(x;]0,\phi])dx < \varepsilon/2 \tag{6.6}$$

c given by (1.5). Let $t > s$ and $t-s < \phi \wedge \psi$. Denote $A = \{|Y_t - Y_s| \geq \varepsilon\}$. We have

$$P\{A\} = P\{A,V_s > t\} + P\{A,V_s \leq t\} = P\{A,s < V_s \leq t\} + P\{A,V_s = s\} \tag{6.7}$$

The first term in the right side of (6.7) is not greater than $P\{V_s \in]s,t]\}$ and by (6.6) does not exceed $\varepsilon/2$. Put $z = (s,s)$. The second term in (6.7) is less than or equal to

$$P\{Y_s=z\}\ p(s,z;t,T_t \times T^{s+\varepsilon})$$

$$= P\{Y_s=z\}\ P_s\{|V_t-s| \geq \varepsilon\} = P\{Y_s=z\}\ P_0\{V_{t-s} > \varepsilon\} \leq \varepsilon/2$$

Now put $u_t = \lim U_r$, $r\downarrow t$; and $v_t = \lim V_r$, $r\uparrow t$, r is rational. Put $z_t = (u_t,v_t)$, $I(t) =]u_t,v_t[$. By Lemma 6.2 the family u_t and v_t satisfy (6.1)-(6.4) for all t a.s.P. It is easy to see that u_t is right and v_t is left continuous a.s.P. Therefore z_t may be considered as a process associated with a set M, which can be defined as

$$M = T - \bigcup_t I(t) \tag{6.8}$$

LEMMA 6.3. The set M defined by (6.8) is a t.i. (α,Π)-generated set.

Proof. 1^0. Let μ_t be the distribution of v_t. Let $t < t_1 < \cdots < t_k$. By Lemma 6.2, $z_t = Y_t$ a.s. Therefore, applying the first part of Lemma 6.2, we get

$$P\{z_{t_1} \in \Gamma_1, z_{t_2} \in \Gamma_2, \ldots , z_{t_k} \in \Gamma_k, u_{t_1} > t\}$$

$$= \int_{-\infty}^{t_1} \mu_t(dy) \int_{\Gamma_1} p(t,(x,y);t_1 dz_1) \int_{\Gamma_2} p(t_1,z_1;t_2 dz_2) \ldots \int_{\Gamma_{k-1}} p(t_{k-1};z_{k-1};t_k,\Gamma_k)$$

$$= \int_{-\infty}^{t_1} \mu_t(dy)\ P_y\{Y_{t_1} \in \Gamma_1, Y_{t_2} \in \Gamma_2, \ldots, Y_{t_k} \in \Gamma_k\} \tag{6.9}$$

The expression (6.9) shows that M^t has the same distribution as the range of the (α,Π)-process, whose initial distribution is equal to μ_t.

2^0. Consider the process $(z_t - t, P)$, which is Markov. Formula (5.2) shows that this process has a stationary transition function, and (5.1) implies that the one-dimensional distributions of $(z_t - t, P)$ are stationary. Therefore $(z_t - t, P)$ is a stationary process, which is equivalent to M being t.i.

The rest of this section is devoted to the proof of the first statement of Theorem 1 (that each t.i.r.r. set is l.r.). We shall prove even more; namely that $-M$ has the same distribution as M; therefore $-M$ is (α,Π)-generated. To this end we consider the jumps of the process $y_t^* = -y_t$, where y_t is an (α,Π)-process. We prove that: (i) the backward transition function of (z_t,P) coincides with the backward transition function of the jumps of y_t^* (Lemma 6.6); (ii) the one-dimensional distributions of M are equal to those of $-M$ (Lemma 6.5).

The process y_t^* is a decreasing process with independent increments with translation constant $-\alpha$ and the Levy measure $\Pi^*(\Gamma) = \Pi(-\Gamma)$. Let P_b^* be the transition probabilities of y_t^*. Put

$$\lambda_b^*(\Gamma) = P_b^* \{\int_0^\infty 1_\Gamma(y_t^*)dt\}$$

It is clear that $\lambda_b^*(\Gamma) = \lambda_{-b}(-\Gamma)$. Let $\Pi^*(x;-)$ and Π_x^* be defined the same way as $\Pi(x;-)$ and Π_x. Put $g = \pi f$ if $g(x) = \Pi(x;f)$; put $h = \Lambda f$, if $h(x) = \lambda_x(f)$. The operators π^* and Λ^* are defined similarly. We denote by (f,g) the integral of fg with respect to the Lebesgue measure.

LEMMA 6.4. If f and g are functions on T_t and T^t respectively then

$$(\Lambda f,g) = (f,\Lambda^* g) \tag{6.10}$$

$$(\pi f,g) = (f,\pi^* g) \tag{6.11}$$

Proof. Let f and g be infinite differentiable with compact support. Consider the sequence of functions q_n such that $(q_n,h) \to \lambda_0(h)$ for each infinitely differentiable h with a compact support. Then,

$$(\Lambda f,g) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f(x+y)\ \lambda_0(dx)\ g(y)dy = \lim_{n\to\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} q_n(z-y)\ f(z)dz\ g(y)dy$$

$$= \lim_{n\to\infty} \{\int f(z)dz \int q_n(z-y)\ g(y)dy\} = \int \lambda_z^*(g)\ f(z)dz = (f,\Lambda^* g)\ .$$

The usual arguments show that (6.10) is true for all f. The proof of (6.1.1) is similar.

For $z = (x,y)$, put $\bar{z} = (y,x)$. If $\Gamma \subset T \times T$ then $\bar{\Gamma}$ must be understood in the same way.

LEMMA 6.5. If M is a t.i. (α,Π)-generated set then the distribution of $-\bar{z}_t$ coincides with that of z_{-t}.

Proof. Put $\nu_t^*(\Gamma) = P\{-\bar{z}_t \in \Gamma\}$. By virtue of (5.1)

$$\nu_t^*(\Gamma) = P\{z_t \in -\bar{\Gamma}\} = c\alpha\ 1_{(t,t)}(-\bar{\Gamma}) + c\int_{-\infty}^{t} \Pi_x(-\bar{\Gamma})dx$$

$$= c\alpha\ 1_{(-t,-t)}(\Gamma) + c\int_{-t}^{\infty} \Pi_x^*(\Gamma)dx\ ,\ \Gamma \subset T_{-t} \times T^{-t}\ . \tag{6.12}$$

By virtue of (6.11) the second summand in (6.12) is equal to $c\int_{-\infty}^{-t} \Pi_x(\Gamma)dx$; therefore (6.12) is equal to $\nu_{-t}(\Gamma)$.

For $t > s \in T$, $z \in T \times T$, $\Gamma \subset T \times T$ put $p^*(t,z;s,\Gamma) = p(-t,-\bar{z};-s,-\bar{\Gamma})$. The function p^* is the backward transition function of the process $Y_t^* = -\bar{Y}_{-t}$, which is the process of jumps of y_t^*.

LEMMA 6.6. If M is a t.i. (α,Π)-generated set, then the backward transition function of (z_t,P) is equal to p^*.

Proof. Let $s < t$ and f, g, h, j be functions on T with supports on $]-\infty,s[$, $]s,t[$, $]s,t[$ and $]t,\infty[$ respectively. Put $F(x,y) = f(x)\, g(y)$; $H(x,y) = h(x)\, j(y)$. By Lemma 2.1

$$P_y\{H(Y_t)\} = P_y\{\sum_{u\in J} f(y_{u-})\, j(y_u)\} = \int_y^t \lambda_y(dx)\, h(x)\, \Pi(x;j) \tag{6.13}$$

By virtue of Lemma 5.3 and (6.13)

$$\begin{aligned} P\{F(z_s)\, H(z_t)\} &= \int \nu_s(dz)\, F(z)\, p(s,z;t,H) \\ &= \int_{-\infty}^{s} dx\, f(x) \int_s^t \Pi(x;dy)\, g(y) \int_y^t \lambda_y(du)\, h(u) \int_t^\infty \Pi(u;dv)\, j(v) \end{aligned} \tag{6.14}$$

Applying successively (6.11), (6.10) and (6.11) we get that (6.14) is equal to

$$\int_t^\infty dv\, j(v) \int_s^t \Pi^*(v;du)\, h(u) \int_s^u \lambda_u^*(dy)\, g(y) \int_{-\infty}^{s} \Pi^*(y;dx)\, f(x)$$

$$= \int \nu_{-t}^*(dz)\, H(z)\, p^*(t,z;s,F) = P\{F(z_s)\, H(z_t)\} \tag{6.15}$$

For arbitrary H and F the equality (6.15) is proved similarly. By Lemma 6.5 $\nu_{-t}^* = \nu_t$; therefore we get the statement of the lemma.

LEMMA 6.7. If M is a t.i. (α,Π)-generated set then so is $-M$.

Proof. Let $\tilde{z}_t$ be the process associated with $-M$. It is easy to see that $\tilde{z}_t = -\bar{z}_{-t}$; therefore $\tilde{z}_t$ is a Markov process. By Lemma 6.5 the one-dimensional distributions of $\tilde{z}_t$ are equal to those of z_t. The transition function of $\tilde{z}_t$ is equal to $\tilde{p}(s,z;t,\Gamma) = p^*(-s,-\bar{z};-t,-\bar{\Gamma}) = p(s,z;t,\Gamma)$. Consequently the distribution of $-M$ is equal to that of M.

The following lemma will be useful in the sequel. Its proof follows from Theorem 1 and Lemma 6.7.

LEMMA 6.8. Let P_y and P^*_y be the transition probabilities of an (α,Π)-process y_t and a $(-\alpha,\Pi^*)$-process y^*_t respectively. For a function F on $(T \times T)^n$ and a function G on $(T \times T)^m$ put

$$g(x) = P_x\{\Sigma\; G(y_{t_{1-}}, y_{t_1}, \dots , y_{t_{m-}}, y_{t_m})\} , \tag{6.16}$$

$$f(y) = P^*_y\{\Sigma\; F(y^*_{s_n}, y^*_{s_{n-}}, \dots, y^*_{s_1}, y^*_{s_{1-}})\} \tag{6.17}$$

where the sum in (6.16) is taken over all $t_1 < t_2 < \cdots < t_m$, $t_1, t_2, \dots, t_m \in J$; and the sum in (6.17) is taken over all $s_1 < s_2 < \cdots < s_n$, $s_1, s_2, \dots , s_n \in J$. Let M be a t.i. (α,Π)-generated set and let $\Sigma^{(k)}$ denote the sum over all k-tuples $\gamma_1, \gamma_2, \dots, \gamma_k$ such that $\gamma_1 < \gamma_2 < \cdots < \gamma_k$. Then

$$P\{\Sigma^{(m+n)} F(\gamma_1,\delta_1,\gamma_2,\delta_2,\dots,\gamma_n,\delta_n)\; G(\gamma_{n+1},\delta_{n+1},\dots,\gamma_{n+m},\delta_{n+m})$$

$$= P\{\Sigma^{(n)} F(\gamma_1,\delta_1,\gamma_2,\delta_2,\dots, \gamma_n,\delta_n)\; g(\delta_n)\}$$

$$= P\{\Sigma^{(m)} f(\gamma_1)\; G(\gamma_1,\delta_1,\dots,\gamma_m,\delta_m) .$$

7. MARKOV T.I. SETS

In this section we prove Theorem 2 and give an example of a t.i. Markov set which is not r.r. The proof of Theorem 2 is based on the following two analytic lemmas.

LEMMA 7.1. If μ is a finite measure on $]0,\infty[$ such that

$$\mu(\Gamma + t) \le \mu(\Gamma) , \qquad t > 0 \tag{7.1}$$

then μ is absolutely continuous with respect to Lebesgue's measure m and $k(x) = \mu(dx)/m(dx)$ can be chosen as a monotone function of x, $x > 0$.

<u>Proof</u>. 1^0. Let μ^t be a measure on $]0,\infty[$ defined by $\mu^t(\Gamma) = \mu(\Gamma + t)$, $\Gamma \subset]0,\infty[$. The relation (7.1) implies

$$\mu^t(\Gamma) \leq \mu^s(\Gamma] \qquad \text{if} \qquad t > s . \tag{7.2}$$

Put

$$\nu(\Gamma) = \int_0^\infty e^{-t} \mu_t(\Gamma)dt = \int_0^\infty \{ \int_0^x e^{-t} 1_\Gamma(x-t)dt\} \mu(dx) . \tag{7.3}$$

It is easy to see that if $f = 0$ m-almost everywhere then

$$\nu(f) = \int_0^\infty \{\int_0^x e^{-t} f(x-t)dt\} \mu(dx) = 0 .$$

Therefore ν is absolutely continuous with respect to the Lebesgue measure m.

On the other hand, if $\nu(\Gamma) = 0$ then for m-a.e. t

$$\mu_t(\Gamma) = 0 \tag{7.4}$$

and there exists a sequence $t_n \downarrow 0$ such that (7.4) holds for each t_n. In view of (7.2), (7.4) holds for all t. Therefore for each t μ_t is absolutely continuous with respect to m, hence μ is absolutely continuous with respect to m on $]t,\infty[$ for each $t > 0$.

2^0. Let $\tilde{k}$ be any density of μ with respect to m and let $L(n,k)$ be defined by (3.6). Put

$$k^n(x) = \sum_{k=1}^\infty 1_{L(k,n)}(x) \, \mu(L(k,n))\cdot 2^n .$$

The function $k^n(x)$ is monotone and so is $k(x) = \limsup_n k^n(x)$, which is equal to $\tilde{k}(x)$ for m-almost all x.

LEMMA 7.2. Let μ be the measure, satisfying the same conditions as in Lemma 7.1. If $f(y)$ is a function on T such that for any fixed $r > 0$ for μ-almost all $y > r$

$$f(y-r) = f(y)$$

then $f(y)$ is a constant μ-a.e.

Proof. Put $a = \inf\{x:k(x) = 0\}$. The measures μ and m are equivalent on $]0,a[$. Consider a set Γ in $T \times T:\Gamma = \{(x,y):f(y) \neq f(x)\}$. By the Fubini theorem

$$\int_0^a \int_0^a 1_\Gamma dx\, dy = 2 \int_0^a dx \int_x^a 1_\Gamma(y,y-x)dy = 0 .$$

Therefore there exists x_0 such that $m\{y:(x_0,y) \in \Gamma\} = 0$. Hence $f(y) = f(x_0)$ for m- almost all y.

LEMMA 7.3. If M is a t.i. set subject to 1.A.a and ξ is $\sigma(\widetilde{M}^s)$-measurable then there exist two constants a and b such that

$$P\{\xi|\sigma(M_{v_s})\} = a 1_{v_s<s} + b\, 1_{v_s=s} \quad \text{a.s.} \tag{7.5}$$

Proof. Let $0 < t_1 < t_2 < \cdots < t_k$ and f be a function on $(T \times T)^k$. For a random set N, put $\zeta(N) = f(z_{t_1}^{(N)}, z_{t_2}^{(N)},\ldots, z_{t_k}^{(N)})$, where $z_t^{(N)}$ is the process associated with the random set N. It is enough to prove (7.5) for ξ of the form $\zeta(\widetilde{M}^s)$.

By 1.A.a M^s and M_s are conditionally independent given v_s. Since $\sigma(M_{v_s}) = \sigma(M_s) \vee \sigma(v_s)$; therefore M_{v_s} and M^s are also conditionally independent given v_s. But $\sigma(\widetilde{M}^s) = \sigma(M^s - v_s)$ is a subfield of $\sigma(M^s)$; consequently for each $s \in T$

$$P\{\zeta(\widetilde{M}^s)/\sigma(M_{v_s})\} = P\{\zeta(\widetilde{M}^s)/v_s\} = g_s(v_s) \quad \text{a.s.}$$

where $g_s(x)$ is a function on T^s. Owing to the fact that M is t.i., we get

$$g_s(x) = g_t(x - s + t) \quad \text{for } \mu_s \text{ a.e. } x, \tag{7.6}$$

μ_s being the distribution of v_s. On the other hand $v_s = v_t$ and $\widetilde{M}^s = \widetilde{M}^t$ a.s. on the set $\{v_s > t\}$, therefore

$$g_s(x) = g_t(x) \qquad \text{a.s. } \mu_s \text{ on the set } \{x > t\} . \tag{7.7}$$

Let μ be the restriction of μ_0 to $]0,\infty[$. Since $\tilde{M}^0 = \tilde{M}^t$ and $v_0 = v_t$ on the set $\{v_0 > t\}$; then for any $\Gamma \subset]0,\infty[$

$$\mu(\Gamma + t) = P\{v_0 \in \Gamma + t\} = P\{v_{-t} \in \Gamma\} \leq P\{v_0 \in \Gamma\} = \mu(\Gamma) . \tag{7.8}$$

The expressions (7.6), (7.7) and (7.8) show that μ satisfies the conditions of Lemma 7.1 and $g_0(x)$, $x > 0$ satisfies the conditions of Lemma 7.2. By virtue of these two lemmas these exists a constant a such that $g_0(x) = a$ for μ-a.e., $x > 0$. Put $b = g_0(0)$. By virtue of (7.6) $g_s(x) = a\, 1_{x<s} + b\, 1_{x=s}$. Hence (7.5) is proved.

COROLLARY (THEOREM 2). Every thin t.i. set M subject to 1.A.a is $(0,\Pi)$-generated.

<u>Proof</u>. Since $P\{v_s = s\} = 0$, the expression $b\, 1_{v_s=s}$ is equal to 0 a.s. Therefore for each $\sigma(\tilde{M}^s)$-measurable ξ the right side of (7.5) is a constant a.s. This is equivalent to M_{v_s} being independent on $\tilde{M}^s$. Therefore M is r.r. by virtue of Theorem 1 M is (α,Π)-generated. Formula (1.3) impiles that $\alpha = 0$.

An example of a t.i. Markov set which is not r.r. is very simple. Consider a t.i. $(0,\Pi)$-generated set M_1 and a $(1,0)$-generated set M_2 (this means that $M_2 \equiv T$). The set M which is the mixture of M_1 and M_2 with the coefficients p and q, $p+q = 1$, $p,q \neq 0$, is the set we are looking for. To prove that M is t.i. and Markov we have to use the same arguments as in proving that the mixture of two stationary Markov processes with singular one-dimensional distributions is stationary and Markov.

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REFERENCES

[1] A. V. Skorohod, Stochastic Processes with Independent Increments (in Russian), "Nauka", Moscow, 1964.

[2] D. R. Cox, H. D. Miller, The Theory of Stochastic Processes, Methuen and Co., Ltd., London 1968.

[3] B. Maisonneuve, Systèms Régénératifs, Asterisque 15, 1974.

[4] E. B. Dynkin, Some Limit Theorems for Sums of Independent Random Variables with Infinite Mathematical Expectations, Selected Translations in Mathematics Statistics and Probability, Vol. 1 (1961), IMS-AMS, 171-189.

[5] W. Feller, An Introduction to Probability Theory and its Application, Volume II, John Wiley and Sons, Inc., New York, London, Sydney, Toronto, 1971.

[6] P. Billingsley, Convergence of Probability Measures, John Wiley and Sons, Inc., New York, London, Sydney, Toronto, 1968.

[7] C. Dellacherie, Capacités et processus stochastique, Springer-Verlag, Berlin-Heidelberg-New York, 1972.