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Introduction

In [3] under mild conditions on the potential kernel, a representation for the equilibrium potential was derived. General as they were, there was some dissatisfaction with these conditions on the grounds that, for example, potential kernels of many Lévy processes failed to fulfil these conditions. In this note, we resolve this problem with a set of conditions which include many Lévy processes and those in [3].

In §1, Revuz measure is examined and we prove that whenever Uf_n increases to a class D potential, f_n converges weakly to its Revuz measure, provided it is finite. We consider only Hunt processes satisfying hypothesis L.

In §2 we introduce the following condition: There is a version $u(x,y)$ of the potential kernel such that

> $y \to u(x,y)$ is lower semi-continuous and there exists a $\varphi > 0$ such that $\int u(x,y)\varphi(x)dx$ is continuous and positive.

If these are satisfied, we show that u can be modified to v so that the relation

$$s(x) = \int v(x,y)\mu(dy)$$

is valid for any class D potential s and its Revuz measure μ. And then μ is unique.

§3 is devoted to showing that the conditions i [3] are included in those of the present article.

We gratefully acknowledge the kindness of Professor P. A. Meyer in accepting to publish this note in spite of the large overlap with [6]. Thanks are due to Professor K. L. Chung for discussions and encouragement.

§1. We use no duality assumptions except Hypothesis L. X_t will denote a Hunt process with a locally compact metric state space.

Proposition 1. Let ξ be an excessive reference measure. Then $f,g \geqq 0$ and $Uf \leqq Ug < \infty$ a.e. imply

(1) $$\int f \leqq \int g$$

where integration is always relative to ξ unless specified.

Proof. We may assume f and g are integrable. Indeed, if $(g,\xi) = \infty$, there is nothing to show. If $(g,\xi) < \infty$, replace f by $f\cdot\varphi$ with $0 \leqq \varphi \leqq 1$ and $(f\varphi,\xi) < \infty$, apply the result to $f\varphi$ and let $\varphi \uparrow 1$.

Suppose first that for some $\alpha > 0$, $U_\alpha f \leqq U_\alpha g$. Since ξ is excessive, $(\beta U_\beta f,\xi)$ increases with β and tends as $\beta \uparrow \infty$ to (f,ξ). From the resolvent equation for $\beta > \alpha$,

$$U_\alpha g - U_\alpha f = U_\beta g - U_\beta f + (\beta-\alpha) U_\beta (U_\alpha g - U_\alpha f)$$

so that

$$(U_\beta g - U_\beta f,\xi) = (U_\alpha g - U_\alpha f,\xi) - (\beta-\alpha)(U_\beta(U_\alpha g - U_\alpha f),\xi).$$

Since $U_\alpha g \geqq U_\alpha f$, by assumption we get

$$(U_\beta g - U_\beta f,\xi) \geqq \alpha(U_\beta(U_\alpha g - U_\alpha f),\xi) \geqq 0.$$

The left side is thus $\geqq 0$. Multiply by β and let $\beta \uparrow \infty$ to conclude $(g,\xi) \geqq (f,\xi)$.

Now suppose $Uf \leqq Ug$. Let $0 < \lambda < 1$. Then $\lambda Uf < Ug$ a.e. Let A_n be the set

$$A_n = \{\lambda U_{\frac{1}{n}} f \leqq U_{\frac{1}{n}} g\}.$$

We claim that $\liminf A_n = E$, ξ-a.e. Indeed, if $x \in A_n^c$ for infinitely many n, then $\lambda U_{\frac{1}{n}} f > U_{\frac{1}{n}} g$ for infinitely many $n \Rightarrow$ $\lambda Uf \geqq Ug$ and this set has ξ-measure 0. Now for any n,

$$\lambda U_{\frac{1}{n}}(f 1_{A_n}) \leqq \lambda U_{\frac{1}{n}} f \leqq U_{\frac{1}{n}} g$$

on A_n and hence $\lambda U_{\frac{1}{n}}(f 1_{A_n}) \leqq U_{\frac{1}{n}} g$ everywhere. From what we have already shown, $\lambda \int f 1_{A_n} \leqq \int g$. This being true for all n, $\lambda \int f \leqq \int g$. Let λ increase to 1.

<u>Proposition 2</u>. Let s be excessive and finite a.e. Let f_n be such that $\lim Uf_n = s$ a.e.. Let $g \geqq 0$ and $Ug \leqq s$. Then

$$\int g \leqq \liminf \int f_n. \tag{2}$$

<u>Proof</u>. Let $\lambda < 1$ and A_n the set

$$A_n = \{Uf_n \geqq \lambda Ug\}.$$

Then $\liminf 1_{A_n} = 1$ a.e.

$$U(f_n) \geqq \lambda U(g 1_{A_n}) \quad \text{on } A_n$$

and hence everywhere. By Proposition 1,

$$\lambda \int g 1_{A_n} \leqq \int f_n.$$

Let $n \uparrow \infty$ and use the Fatou lemma and note that $\lambda < 1$ is arbitrary. Q.e.d.

Corollary 3. Let s be excessive and finite a.e. Suppose $Uf_n \leqq s$, $Ug_n \leqq s$ and $\lim Uf_n = \lim Ug_n = s$. Then

$$\lim \int g_n = \lim \int f_n. \tag{3}$$

Proof. From Proposition 2, for all n,

$$\int g_n \leqq \liminf \int f_m,$$

i.e.

$$\limsup \int g_n \leqq \liminf \int f_m.$$

By the same reasoning can be applied $\limsup \int f_m \leqq \liminf \int g_n$. Q.e.d.

Proposition 4. Let A be a natural additive functional with potential s. Let $Uf_n \leqq s$ and $\lim Uf_n = s$. Suppose $\lim \int f_n < \infty$. Then for all bounded continuous φ

$$\lim \int f_n \varphi \text{ exists.} \tag{4}$$

Proof. Note that $\lim \int f_n$ exists because from Corollary 3, if $Ug_n \uparrow U_A = s$, then $\lim \int g_n = \lim \int f_n$ and $\int g_n \uparrow$. Suppose $0 \leqq \varphi \leqq 1$ is continuous. We know $U[f_n \varphi] \to U_A \varphi = E.[\int_0^\infty \varphi(x_t) dA_t]$ and $U[f_n(1-\varphi)] \to U_A(1-\varphi)$. Let g_n and h_n be such that $Ug_n \uparrow U_A \varphi$ and $Uh_n \uparrow U_A(1-\varphi)$. Then since $\lim Uf_n = s = U_A 1$, $\lim Ug_n = U_A$, $\lim Uh_n = U_A(1-\varphi)$. From Proposition 2

$$\lim \int g_n \leqq \liminf \int f_n \varphi$$

$$\lim \int h_n \leqq \liminf \int f_n (1-\varphi).$$

But

$$U(g_n + h_n) \uparrow U_A = \lim Uf_n \Rightarrow \lim \int f_n = \lim \int g_n + \lim \int h_n$$

$$\leq \liminf \int f_n\varphi + \liminf \int f_n(1-\varphi)$$

$$\leq \limsup \int f_n\varphi + \liminf \int f_n(1-\varphi) \leq \lim \int f_n .$$

Showing that $\lim \int f_n\varphi$ exists. Q.e.d.

Proposition 5. Let A be natural additive and ν its Revuz measure [1]. Then

1. For every f with $Uf \leq U_A$ we have $\int f \leq \nu(1)$. If B is a natural additive functional with $U_B \leq U_A$, then $\mu(1) \leq \nu(1)$ where μ is the Revuz measure of B.

2. Let f_n be such that $Uf_n \leq U_A$ and $\lim Uf_n = U_A$. Then for all bounded positive continuous φ
$$\nu(\varphi) \leq \liminf \int f_n\varphi .$$

3. If $\nu(1) < \infty$, $Uf_n \leq U_A$ and $\lim Uf_n = U_A$, then
$$\lim \int f_n\varphi = \nu(\varphi)$$
for all bounded continuous φ, i.e. f_n converges to ν weakly.

Proof. 1. If $Uf \leq U_A$, then $\int f \leq \nu(1)$ is proved as in Proposition 1.

2. If $\varphi \geq 0$ is bounded and continuous, $U_A^\alpha\varphi = \lim U^\alpha(f_n\varphi)$. So by Fatou,

$$(\xi, \alpha U_A^\alpha\varphi) \leq \liminf \alpha(\xi, U^\alpha(f_n\varphi)) \leq \liminf \int f_n\varphi \quad (\xi \text{ is excessive}).$$

As $\alpha \uparrow \infty$, the left side above increases to (ν,φ). That proves 2.

3. From 1, $\nu(1) \geq \int f_n$ and from 2, $\liminf \int f_n\varphi \geq \nu(\varphi)$. These obviously give 3.

Corollary 6. Let D be relatively compact open and $s = P_D 1$. Suppose the process is transient so that $P_D 1$ is given by a natural additive functional A. Then the Revuz measure of A is concentrated on $\overline{D}$.

Proof. By (4.15), p. 88 of [7] we can choose functions f_n with support $(f_n) \subset D$ so that $Uf_n \uparrow s$. If φ is continuous $\geqq 0$ and vanishes on $\overline{D}$, then from 2 of Proposition 5 we see

$$\nu(\varphi) \leqq \liminf \int f_n \varphi = 0. \qquad \text{Q.e.d.}$$

Remark. The same proof shows that if $s = U_A$ satisfies $s = P_D s$ for an open set D, then the Revuz measure of s is concentrated on $\overline{D}$.

Proposition 7. Let A be a natural additive functional with Revuz measure ν. Then we can write $A = \sum_1^\infty A_n$, A_n natural such that the Revuz measure of A_n is finite.

Proof. Let f be integrable so that $Uf > 0$ everywhere. Put $s = U_A 1$. Define s_n inductively as follows:

Put $s_0 = s$. Now $s_0 = s_0 \wedge Uf + x_1$ where $x_1 = s_0 - s_0 \wedge Uf$. By a theorem of Mokobodzki [2],

$$s_0 = s_1 + Rx_1 = s_1 + s^1, \quad \text{say},$$

where $s_1 \leqq s_0 \wedge Uf$ and Rx_1 is the smallest excessive function dominating x_1. Similarly we can write

$$s_1^1 = s_2 + Rx_2 = s_2 + s_2^1, \quad \text{say},$$

where $s_2 \leq s_1^1 \wedge 2Uf$ and $x_2 = s_1^1 - 2Uf$. We then have $s = s_1 + s_2 + s_2^1$. In general, if $s = s_1 + \ldots + s_n + s_n^1$, we write

$$s_n^1 = s_{n+1} + s_{n+1}^1,$$

where $s_{n+1} \leq (n+1)Uf \wedge s_n^1$ and $s_{n+1}^1 = R\{(s_n^1 - (n+1)Uf)^+\}$.

We claim that s_n^1 decreases to zero. It is clear that $R(s_n^1 - (n+1)Uf)$ decreases. Further $s_n^1 - (n+1)Uf \leq s - (n+1)Uf$. If $D_n = \{s > (n+1)Uf\}$, then $P_{D_n} s \geq s.1_{D_n} \geq (s - (n+1)Uf)^+$ so that $s_{n+1}^1 \leq P_{D_n} s$. So it suffices to show that $P_{D_n} s$ tends to zero. Because $s(X_t)$ is a class D potential, we need only show that $T_{D_n} \uparrow \infty$ a.s. If $T = \lim T_{D_n}$, on the set $T < \infty$, $s(x_{T_{D_n}}) \geq (n+1)Uf(X_{T_{D_n}})$. Now $\lim s(X_{T_{D_n}})$ and $\lim Uf(X_{T_{D_n}})$ exist and both are finite almost surely and the latter $\geq Uf(X_T) > 0$. Therefore $T = \infty$ a.s.

Thus we have written

$$s = \sum_1^\infty s_i$$

where $s_i \leq i.Uf$. If A_i is the natural additive functional of s_i, then Revuz measure of A_i is finite and $A = \Sigma A_i$ as desired. Q.e.d.

§2. In [3] the starting point was the following conditions on the potential kernel u:

1) $y \to u(x,y)^{-1}$ is finite and continuous

2) $u(x,y) = \infty$ iff $x = y$.

Using these it was shown that there is a σ-finite measure n such that

(1) $$P_K 1(x) = \int u(x,y)\, n(dy) \qquad (K \text{ compact}).$$

There has been dissatisfaction with these conditions on the grounds that these do not cover, for example, many Lévy processes. Now we give a set of conditions - such that (1) is still true - which are more general that that of [3] and which include the case of Lévy processes. In the next section we shall show that these conditions are indeed more general than those in [3].

More precisely, we shall prove the following: Suppose $y \to u(x,y)$ is l. s. c. for fixed x and there exists a $\varphi > 0$ such that

$$\int u(x,y)\varphi(x)\, dx$$

is continuous. Then we can find a version v of u so that $y \to v(x,y)$ is l.s.c., $x \to v(x,y)$ is excessive and for every natural additive functional A with Revuz measure ν:

$$U_A f = \int v(x,y) f(y)\, \nu(dy).$$

So suppose that we denote by u a density for U:

(2) $$Uf(x) = \int u(x,y) f(y)\, dy.$$

Proposition 1. Suppose $y \to u(x,y)$ is l.s.c. Let A be a natural additive functional with Revuz measure μ. Then

$$(3) \qquad U_A f \geqq U(f\mu) = \int u(x,y) f(y) \mu(dy), \qquad f \geqq 0.$$

Proof. It is enough to prove (3) when $f \geqq 0$ is bounded and continuous. Let $Uf_n \uparrow U_A 1$. From part 2 of Proposition 5, §1,

$$\liminf \int f_n \varphi = \nu(\varphi)$$

for all bounded continuous $\varphi \geqq 0$. Since $u(x,\cdot)$ is l.s.c. and f is continuous, this implies

$$\liminf U(f_n f) \geqq \int u(x,y) f(y) \mu(dy).$$

The left side above is just $U_A f$. Q.e.d.

Remark. If for each compact K there exists x such that $\inf_{y \in K} u(x,y) > 0$, the above implies, taking $f = 1$, that μ is a Radon measure. Also, we cannot claim equality in (3), in general. For example, let $D \subset R^n$ be an open set and G its Green function. Choose a compact set K of zero measure and put $u(x,y) = G(x,y)$, if $y \notin K$, and $u(x,y) = 0$, if $y \in K$. Then $y \to u(x,y)$ is l.s.c. if μ is the equilibrium measure of K, $\int u(x,y) \mu(dy) \equiv 0$.

Proposition 2. Suppose $y \to u(x,y)$ is l.s.c. and for some $0 \leqq \varphi$, $\hat{U}\varphi(y) = \int u(x,y) \varphi(x) dx$ is strictly positive. Then the Revuz measure of a natural additive functional is a Radon measure.

Proof. If $s = U_A 1$, we masy assume $\int s\varphi dx < \infty$. Let $Uf_n \uparrow s$. By l.s.continuity,

$$\int \hat{U}\varphi d\mu \leqq \liminf \int (\hat{U}\varphi) f_n dx = \liminf \int \varphi U f_n dx \leqq \int \varphi s \, dx < \infty. \qquad \text{Q.e.d.}$$

Theorem 3. Let $y \to u(x,y)$ be l.s.c. and suppose that there is $\varphi > 0$ such that $\hat{U}\varphi(y)$ is continuous. Let A be natural additive with Revuz measure μ and $s = U_A 1$. If $x \to u(x,y)$ is super median, then

$$(4) \qquad s = U\mu = \int u(x,y)\mu(dy).$$

In any case there is a version v of u such that $x \to v(x,y)$ is excessive, $y \to v(x,y)$ is l.s.c. and

$$(5) \qquad s = V\mu = \int v(x,y)\mu(dy).$$

Proof. If $\hat{U}\varphi$ is strictly positive, we have seen in Proposition 2 above that μ is a Radon measure. In all cases we can write $s = \Sigma s_n$ where each s_n has finite Revuz measure, by Proposition 7, §1. Thus there is no loss of generality in assuming that μ is finite.

It is given that $\varphi > 0$ and $\hat{U}\varphi$ is continuous. We will show in §4 that we may assume that $\hat{U}\varphi$ is bounded. If $\varphi = a+b$, because $\hat{U}a$ and $\hat{U}b$ are ls.c. with sum continuous, both have to be continuous. So making φ smaller does not affect continuity. We may thus assume $\int \varphi s < \infty$.

Let $Uf_n \uparrow s$. Then by Proposition 5, §1, $f_n dx$ converges weakly to μ.

$$\int \varphi U\mu = \int \hat{U}\varphi d\mu = \lim \int (\hat{U}\varphi) f_n = \int \varphi s < \infty.$$

On the other hand, by Proposition 1, $U\mu \leqq s$. We deduce

(6) $$U\mu = s \quad \text{a.e.}$$

If $x \to u(x,y)$ is super median and $\underline{u}$ its excessive regularization, then $\underline{u} \leqq u$, $U\mu = \underline{U}\mu$ a.e., $\underline{U}\mu = s$ a.e. Hence $\underline{U}\mu = s$ everywhere. Since $\underline{U}\mu \leqq U\mu \leqq s$, we also have $U\mu \equiv s$.

It remains to prove the last claim.

Claim 1. Let U^α denote the resolvent corresponding to U. Then for every y

(7) $$\alpha U^\alpha u(x,y) \leqq u(x,y) \qquad \text{almost all } x.$$

To prove this, note that $\alpha U^\alpha u(x,y)$ is l.s.c. in y. Since for each $f \geqq 0$, $\alpha U^\alpha Uf(x) \leqq Uf(x)$:

(8) $$\alpha U^\alpha u(x,y) \leqq u(x,y) \quad \text{for each } x \text{ for almost all } y.$$

Let $\varphi > 0$ be such that $\hat{U}\varphi(y)$ is continuous. As we observed before, for each Borel function $0 \leqq \rho \leqq 1$, the same is true of $\hat{U}(\rho\varphi)$. $\alpha U^\alpha u(x,y)$ being l.s.c., the same is true of the left side of

(9) $$\int \alpha U^\alpha u(x,y)\rho(x)\varphi(x)\,dx \leqq \int u(x,y)\rho(x)\varphi(x)\,dx,$$

the inequality in (9) holding for almost all y as seen from (8). The right side of (9) being continuous, (9) holds for all y. Since $\varphi > 0$ and $0 \leqq \rho \leqq 1$, arbitrary (7) is established.

Claim 2. For each y, $x \to U^\beta u(x,y)$ is super median for each β.

Indeed, applying U^β to both sides of (7), we see that $\alpha U^\alpha(U^\beta u) \leqq U^\beta u$ for all α, i.e. that $U^\beta u(x,y)$ is super median in x for each y.

Claim 3. $\alpha U^{\alpha}u(x,y)$ is increasing in α. Indeed, if g is function such that $U^{\beta}g$ is super median for each $\beta > 0$, then, as seen by using the resolvent equation $\beta U^{\beta}g$ is increasing in β.

Claim 4. Put

$$\tilde{u}(x,y) = \lim_{\alpha\uparrow\infty} \alpha U^{\alpha}u(x,y). \tag{10}$$

Then $\tilde{u}(x,\cdot) = u(x,\cdot)$ almost everywhere, $y \to \tilde{u}(x,y)$ is l.s.c. and $x \to \tilde{u}(x,y)$ is super median. If $v(x,y)$ is the excessive regularization of $\tilde{u}$, then for every class (D) potential s with Revuz measure μ

$$s = V\mu = \int v(x,y)\mu(dy). \tag{11}$$

The limit in (10) exists by Claim 3, and is super median in x by Claim 2. It is also l.s.c. in y since $\alpha U^{\alpha}u(x,y)$ is l.s.c. in y for each α. Operating both sides of (6) by αU^{α} and taking limits, we see

$$\int \tilde{u}(\cdot,y)\mu(dy) = s(\cdot). \tag{12}$$

Since s is excessive, we can replace $\tilde{u}$ by v in (12) to get (11). Since Uf is excessive of class (D) with Revuz measure f, we get

$$\tilde{U}f = Vf = Uf,$$

i.e. $\tilde{\tilde{u}}(x,y) = v(x,y) = u(x,y)$ almost every y it is also clear that $v(x,y)$ is l.s.c. in Y being the increasing limit of $\alpha U^{\alpha}\tilde{u}(x,y)$. That completes the proof of the theorem.

A simple consequence of Proposition 1 is that if u is infinite on the diagonal, then points are polar. Let us assume that points are polar and $u(\cdot,y)$ is finite almost everywhere for each y. It is clear from the proof of the above theorem that $v(\cdot,y)$ is also finite almost everywhere. The construction of §2 in [4] when applied to v gives us a kernel w satisfying

$$P_D w(x,y) = w(x,y), \qquad y \in D, \tag{13}$$

is valid for all open sets D and $x \to w(x,y)$ is excessive for every y.

The proof of the following proposition is the same as the proof of Theorem 5, §3 of [4]. Only small changes are needed and we will only indicate these.

Proposition 4. The set of all y such that

$$v(\cdot,y) \not\equiv w(\cdot,y) \tag{14}$$

has measure zero for any Revuz measure associated to a natural additive functional.

Proof. Let $Uf_n = Vf_n$ increase to $s = U_A 1$. Then for all bounded continuous φ, $V(f_n\varphi)$ tends to $U_A\varphi = V(\varphi\mu) = \int v(x,y)\varphi(y)\mu(dy)$. This means that $v(x,y)f_n(y)dy$ tends weakly to $v(x,y)\mu(dy)$. The rest of the proof is verbatim the same as that of Theorem 5, §3 of [4]. Q.e.d.

The above proposition implies: If s is a class D potential with Revuz measure μ, then

$$V\mu = W\mu = s. \tag{15}$$

As in Theorem 8 of [4] we have uniqueness with regard to w. Since the proof is similar and simpler, we will only outline the proof.

Corollary 5. The set of y such that $v(\cdot,y) \neq w(\cdot,y)$ is left polar, i.e. X_{t-} never hits this set

Theorem 6. Let s be a class D potential. If m is a Radon measure such that

$$s = Wm,$$

then $W(\varphi m) = W(\varphi\mu)$ for all Borel φ, where μ is the Revuz measure of s.

Proof. Step 1. Suppose first that m is concentrated on a compact set K. Then $P_D s = s$ for each open neighbourhood D of K. There is a sequence f_n which vanish off D such that Uf_n increase to s. The Revuz measure of s is then concentrated on $\bar{D}$. This being true for all open D containing K, μ, the Revuz measure of s is also concentrated on K.

Step 2. Suppose m is concentrated on a compact set K. Then $wdm = wd\mu$, μ = Revuz measure of s. The proof is verbatim the same as that of Step 2 in Theorem 8, §3 of [4].

Step 3. The general case. Let μ be the Revuz measure of s. For any compact set K, if $s_K(x) = \int_K w(x,y)m(dy)$, then s dominates s_K in the strong order. So μ dominates the Revuz measure of s_K, so from Step 2, $w(x,y)\mu(dy)$ dominates $w(x,y)1_K(y)m(dy)$.

This is true for all compacts K and $Wm = W\mu$, so we must have

$$w(x,y)m(dy) \equiv w(x,y)\mu(dy).$$

The proof is thus complete.

§3. In this article we compare the conditions in [3] with those of this note.

Let $\{U^\alpha, \alpha \geqq 0\}$ denote the resolvent for the Markov process. It is well-known that there is a dual resolvent $\{\hat{V}^\alpha \ \alpha \geqq 0\}$ - with respect to the excessive reference measure ξ. We will simply write U and $\hat{V}$ when $\alpha = 0$.

Proposition 1. Suppose u is a density for U such that $y \to u(x,y)$ is l.s.c. Put

$$(1) \qquad \hat{U}f(y) = \int u(x,y)f(x)dx, \qquad f \geqq 0.$$

Then $\hat{U}$ satisfies the maximum principle:

$$(2) \qquad \sup(\hat{U}f(y)) = \sup(\hat{U}f(y): f(y) > 0).$$

Proof. By Fubini, $\hat{U}f(y) = \hat{V}f(y)$ for almost all y. (2) holds with $\hat{U}$ replaced by $\hat{V}$ since the latter corresponds to a sub-Markov resolvent. Let $E = (\hat{U}f = \hat{V}f)$ and $g = 1_E.f$. Then $\hat{U}f \equiv \hat{U}g$ and $\hat{U}g = \hat{V}g$ on $(g>0)$. Thus $\sup(\hat{U}f: f>0) \geqq \sup(\hat{U}g: g>0) = \sup(\hat{V}g: g>0) = \sup(\hat{V}g) = \sup \hat{V}f$. Lower semi continuity takes care of the rest. Q.E.D.

Corollary 2. Suppose the assumptions of Proposition 1 hold. If there is an f strictly positive such that $\hat{U}f$ is continuous and positive, then there is a strictly positive g such that Ug is bounded continuous and strictly positive.

Proof. If $0 \leqq \varphi \leqq f$, then $\hat{U}f = \hat{U}^{\varphi} + \hat{U}(f-\varphi)$. Therefore continuity of the left side implies the continuity of each summand on the right because each is lower semi continuous. So let us show that there is a strictly positive $g \leqq f$ such that $0 < \hat{U}g \leqq 1$.

On each compact set K, $\hat{U}f$ is bounded. So by the maximum principle $\hat{U}(f 1_K)$ is bounded everywhere. Also as K increases to the state space E, these functions increase to $\hat{U}f$, which is strictly positive. Thus by Dini, to each compact set K corresponds a compact set L - which we may assume contains K - such that $\hat{U}(f 1_L)$ is strictly positive on K and bounded elsewhere.

A sum of suitable multiples of these functions gives us the desired function g. That completes the proof.

In the rest of this article we assume that u satisfies the conditions in [3], namely that

$$u(x,y) = \infty \quad \text{iff} \quad x = y$$

and $y \to u(x,y)^{-1}$ is finite and continuous.

Proposition 3. There is a strictly positive function b such that $0 < \hat{U}b \leq 1$ everywhere.

Proof. For any fixed x, the measure $U^1(x,dz)$ is absolutely continuous relative to dz. We claim it is equivalent to dz. Indeed, $U^1(x,f) = 0$ implies $P_t f(x) = 0$ almost all t, which in turn implies that $Uf(x) = 0$. Since $u(x,y) > 0$, this can only happen if $f = 0$.

Also $U^1 Uf(x) \leqq Uf(x)$ for all $f \geqq 0$ and hence $\hat{U}\varphi(y) \leqq u(x,y)$ almost all y where φ is the density of $U^1(x,dz)$. In particular we have a strictly positive function φ such that $\hat{U}\varphi$ is finite almost everywhere.

The required function b can be constructed, using the maximum principle as in Corollary 2. Q.e.d.

Proposition 4 (The continuity principle). Let $f \geqq 0$ and $\hat{U}f$ be finite and continuous on support (f), which is assumed compact. Then $\hat{U}f$ is continuous everywhere.

Proof. Let K be the support of f. Note that dominated convergence cannot be used to conclude the continuity of $\hat{U}f$ off K. We proceed as follows.

First Uf being continuous on K is also bounded and hence bounded by the same constant everywhere, by the maximum principle. The continuity of $\hat{U}f$ on K and the continuity of $u(x,\cdot)$ imply that the set of functions $\{u(x,y)f(x),\ y \in K\}$ is uniformly integrable on K.

Therefore, given $\varepsilon > 0$, there is a $\delta > 0$ such that, $A \subset K$, $\xi(A) < \delta$ imply

$$\int_A u(x,y)f(x)dx < \varepsilon$$

for all $y \in K$ and hence everywhere by the maximum principle.

This fact together with the boundedness of Uf on E imply that the family $\{u(\cdot,y)f(\cdot),\ y \in E\}$ is uniformly integrable on K. See T19, p. 17 of Meyer [5]. Since $u(x,\cdot)$ is continuous, the proof is complete.

A standard argument using the above proposition and Lusin's theorem shows the following: If $\hat{U}f$ is finite almost everywhere, then we can write $f = \Sigma f_n$ so that for every n, $\hat{U}f_n$ is bounded and continuous on E.

We have shown in Proposition 3 that there is a strictly positive function b such that $\hat{U}b \leqq 1$. We can write $b = \Sigma b_n$ so that $\hat{U}b_n \leqq 1$ and is continuous everywhere. The function $a = \Sigma 2^{-n} b_n$ is strictly positive everywhere and $0 < \hat{U}a \leqq 1$ and is continuous. Thus we have

Theorem 5. There is a strictly positive function a such that $0 < \hat{U}a \leqq 1$ and Ua is continuous everywhere.

Therefore the conditions in [3] imply the conditions here.

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