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Séminaire de probabilités (Strasbourg), tome 12 (1978), p. 428-445
<http://www.numdam.org/item?id=SPS_1978__12__428_0>

Université de Strasbourg
Séminaire de Probabilités 1977/78

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Introduction

The sojourn (or occupation) times of the standard Brownian motion $B(t)$ have long been investigated using the method of M. Kac and its variants. For $B(t)$ killed at a point $\alpha > 0$, for example, the total sojourn in an interval has a distribution whose Laplace transform is available but somewhat complicated. Thus, for $a < b < 0$ and $B(0) = 0$, it is given by

$$(0.1) \qquad \frac{1 - b\sqrt{2\lambda}\,\tanh\,(b-a)\sqrt{2\lambda}}{1 + (\alpha - b)\sqrt{2\lambda}\,\tanh\,(b-a)\sqrt{2\lambda}},$$

a result which is obtained below as Example 2.4, but can also be obtained by Kac's method as in [2, 2.b].

In the present paper, such problems are treated by utilizing the local time method of [3] and [4]. Perhaps the most significant advantage of this method is its adaptability to the multivariate case. However, even for the case of a single M-dependent boundary (where $M(t) = \max_{0 \le s \le t} B(s)$) it seems to be the more natural approach. It turns out that for a fairly wide class of such boundaries the Laplace transforms of the sojourn times can be given explicitly in simple form. While these results, again, can probably also be obtained by combining Kac's method with a suitable limit procedure, it seems better to use Brownian local times. Then, in the multivariate cases, the Markov property of the local time circumvents complications which can easily become prohibitive.

Our main result is to obtain, in the case of a single boundary, expressions which are completely general, insofar as they apply to all measurable boundary functions $f\ (M(t))$. Analogous expressions can be obtained when we have several such boundaries, and seek the joint Laplace transforms of the sojourn times in the several bounded intervals. The formulas become longer to state, and we limit our treatment to two such boundaries. But even in the case of constant boundaries the formulas, although presumably known, do not seem to be in the literature.

In Section 1 we deal with a single boundary, where our method requires only relatively simple information. Some of the preliminaries can also be obtained by Kac's method, but we proceed here without this additional prerequisite. We include a zero-one law and an absorbtion probability formula which are easy consequences, as well as four explicit examples in which the transforms are elementary functions.

In Section 2 we present the cases of two and three bounded intervals, which require two formulas from [4]. As a check on these results, we rederive the stopped Brownian formula of H. M. Taylor [5] and D. Williams [6] by our method. We also give some results for constant boundaries, and one further example. Finally, we specialize to the case of one interior boundary point with two killing points, and extend the Taylor formula to encompass two sojourn intervals.

Except in a couple of instances we have made no attempt to invert the transforms or to obtain further information from them. Nor have we sought to extend the method to more than two boundaries. Such matters can perhaps be dealt with better when (and if) the need arises.

Section 1. A Semi-infinite Boundary.

The problem described in the introduction has a "disguised form" which is treated first. We write P^x or E^x for the process $B(t)$, $B(0) = x$, but simply P and E when $x = 0$. Let $s^+(t,x) = s^+(t, x, w) = \frac{1}{2} \frac{d}{dx} \int_0^t I_{(0,x)}(|B(t)|)\, dt$ denote the local time at x of the reflected Brownian motion $|B|$, and set $s^+(t, 0) = s^+(t)$ [2, 2.2]. For $\alpha \geq 0$, let $T^+(\alpha) = \inf \left\{ t: s^+(t) > \alpha \right\}$ be the right-continuous inverse local time. By a well-known result of P. Levy we have the P - equivalence $(M(t) - B(t), M(t)) \equiv (|B(t)|, s^+(t))$, so that in particular $T^+(\alpha)$ is a stable process of exponent $\frac{1}{2}$, equivalent to the inverse maximum $M^{-1}(t)$.

Lemma 1.1. For $x > 0$ and $\lambda > 0$,

$$E^x \exp - \lambda \int_0^{T^+(0)} I_{(x,\infty)}(|B(t)|)\, dt = (1 + x\sqrt{2\lambda})^{-1} .$$

Remark: By use of tables, this is the transform of

$$(2x^2 \pi y)^{-\frac{1}{2}} - (x^3\sqrt{2\pi})^{-1}(\exp - \frac{y}{2x^2}) \int_0^{\sqrt{y}} \exp - \frac{z^2}{2x^2}\, dz, \quad 0 < y.$$

Proof. Although this result is quite certainly known, we will use a method which introduces the sequel. We note first that $T^+(0)$ is equal P^x - a.s. to the passage time to 0. Now for P^x, the process $s^+(T^+(0), y)$ is a Markov process in the parameter $y \geq 0$ with continuous trajectories. In fact, in $0 \leq y \leq x$ it is the diffusion with generator $z\frac{d^2}{dz^2} + \frac{d}{dz}$, while in (x, ∞) it is the diffusion with generator $z\frac{d^2}{dz^2}$, in accordance with [3, Theorem 2.2].* In particular, if $s^+(T^+(0), x) = \beta$ is given $(\beta > 0)$ then

$$\int_0^{T^+(0)} I_{(x,\infty)}(|B(t)|)\, dt = 2\int_x^\infty s^+(T^+(0), y)\, dy,$$

* In the introduction to [3] the factor 4 should be 2, and in Theorem 2.2, line 2, $T(x_0, \alpha_0)$ should be as in line 14.

which is P^x-independent of $T^+(0)$ and has the same distribution as $2\int_0^\infty s^+(T^+(\beta), y)\,dy$ for P. Setting $s(t, y) = \frac{1}{2}\frac{d}{dy}\int_0^t I_{(-\infty, y)}(B(s))\,ds$ (the local time of B(t)) and $T(\beta) = \inf\left\{t : s(t, 0) > \beta\right\}$, this is P-equivalent to $2\int_0^\infty s(T(\beta), y)\,dy$. To see this, we recall [2, 2.11] that a process equivalent to $|B|$ may be constructed from B by deleting the set $\left\{ t : B(t) < 0 \right\}$ and telescoping the remainder of the time axis to restore the continuum. But since we have

$$T(\beta) = 2\int_0^\infty s(T(\beta), y)dy + 2\int_{-\infty}^0 s(T(\beta), y)dy,$$

where the two terms on the right are independent and identically distributed, their Laplace transform is ([2, p. 26])

$$\begin{aligned}(E \exp - \lambda T(\beta))^{\frac{1}{2}} &= (E \exp - \lambda T^+(2\beta))^{\frac{1}{2}} \\ &= (E \exp - \lambda M^{-1}(2\beta))^{\frac{1}{2}} \\ &= \exp - \beta\sqrt{2\lambda}.\end{aligned}$$

But, finally, the distribution of $s^+(T^+(0), x)$ for P^x is the same as that of $s(T(0), x)$ for P^x, which is known to be exponential, with density $x^{-1} \exp - \beta x^{-1}$ [3, p.74]. Thus, the transform of the Lemma is given by

$$\begin{aligned}E^x \exp - 2\lambda\int_x^\infty s^+(T^+(0), \beta)\,d\beta &= \int_0^\infty x^{-1} \exp - \beta(x^{-1} + \sqrt{2\lambda})\,d\beta \\ &= (1 + x\sqrt{2\lambda})^{-1}.\end{aligned}$$

Using this result, we next establish

Lemma 1.2. $E \exp -\lambda\int_0^{T^+(\alpha)} I_{(a,\infty)}(|B(t)|)dt = \exp - \alpha\sqrt{2\lambda}\,(1 + a\sqrt{2\lambda})^{-1}$.

Remark. This transform can be inverted explicitely in terms of the Bessel function I_1 by using [1, Chap.XIII. 11, Exercise 13] to introduce $\sqrt{\lambda}$. The result is complicated.

Proof. Since $B(T^+(\frac{k\alpha}{n})) = 0$ a.s., the sojourn times $\int_{T^+(\frac{(k-1)\alpha}{n})}^{T^+(\frac{k\alpha}{n})} I_{(a,\infty)}(|B(t)|)\,dt$ are independent and identically distributed $1 \le k \le n$. With probability one, there is for large n at most one return from a to 0 in each such interval. Finally we have $P\left\{\max_{0 \le t \le T^+(\alpha)} |B(t)| < a\right\} = \exp - \alpha a^{-1}$, since $s^+(T^+(\alpha), y)$ for P is the diffusion with generator $z\frac{d^2}{dz^2}$ and initial value α, which is killed before $y = a$ with probability $\exp - \alpha a^{-1}$ [3, Corollary 1.2]. Combining these observations with Lemma 1.1, the desired transform can be written as

$$\lim_{n\to\infty}\left[(1 - \exp - \alpha(na)^{-1})\,(1 + a\sqrt{2\lambda})^{-1} + \exp - \alpha(na)^{-1}\right]^n$$

$$= \lim_{n\to\infty}\left[\alpha(na)^{-1}\,((1 + a\sqrt{2\lambda})^{-1} - 1) + 1\right]^n$$

$$= \lim_{n\to\infty}\left[1 - n^{-1}\alpha\sqrt{2\lambda}(1 + a\sqrt{2\lambda})^{-1}\right]^n$$

$$= \exp - \alpha\sqrt{2\lambda}\,(1 + a\sqrt{2\lambda})^{-1}, \text{ as asserted.}$$

Theorem 1.1. For Borel measurable $f(x) \ge 0$,

$$E \exp - \lambda\int_0^{T^+(\alpha)} I_{(f(s^+(t)),\infty)}(|B(t)|)\,dt$$

$$= \exp - \sqrt{2\lambda}\int_0^{\alpha}(1 + \sqrt{2\lambda}\,f(x))^{-1}\,dx\ ;\ \lambda > 0,\ \alpha > 0.$$

Proof. Suppose first that for constants c_k,

$f(x) = c_k$ for $(k-1)\,\alpha n^{-1} < x \le k\alpha n^{-1}$, $1 \le k \le n$.

Then since $T^+(k\alpha n^{-1})$ is a stopping time with $P\left\{ B(T^+(k\alpha n^{-1})) = 0 \right\} = 1$, and $s^+(t)$ is a strong additive functional of $B^+(t)$, the strong Markov property yields

$$E \exp - \lambda \int_0^{T^+(\alpha)} I_{(f(s+(t)),\, \infty)} (|B(t)|)\, dt$$

$$= \prod_{k=1}^{n} E \exp - \lambda \int_0^{T^+(\alpha n^{-1})} I_{(c_k,\infty)} (|B(t)|)\, dt$$

$$= \prod_{k=1}^{n} \exp - \alpha n^{-1} \sqrt{2\lambda}\, (1 + c_k \sqrt{2\lambda})^{-1}$$

$$= \exp - \sqrt{2\lambda} \int_0^{\alpha} (1 + f(x) \sqrt{2\lambda})^{-1}\, dx.$$

For the general case, we need only remark that, by the monotone convergence theorem, passage to monotone limits in f is justified on both sides of the equation. Hence the assertion is valid for the monotone bounded closure of the non-negative step functions. Since this clearly contains the indicator functions of disjoint finite unions of intervals, it also contains the Borel indicators. But it is not hard to see that the closure is likewise closed under linear combination with non-negative coefficients. Hence it contains the non-negative Borel simple functions, and the assertion follows.

Remark. For given $\alpha > 0$, we note that the theorem involves $f(x)$ only for $0 < x < \alpha$.

We now obtain the main result for single boundaries as

Corollary 1.1. For any continuous $g(x) \leq x$, we have for $\alpha > 0$, $\lambda > 0$

$$E \exp - \lambda \int_0^{M^{-1}(\alpha)} I_{(-\infty,\, g(M(t)))} (B(t))\, dt$$

$$= \exp - \sqrt{2\lambda} \int_0^{\alpha} (1 + \sqrt{2\lambda}\, (x - g\,(x))^{-1}\, dx,$$

where $M^{-1}(\alpha)$ is a.s. equal to the first passage time to α.

Proof. We have

$$\exp - \lambda \int_0^{M^{-1}(\alpha)} I_{(-\infty,\ g(M(t)))}(B(t))\,dt$$

$$= \exp - \lambda \int_0^{M^{-1}(\alpha)} I_{(M(t)\ -\ g(M(t)),\ \infty)}(M(t) - B(t))\,dt.$$

Now, Lévy's equivalence $(M - B, M) \equiv (|B|, s^{+})$ reduces Corollary 1.1 to Theorem 1.1.

Let us state three simple examples.

Example 1.1. Letting $g(x) = x$ for $0 < x < c$ and $= c$ for $c < x < \alpha$,

$$E \exp - \lambda \int_0^{M^{-1}(\alpha)} I_{(-\infty,\ c)}(B(t))\,dt$$

$$= \begin{cases} (\exp - \sqrt{2\lambda}\ c)(1 + (\alpha - c)\sqrt{2\lambda})^{-1} & \text{if } 0 < c < \alpha, \\ (1 + \sqrt{2\lambda}\ c)(1 + \sqrt{2\lambda}\ (\alpha + c))^{-1} & \text{if } c < 0. \end{cases}$$

Example 1.2. For $c_1 \leq 1$ and $c_2 \geq 0$,

$$E \exp - \lambda \int_0^{M^{-1}(\alpha)} I_{(-\infty,\ c_1 M(t) - c_2)}(B(t))\,dt$$

$$= (1 + \alpha\sqrt{2\lambda}(1 - c_1)(1 + \sqrt{2\lambda}\ c_2)^{-1})^{-(1-c_1)^{-1}}.$$

Example 1.3. For $0 < c \leq 1$ and $\alpha \leq 1$,

$$E \exp - \lambda \int_0^{M^{-1}(\alpha)} I_{(-\infty,\ cM^2(t))}(B(t))\,dt$$

$$= \left| \frac{4c^2\gamma\alpha + (1 - 4c^2\gamma^2 - 2c\alpha)}{4c^2\gamma\alpha - (1 - 4c^2\gamma^2 - 2c\alpha)} \right|^{\frac{1}{2c\alpha}}, \text{ where}$$

$\gamma = \left(\dfrac{c\sqrt{2\lambda} + 4c^2}{4c^3\sqrt{2\lambda}} \right)^{\frac{1}{2}}$. In particular, for $c = \alpha = 1$ this

reduces to $\left(\dfrac{2\gamma - 1}{2\gamma + 1} \right)$, where $\gamma = \left(\dfrac{\sqrt{2\lambda} + 4}{4\sqrt{2\lambda}} \right)^{\frac{1}{2}}$

One general consequence of Corollary 1.1 which seems of interest is the following:

Corollary 1.2. For $g(x) \leq x$, $P\left\{\int_0^\infty I_{(-\infty,\ g(M(t)))}(B(t))dt < \infty\right\} = 1$ or 0 according as $\int_0^\infty \left(\frac{1}{1 + x - g(x)}\right) dx < \infty$ or $= \infty$.

Proof. Since the finiteness of total sojourn is a tail event of $B(t)$, and $B(t)$ satisfies the $0 - 1$ Law as $t \to \infty$ (by use of the equivalence $B(t) \equiv t\,B(\frac{1}{t})$, for instance), we conclude that the probability in Corollary 1.2 is always zero or one. Then the assertion fol ows from Corollary 1.1 by letting $\alpha \to \infty$ and $\lambda \to 0$. By applying this criterion, we obtain easily

Example 1.4. The total sojourn in $(-\infty, -M(t)|\log M(t)|^c)$ is a.s. finite for $c > 1$, infinite for $0 \leq c \leq 1$.

Another general result, even more easily derived, is

Corollary 1.3. For fixed $\alpha > 0$, and Borel measurable $g(x) \leq x$, $0 < x < \alpha$,

$$P\left\{ B(t) \geq g(M(t)),\ 0 < t < M^{-1}(\alpha) \right\} = \exp - \int_0^\alpha (x - g(x))^{-1} dx.$$

Proof. Let $\lambda \to \infty$ in Corollary 1.1.

Section 2. Two and Three Intervals.

Continuing the notation of Section 1, let

$$X(a) = \int_0^{T^+(\alpha)} I_{(o,\ a)}(|B(t)|)\, dt = 2\int_0^a s^+ (T^+(\alpha), y)\, dy,$$

and similarly $Y(a) = 2\int_a^\infty s^+(T^+(\alpha), y)\, dy$.

The counterpart of Lemma 1.2 for the two intervals (o, a) and (a, ∞) is given by

<u>Lemma 2.1.</u> $E \exp - (\lambda X(a) + \mu Y(a))$

$$= \exp - \left\{ \alpha\sqrt{2\lambda} \left(\frac{\sqrt{\lambda}\tanh a\sqrt{2\lambda} + \sqrt{\mu}}{\sqrt{\mu}\tanh a\sqrt{2\lambda} + \sqrt{\lambda}} \right) \right\} .$$

<u>Proof</u>. The Laplace Transform in λ of the density of $X(a)$, conditional upon $2s^{+}(T^{+}(\alpha), a) = \beta > 0$, is given by [4, Theorem 2.2] in the form (after trivial adjustment)

$$\left(\frac{\lambda\alpha}{\beta}\right)^{\frac{1}{2}} \frac{\operatorname{cosech} a\sqrt{2\lambda}\ J_1(2i\sqrt{\lambda\alpha\beta}\ \operatorname{cosech} a\sqrt{2\lambda})}{ip_0(a;2\alpha,\beta)\exp((2\alpha+\beta)\sqrt{\frac{\lambda}{2}}\ \operatorname{cotanh} a\sqrt{2\lambda})} ,$$

where $p_0(a;2\alpha,\beta)$ is the density of $2s^{+}(T^{+}(\alpha), a)$. When $\beta > 0$ is given, $X(a)$ and $Y(a)$ are independent and the latter has transform $\exp - \beta\sqrt{\frac{\mu}{2}}$, as seen in the proof of Lemma 1.1. Finally, we have

$P\left\{ s^{+}(T^{+}(\alpha), a) = 0 \right\} = \exp - \alpha a^{-1}$, and conditional upon this event, $Y(a) = 0$ a.s. and $X(a)$ has transform $\exp(\alpha a^{-1} - \sqrt{2\lambda}\ \operatorname{cotanh} a\sqrt{2\lambda})$, as given by [4, Theorem 2.1]. Combining these observations, and integrating term by term the series obtained by cancelling p_0 and substituting

$$J_1(x) = \sum_{j=0}^{\infty} \frac{(-1)^j}{j!(j+1)!} (x/2)^{1+2j} ,$$ we have

$$(2.1)\quad E\exp - (\lambda X(a) + \mu Y(a)) = \exp - \alpha\sqrt{2\lambda}\ \operatorname{cotanh} a\sqrt{2\lambda}$$

$$+ \int_0^{\infty} \left(\frac{\lambda\alpha}{\beta}\right)^{\frac{1}{2}} \frac{\operatorname{cosech} a\sqrt{2\lambda}\ J_1(2i\sqrt{\lambda\alpha\beta}\ \operatorname{cosech} a\sqrt{2\lambda})}{i\exp((2\alpha+\beta)\sqrt{\frac{\lambda}{2}}\ \operatorname{cotanh} a\sqrt{2\lambda})} \exp -\beta\sqrt{\frac{\mu}{2}}\ d\beta$$

$$= (\exp - \alpha\sqrt{2\lambda}\ \operatorname{cotanh} a\sqrt{2\lambda}) \left(\frac{\exp\sqrt{2}\ \alpha\lambda\ \operatorname{cosech}^2 a\sqrt{2\lambda}}{\sqrt{\mu} + \sqrt{\lambda}\ \operatorname{cotanh} a\sqrt{2\lambda}} \right).$$

This reduces easily to the assertion of the Lemma.

It now follows exactly as in the proof of Theorem 1.1 and Corollary 1.1 that we have

Theorem 2.1. For Borel measurable $f(x) \geq 0$,

$$E \exp - \left(\lambda \int_0^{T^+(\alpha)} I_{(0,\ f(s^+(t)))}(|B(t)|)\,dt + \mu \int_0^{T^+(\alpha)} I_{(f(s^+(t)),\ \infty)}(|B(t)|)\,dt\right)$$

$$= \exp - \sqrt{2\lambda} \int_0^{\alpha} \frac{\sqrt{\lambda}\ \tanh\sqrt{2\lambda}\ f(x) + \sqrt{\mu}}{\sqrt{\mu}\ \tanh\sqrt{2\lambda}\ f(x) + \sqrt{\lambda}}\,dx\ ;\ \lambda,\ \mu > 0.$$

Corollary 2.1. For Borel measurable $g(x) \leq x$,

$$E \exp - \left(\lambda \int_0^{M^{-1}(\alpha)} I_{(g(M(t)),\alpha)}(B(t))\,dt + \mu \int_0^{M^{-1}(\alpha)} I_{(-\infty,g(M(t))}(B(t))\,dt\right)$$

$$= \exp - \sqrt{2\lambda} \int_0^{\alpha} \frac{\sqrt{\lambda}\ \tanh\sqrt{2\lambda}\ (x - g(x)) + \sqrt{\mu}}{\sqrt{\mu}\ \tanh\sqrt{2\lambda}\ (x - g(x)) + \sqrt{\lambda}}\,dx.$$

As a first example, we can treat the case of a constant boundary.

Example 2.1. For $g(x) = c \leq 0$ in Corollary 2.1, we obtain by direct integration (setting $y = \exp\sqrt{2\lambda}\ (x - c)$) the expression

$$\frac{\sqrt{\lambda}\ \cosh c\sqrt{2\lambda} - \sqrt{\mu}\ \sinh c\sqrt{2\lambda}}{\sqrt{\lambda}\ \cosh(\alpha - c)\sqrt{2\lambda} + \sqrt{\mu}\ \sinh(\alpha - c)\sqrt{2\lambda}}$$

To obtain the formula of Taylor [5], we need to apply the following extension of Corollary 1.3:

Corollary 2.2. $E\left[\exp - \lambda M^{-1}(\alpha);\ B(t) \geq g(M(t)),\ 0 < t < M^{-1}(\alpha)\right]$

$$= \exp - \sqrt{2\lambda} \int_0^{\alpha} \operatorname{cotanh}\sqrt{2\lambda}\ (x - g(x))\,dx.$$

Proof. We first replace μ by $\lambda + \mu$ in Corollary 2.1, and then let $\mu \to \infty$.

Example 2.2. (Taylor [5], Williams [6]). For $c > 0$, let $T_c = \inf\{t: M(t) - B(t) > c\}$. Then $E \exp - (\lambda T_c + \nu M(T_c))$

$$= \delta\,[\,\delta \cosh \delta c + \nu \sinh \delta c]^{-1}; \quad \delta = \sqrt{2\lambda},\ \nu > -\delta \operatorname{cotanh} \delta c.$$

Remark: This is the expression (1.5) from [6]. As noticed there, the general case of [5] follows from this by using the Cameron-Martin formula.

Proof. We first invert this transform in ν with λ fixed, to write equivalently

$$(2.2)\quad E\left[\ \exp - \lambda T_c;\ M(T_c) > \alpha\ \right]$$

$$= (\cosh \delta c)^{-1} \exp - (\alpha\delta \operatorname{cotanh} c\,\delta)$$

$$= E\left[\ \exp - \lambda T_c;\ M(t) - B(t) \leq c,\ 0 < t < T^{+}(\alpha)\ \right].$$

Now since $M^{-1}(\alpha)$ is a stopping time with $\alpha = B(M^{-1}(\alpha))$, it is seen that the last expression in (2.2) becomes

$$(2.3)\quad E\left[\ \exp - \lambda M^{-1}(\alpha);\ M(t) - B(t) \leq c,\ 0 < t < M^{-1}(\alpha)\ \right]$$

$$\cdot\ E\,(\exp - \lambda T_c).$$

Substituting $g(x) = x - c$ in Corollary 2.2, the first term of (2.3) is $\exp - \alpha\,\delta \operatorname{cotanh} c\,\delta$. The second factor is well-known to be $(\cosh \delta\, c)^{-1}$ (see for example [2, Section 2.6, Problem 2]), hence the derivation is complete.

Example 2.3. For $0 < \gamma < 1$ and $c, \alpha < 0$,

$$E\left[\ \exp - \lambda M^{-1}(\alpha);\ B(t) > \gamma M(t) - c,\ 0 < t < M^{-1}(\alpha)\ \right]$$

$$= \left(\frac{\sinh \sqrt{2\lambda}\ c}{\sinh \sqrt{2\lambda}\ (c + (1 - \gamma)\alpha)}\right)^{\frac{1}{1-\gamma}}$$

Proof. By Corollary 2.2 with $g(x) = \gamma x - c$.

We proceed next to the case of two boundaries and three bounded intervals. Except for an inevitable increase of complexity, this reduces directly to the former situation. For $0 < a < b$, we set

$$X = X(a,b) = 2\int_0^a s^+(T^+(\alpha),\ y)\, dy,$$

$$Y = 2\int_a^b s^+(T^+(\alpha),\ y)\, dy, \text{ and}$$

$$Z = 2\int_b^\infty s^+(T^+(\alpha),\ y)\, dy.$$

Now by the Markov property of $s^+(T^+(\alpha),\ y)$, we have from Lemma 2.1,

$$\begin{aligned} E \exp - (\lambda X + \mu Y + \nu Z) &= E\left[(\exp -\lambda X)(E(\exp - (\mu Y + \nu Z) \mid s^+(T^+(\alpha),a))\right] \\ &= E \exp - \left\{ \lambda X + s^+(T^+(\alpha),a)\sqrt{2\mu}\left(\frac{\sqrt{\mu}\tanh (b-a)\sqrt{2\mu} + \sqrt{\nu}}{\sqrt{\nu}\tanh (b-a)\sqrt{2\mu} + \sqrt{\mu}}\right)\right\} \end{aligned}$$

This simply has the effect of replacing the factor $\exp - \beta\sqrt{\frac{\mu}{2}}$ in the integrand of (2.1) by

$\exp - \beta\sqrt{\frac{\mu}{2}}\left(\frac{\sqrt{\mu}\tanh (b-a)\sqrt{2\mu} + \sqrt{\nu}}{\sqrt{\nu}\tanh (b-a)\sqrt{2\mu} + \sqrt{\mu}}\right)$, or in other words of substituting a different value of μ in the right side of Lemma 2.1 In short, we have

Lemma 2.2

$$E \exp - (\lambda X + \mu Y + \nu Z) = \exp - \left\{ \alpha\sqrt{2\lambda}\left(\frac{\sqrt{\lambda}\tanh a\sqrt{2\lambda} + \sqrt{\zeta}}{\sqrt{\zeta}\tanh a\sqrt{2\lambda} + \sqrt{\lambda}}\right)\right\},$$

where $\sqrt{\zeta} = \sqrt{\mu}\left(\frac{\sqrt{\mu}\tanh (b-a)\sqrt{2\mu} + \sqrt{\nu}}{\sqrt{\nu}\tanh (b-a)\sqrt{2\mu} + \sqrt{\mu}}\right)$.

As before, we have immediately

Theorem 2.2. For Borel measurable $f_2(x) \geq f_1(x) \geq 0$,

$$\begin{aligned} E \exp - \Big(\lambda \int_0^{T^+(\alpha)} & I_{(0,f_1(s^+(t))}|B(t)|dt + \mu\int_0^{T^+(\alpha)} I_{(f_1(s^+(t)),f_2(s^+(t))}|B(t)|dt + \nu\int_0^{T^+(\alpha)} I_{(f_2(s^+(t)),\infty)}|B(t)|dt\Big) \\ &= \exp - \sqrt{2\lambda}\int_0^\alpha \frac{\sqrt{\lambda}\tanh\sqrt{2\lambda}\, f_1(x) + \sqrt{\zeta(x)}}{\sqrt{\zeta(x)}\tanh\sqrt{2\lambda}\, f_1(x) + \sqrt{\lambda}}\, dx, \text{ where} \end{aligned}$$

$$\sqrt{\zeta(x)} = \sqrt{\mu}\left(\frac{\sqrt{\mu}\ \tanh\ (f_2(x) - f_1(x))\sqrt{2\mu} + \sqrt{\nu}}{\sqrt{\nu}\ \tanh\ (f_2(x) - f_1(x))\sqrt{2\mu} + \sqrt{\mu}}\right).$$

Similarly, we have

Corollary 2.3. For Borel measurable $g_2(x) \le g_1(x) \le x$,

$$E \exp - \left(\lambda \int_0^{M^{-1}(\alpha)} I_{(g_1(M(t)),\alpha)}(B(t))\,dt + \mu \int_0^{M^{-1}(\alpha)} I_{(g_2(M(t)),\ g_1(M(t)))}(B(t))\,dt + \nu \int_0^{M^{-1}(\alpha)} I_{(-\infty, g_2(M(t)))}(B(t))\,dt\right)$$

$$= \exp - \sqrt{2\lambda}\int_0^{\alpha} \frac{\sqrt{\lambda}\ \tanh\sqrt{2\lambda}\ (x - g_1(x)) + \sqrt{\zeta(x)}}{\sqrt{\zeta(x)}\ \tanh\sqrt{2\lambda}\ (x - g_1(x) + \sqrt{\lambda}}\,dx\ ,$$

where $\sqrt{\zeta(x)} = \sqrt{\mu}\left(\dfrac{\sqrt{\mu}\ \tanh\ (g_1(x) - g_2(x))\sqrt{2\mu} + \sqrt{\nu}}{\sqrt{\nu}\ \tanh\ (g_1(x) - g_2(x))\sqrt{2\mu} + \sqrt{\mu}}\right).$

As a first example, we can obtain the result quoted in the introduction.

Example 2.4.

$$E \exp - \lambda\int_0^{M^{-1}(\alpha)} I_{(a,b)}(B(t))dt = \frac{1 - b\sqrt{2\lambda}\ \tanh\ (b - a)\sqrt{2\lambda}}{1 + (\alpha - b)\sqrt{2\lambda}\ \tanh\ (b - a)\sqrt{2\lambda}}\ ;$$

$a < b < 0$.

Proof. In Corollary 2.3, we let $\lambda = \nu \to 0$, and then replace μ by λ. With $g_2(x) = a$ and $g_1(x) = b$, we are left with

$$\exp - \int_0^{\alpha} \frac{\sqrt{2\lambda}\ \tanh\ (b - a)\sqrt{2\lambda}}{(\sqrt{2\lambda}\ \tanh\ (b - a)\sqrt{2\lambda})(x - b) + 1}\,dx,$$ which is

integrated by an exponential substitution to give the result.

For a second example, we observe that if we let $v \to \infty$ in Corollary 2.3 with $g_2(x) = -\beta < 0$, we obtain the joint Laplace Transform in (λ, μ) of the sojourn times in $(g_1(M(t)), \alpha)$ and $(-\beta, g_1(M(t)))$ for the Brownian motion killed at both α and $-\beta$, over the set where this killing occurs at α. In this way we can obtain, for instance,

<u>Example 2.5</u>. For $T(\alpha,\beta) = \inf\left\{t : B(t) = \alpha \text{ or } -\beta\right\}$, $-\beta < 0 < \alpha$,

$$E \exp - \left(\lambda \int_0^{T(\alpha,\beta)} I_{(0,\alpha)}(B(t))dt + \mu \int_0^{T(\alpha,\beta)} I_{(-\beta,0)}(B(t))dt\right)$$

$$= I(\alpha,\beta,\lambda,\mu) + I(\beta,\alpha,\mu,\lambda) \text{ where}$$

$$I\,(\alpha,\beta,\lambda,\mu) = \sqrt{\lambda}\,\left(\sqrt{\lambda}\,\cosh \alpha\sqrt{2\lambda} + \sqrt{\mu}\,\sinh \alpha\sqrt{2\lambda}\,\operatorname{cotanh} \beta\sqrt{2\mu}\right)^{-1}.$$

<u>Proof.</u> We set $g_1(x) = 0$, $g_2(x) = -\beta$, and let $v \to \infty$ in Corollary 2.3 to obtain

$$\exp - \sqrt{2\lambda}\int_0^{\alpha} \frac{\sqrt{\lambda}\,\tanh\sqrt{2\lambda}\,x + \sqrt{\zeta}}{\sqrt{\zeta}\,\tanh\sqrt{2\lambda}\,x + \sqrt{\lambda}}\,dx, \text{ where } \sqrt{\zeta} = \sqrt{\mu}\,\operatorname{cotanh}\,(\beta\sqrt{2\mu}).$$

The integral is evaluated by setting $c = 0$ and replacing $\sqrt{\mu}$ by $\sqrt{\zeta}$ in Example 2.1. This yields $I\,(\alpha,\beta,\lambda,\mu)$ for the contribution over the set where $B(t)$ reaches α before $-\beta$. The other term is obtained by the obvious symmetry.

As a last example, we extend the formula of H. M. Taylor (Example 2.2) to the joint sojourn times within two intervals of $M(t)$, for the process killed when $M(t) - B(t)$ reaches c. It is thought, with reference to the stock market application of [5], that this might have a bearing on the question of when to "sell early."

<u>Example 2.6.</u> In the notation of Example 2.2, for $0 < a < c$ we have

$$(2.4)\quad E \exp - \left(\lambda \int_0^{T_c} I_{(M(t) - a,\ M(t))}(B(t))dt + \mu \int_0^{T_c} I_{(M(t) - c,\ M(t) - a)}(B(t))dt + \nu M(T_c)\right)$$

$$= \sqrt{2\lambda\mu} \Big[\sqrt{2\lambda}\ (\sqrt{\lambda} \sinh a\sqrt{2\lambda} \sinh (c-a)\sqrt{2\mu}$$

$$+ \sqrt{\mu} \cosh a\sqrt{2\lambda} \cosh (c - a)\sqrt{2\mu})$$

$$+ \nu\ (\sqrt{\lambda} \cosh a\sqrt{2\lambda} \sinh (c - a)\sqrt{2\mu}$$

$$+ \sqrt{\mu} \sinh a\sqrt{2\lambda} \cosh (c - a)\sqrt{2\mu}) \Big]^{-1}.$$

Proof. We proceed as in Example 2.2 to obtain the inversion of the transform in ν, integrated from α to ∞. The same argument as there shows that this factors into

$$A \cdot E\Big[\exp - \left(\lambda \int_0^{M^{-1}(\alpha)} I_{(M(t) - a,\ M(t))}(B(t))dt + \mu \int_0^{M^{-1}(\alpha)} I_{(M(t) - c,\ M(t) - a)}(B(t))\right);$$

$$M(t) - B(t) \le c,\ 0 < t < T^{+}(\alpha)\Big],$$

where A is (2.4) with $\nu = 0$. Now the last factor is obtained from Corollary 2.3 by setting $g_1(x) = x - a$, $g_2(x) = x - c$, and letting $\nu \to \infty$. Since $\lim_{\nu\to\infty} \zeta(x) = \mu \operatorname{cotanh}^2 (c - a)\sqrt{2\mu}$, this leads to the expression

$$\exp - \alpha\sqrt{2\lambda}\left(\frac{\sqrt{\lambda} \tanh a\sqrt{2\lambda} + \sqrt{\mu} \operatorname{cotanh} (c - a)\sqrt{2\mu}}{\sqrt{\mu} \operatorname{cotanh} (c - a)\sqrt{2\mu} \tanh a\sqrt{2\lambda} + \sqrt{\lambda}}\right).$$

Since A does not involve α, we can introduce the transform variable ν of (2.4) by differentiating the above and then forming the transform in v. This leads to an expression with denominator given by the bracket on the right of (2.4), and numerator

(2.5) $\sqrt{2\lambda}\,(\sqrt{\lambda}\sinh a\sqrt{2\lambda}\,\sinh (c-a)\sqrt{2\mu}$
$+\sqrt{\mu}\cosh a\sqrt{2\lambda}\,\cosh (c-a)\sqrt{2\mu})$.

Turning to the factor A, by Lévy's equivalence $M - B \equiv |B|$ we have

$$A = E\exp -\left(\lambda\int_0^{T(-c,c)} I_{(0,a)}|B(t)|dt + \mu\int_0^{T(-c,c)} I_{(a,c)}|B(t)|dt\right),$$

with $T(-c,c)$ as in Example 2.5. Letting $A(x)$ denote the same expression with E^x in place of E $(A = A(0))$, we can easily compute $A(x)$ by the method of [2, 2.6]. We write $A(x) = 1 - F(x)$, where F is the continuously differentiable solution on $(0,c)$ of

$$\left(\tfrac{1}{2}\frac{d^2}{dx^2} - f_1\right)F = -f \;;\quad f(x) = \lambda I_{(0,a)}(x) + \mu I_{(a,c)}(x),$$

with $F(0) = 0$ and $F(c) = 0$. Then we set

$$F = \begin{cases} 1 + c_1\cosh a\sqrt{2\lambda}\,x & \text{for } 0 \le x < a \\ 1 + c_2\sinh (c-x)\sqrt{2\mu} + c_3\cosh (c-x)\sqrt{2\mu} & \text{for } a \le x \le c. \end{cases}$$

From $F(c) = 0$ we have $c_3 = -1$, whence using continuity of F' at $x = a$ to eliminate c_2 we obtain finally

$$c_1 = -\sqrt{\mu}\,(\sqrt{\lambda}\sinh a\sqrt{2\lambda}\,\sinh (c-a)\sqrt{2\mu} + \sqrt{\mu}\cosh a\sqrt{2\lambda}\,\cosh(b-a)\sqrt{2\mu}).$$

Evaluating $A(0)$ and multiplying by (2.5) now completes the proof.

Addendum

An interesting application of Example 1.2 was brought to my attention by Professor D.L. Burkholder. In Theorem 5.2 of [7], he showed the existence of constants $\beta_p > 0$, $0 < p < 1$, such that if

$$\tau = \inf\{t: B(t) < c_1 M(t) - c_2\},$$

$c_1 < 1$, $c_2 > 0$, then $E\,\tau^{p/2}$ is finite or infinite according as $c_1 > -\beta_p$ or $c_1 \leq -\beta_p$. Using Example 1.2 we will show that $\beta_p = p^{-1}-1$. First we let $\lambda \to \infty$ to obtain

$$\begin{aligned} &P\{B(t) \geq c_1 M(t) - c_2,\ 0 \leq t < M^{-1}(\alpha)\} \\ &= (1 + \alpha \frac{(1-c_1)}{c_2})^{-(1-c_1)^{-1}} \\ &= P\{M(\tau) \geq \alpha\}. \end{aligned}$$

Differentiating yields the density of $M(\tau)$ to be $c_2^{(1-c_1)^{-1}}(c_2 + (1-c_1)\alpha)^{-(2-c_1)(1-c_1)^{-1}}$. Now it is shown in [7] that there are constants c_p and C_p such that, for $0 < p < 1$, $c_p E\,\tau^{p/2} \leq EM^p(\tau) \leq C_p E\tau^{p/2}$. Since $EM^p(\tau) < \infty$ if and only if $p - (2-c_1)(1-c_1)^{-1} < -1$, or $c_1 > 1-p^{-1}$, the same applies to $E\,\tau^{p/2}$.

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