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Stopping times with given laws

by R. M. Dudley[1] and Sam Gutmann

Abstract. Given a stochastic process X_t, $t \in T \subset R$, and $s \in R$, then a) iff b): a) For every probability measure μ on $]s,\infty]$, there is a stopping time τ for X_t with law $L(\tau) = \mu$; b) If A_t is the smallest σ-algebra for which X_u are measurable for all $u \leq t$, then P restricted to A_t is nonatomic for all $t > s$.

This note began with a question of G. Shiryaev, connected with the following example. Let W_t be a standard Wiener process, $t \in T = [0,\infty]$. Any exponential distribution on $]0,\infty]$ will be shown to be the law of a stopping time. Using this, one can obtain a standard Poisson process P_t from W_t by a non-anticipating transformation, $P_t = g(\{X_s: s \leq t\})$.

Definitions. A probability space (Ω,A,P), or A (for P), is nonatomic iff for every $A \in A$ and $0 < p < P(A)$ there is a $B \subset A$, $B \in A$, with $P(B) = p$.

A stochastic process (here) is a map $X: (t,\omega) \longrightarrow X_t(\omega)$, $t \in T \subset R$, $\omega \in \Omega$, where (Ω,A,P) is a complete probability space. Each X_t has values in some measurable space $(S_t, \mathcal{F}_t)$ where S_t is a set, $\mathcal{F}_t$ is a σ-algebra of subsets of S_t, and X_t is measurable from $\mathcal{A}$ to $\mathcal{F}_t$. Let $\mathcal{A}_t$ be the smallest sub-σ-algebra of $\mathcal{A}$ for which X_s is measurable for all $s \leq t$ and for which $A \in \mathcal{A}_t$ whenever $A \subset B$ and $P(B) = 0$. Let $NA(X) := \inf\{t: \mathcal{A}_t$ is nonatomic$\}$.

Note. X_t is said to be nonatomic if $\mathcal{F}_t$ is nonatomic for $P \circ X_t^{-1}$. Then if X_t (or any other $\mathcal{A}_t$-measurable random variable) is nonatomic, $\mathcal{A}_t$ is nonatomic. After R. Dudley proved Theorem 2 below, and a weaker form of Theorem 1 considering only nonatomicity of individual X_t, S. Gutmann found the present Theorem 1.

A stopping time for the process X_t is a random variable τ on Ω with values in $]-\infty,\infty]$ such that for any $t \in T$, $\{\omega: \tau(\omega) < t\} \in \mathcal{A}_t$.

Theorem 1. For any stochastic process X_t and $s \in R$, $s \geq NA(X)$ iff for every Borel probability measure (law) μ on $]s,\infty]$, there is a stopping time τ for X_t with $\mathcal{L}(\tau) = \mu$. If $s \in T$ and $\mathcal{A}_s$ is nonatomic, the same holds for any μ on $[s,\infty]$.

Proof. If $\mathcal{A}_s$ is nonatomic, and μ is any law on $[s,\infty]$, then there is an $\mathcal{A}_s$-measurable random variable g with $\mathcal{L}(g) = \mu$, as follows. We take a nonatomic countably generated sub-σ-algebra

$\mathcal{B}$ of $\mathcal{A}_s$. Then there is a measure-preserving map ϕ of $(\Omega, \mathcal{B}, P)$ into $[0,1]$ with Lebesgue measure (Halmos, 1950, p. 173). Its range has outer measure 1. Let $F_\mu(t) := \mu(]-\infty, t])$, $F_\mu^{-1}(x) := \inf\{t: F_\mu(t) \geq x\}$. Then $g = F_\mu^{-1} \circ \phi$ is as desired.

Now $\{\omega: g(\omega) < t\}$ is empty for $t \leq s$, and belongs to $\mathcal{A}_s \subset \mathcal{A}_t$ for $t > s$. Thus, g is a stopping time, as desired. If for all $\epsilon > 0$ there is a stopping time τ with uniform distribution on $(s, s+\epsilon)$ then τ is $\mathcal{A}_{s+\epsilon}$-measurable, hence $\mathcal{A}_{s+\epsilon}$ is nonatomic and $s \geq NA(X)$.

Now suppose $\mathcal{A}_s$ has an atom, $t(n) \downarrow s$ with $\mathcal{A}_{t(n)}$ nonatomic, and μ is any law on $]s, \infty]$. Let $t(0) = +\infty$, $p_n := \mu(]t(n), t(n-1)])$, $n = 1, 2, \ldots$. By assumption, $\Sigma_{n \geq 1} p_n = 1$. Suppose there is a stopping time $\mathcal{T}$ with $P(\mathcal{T} = t(n)) = p_n$ for all n, and $\{\mathcal{T} = t(n)\} \in \mathcal{A}_{t(n)}$.

Whenever $p_n > 0$, the conditional law of P restricted to $\mathcal{A}_{t(n)}$, given $\mathcal{T} = t(n)$, is nonatomic. Thus for each n there is a real $\mathcal{A}_{t(n)}$-measurable random variable g_n such that

$$P(g_n \in A | \mathcal{T} = t(n)) = \mu(A \cap]t(n), t(n-1)])/p_n.$$

Let $\tau := g_n$ iff $\mathcal{T} = t(n)$. Then τ is measurable and $\mathcal{L}(\tau) = \mu$. If $t \in T$ and $t \leq s$, $\{\tau < t\}$ is empty. If $t > s$,

$$\{\tau < t\} = \left(\bigcup_n \{\mathcal{T} = t(n) < t(n-1) < t\}\right) \cup \{\mathcal{T} = t(n) < t \leq t(n-1) \text{ and } g_n < t\} \in \bigcup_{t(n) < t} \mathcal{A}_{t(n)} \subset \mathcal{A}_t.$$

Then τ is a stopping time with law μ. The problem is now reduced to the case $T = \{t(n)\}$ or equivalently where T is the set of negative integers and all A_t are nonatomic. This will be treated in the following Lemma and Theorem 2.

<u>Lemma</u>. Given a nonatomic probability space (Ω, A, P) and events A, B, D with $A \subset B$, $P(B) > 0$ and $P(D) > 0$, there is an event $C \subset D$ such that $P(C|D) = P(A|B)$ and $P(C \Delta A) \leq 2P(B \Delta D)$, where $C \Delta A := (C \setminus A) \cup (A \setminus C)$.

<u>Proof</u>. Let $p := P(D)P(A)/P(B)$, $E := A \cap D$. If $p \leq P(E)$, choose $C \subset E$ with $P(C) = p$. Then $P(C \Delta A) = P(A \setminus C)$ $= P(A) - p \leq P(B \setminus D)$ since

$$P(A)P(B) \leq P(A)P(D) + P(A)P(B \setminus D).$$
$$\leq P(A)P(D) + P(B)P(B \setminus D).$$

If $p > P(E)$, choose C with $E \subset C \subset D$ and $P(C) = p$. Then $P(A \Delta C) = P(A \setminus D) + p - P(E)$.

We need to prove

$P(A \setminus D)P(B) + P(A)P(D) \leq P(B)P(E) + 2P(B)P(B \Delta D)$. Now $P(A \setminus D) \leq P(B \setminus D)$, and $P(A)P(D) \leq P(A)P(B) + P(A)P(D \setminus B)$

$\leq P(B)P(E) + P(B)P(A \setminus D) + P(B)P(D \setminus B)$

$\leq P(B)P(E) + P(B)P(B \Delta D)$, as desired. In either case $C \subset D$ and $P(C|D) = P(A|B)$, Q.E.D.

<u>Note</u>. If $B = \Omega$ and $A = B \setminus D$, then $P(C \Delta A) = P(A) + P(D)P(A)$ $= 2P(A) - P(A)^2 \sim 2P(B \Delta D)$ as $P(A) \longrightarrow 0$. In this case, the constant 2 is best possible.

Theorem 2. Given a probability space $(\Omega, \mathcal{A}, P)$ and non-increasing sub-σ-algebras $\mathcal{A}_n$, $n = 1,2,\ldots$, $\mathcal{A} \supset \mathcal{A}_1 \supset \mathcal{A}_2 \supset \cdots$, such that P is nonatomic on each $\mathcal{A}_n$, and given any $p_n \geq 0$ with $\Sigma_{n\geq 1} p_n = 1$, there exist disjoint $A_n \in \mathcal{A}_n$ with $P(A_n) = p_n$.

Proof. Let $n(0) := 1$, choose $n(1)$ large enough so that $r_1 := \Sigma_{j<n(1)} p_j > 0$, and let $n(k) \uparrow +\infty$ fast enough so that $\Sigma_{n\geq n(k)} p_n \leq 4^{-k}$ for all $k \geq 2$. Let $r_k := \Sigma_{n(k-1)\leq n<n(k)} p_n$. If we can find disjoint $B_k \in \mathcal{A}_{n(k)}$ with $P(B_k) = r_k$ for all k, then we can choose A_n for $n(k-1) \leq n < n(k)$ as disjoint subsets of B_k with $P(A_n) = p_n$, $A_n \in \mathcal{A}_{n(k)} \subset \mathcal{A}_n$. Thus, we may assume $p_1 > 0$ and $\Sigma_{n\geq 1} 3^n p_n < \infty$.

Let $\pi_n := p_n / \Sigma_{1\leq j\leq n} p_j$. Take $A_{n1} \in \mathcal{A}_n$ with $P(A_{n1}) = \pi_n$ for each n. Given A_{nj} for all n and for $j < k$, let $B_{n1} := \Omega$ and for $k \geq 2$ let $B_{nk} := \Omega \setminus \bigcup_{1\leq j<k} A_{n+j,k-j}$. We choose A_{nk} for each n by the Lemma so that $A_{nk} \in \mathcal{A}_n$, $A_{nk} \subset B_{nk}$, $P(A_{nk} | B_{nk}) = \pi_n$ (or if $P(B_{nk}) = 0$, $A_{nk} = \phi$), and

$$P(A_{nk} \Delta A_{n,k-1}) \leq 2p_{nk} := 2P(B_{nk} \Delta B_{n,k-1}).$$ Then

$$(*) \qquad p_{nk} \leq \pi_{n+k-1} + \Sigma_{1\leq j<k-1} 2p_{n+j,k-j}.$$

Claim: $p_{nk} \leq 3^{k-2} \pi_{n+k-1}$ for all $k \geq 2$.

This will be proved by induction on k. For $k = 2$, (*) gives $p_{n2} \leq \pi_{n+1}$ as desired. For the induction step, (*) gives

$$p_{n,k+1} \le \pi_{n+k} + 2\Sigma_{1\le j<k} 3^{k-j-1}\pi_{n+k}$$

$$= \pi_{n+k}[1 + 2(1 + 3 + \cdots + 3^{k-2})]$$

$$= \pi_{n+k}[1 + 2(3^{k-1} - 1)/(3-1)] = 3^{k-1}\pi_{n+k},$$

proving the Claim.

Now $\Sigma 3^n \pi_n \le \Sigma 3^n p_n/p_1 < \infty$. So A_{nk} converges to some event A_n as $k \longrightarrow \infty$, specifically

$$P(A_n \Delta A_{nk}) \le \Sigma_{j>k} P(A_{nj} \Delta A_{n,j-1})$$

$$\le 2\Sigma_{j>k} 3^{j-2}\pi_{n+j-1} = 2\Sigma_{i\ge k} 3^{i-1}\pi_{n+i}.$$

Since A_{nk} is disjoint from $A_{n+j,k-j}$ for all $j < k$, we can let $k \longrightarrow \infty$ for fixed j to obtain $P(A_n \cap A_{n+j}) = 0$ for all $j \ge 1$. Thus, we may take all the A_n to be disjoint. Let $B_n := \Omega \setminus \bigcup_{m>n} A_m$. Then

$$P(B_n \Delta B_{nk}) \le (\Sigma_{1\le j<k} P(A_{n+j} \Delta A_{n+j,k-j})) + \Sigma_{j\ge k} P(A_{n+j})$$

$$\le 2\Sigma_{1\le j<k}\Sigma_{i\ge k-j} 3^{i-1}\pi_{n+j+i} + \Sigma_{j\ge k}\pi_{n+j}$$

$$\le \Sigma_{j\ge k}\pi_{n+j} + 2\Sigma_{r\ge k}\pi_{n+r}\Sigma_{1\le j<k} 3^{r-j-1}$$

$$\le \Sigma_{j\ge k}\pi_{n+j} + \Sigma_{r\ge k} 3^{r-1}\pi_{n+r} \longrightarrow 0 \text{ as } k \longrightarrow \infty.$$

Thus, $B_{nk} \longrightarrow B_n$. For each n, $P(A_n) \le \pi_n$. So, at least for n large enough, $P(B_n) > 0$ and

$$P(A_n | B_n) = \lim_{k\longrightarrow\infty} P(A_{nk} | B_{nk}) = \pi_n.$$

For such n, $P(A_n) = \pi_n(1 - \Sigma_{k>n}P(A_k))$. Then for $m \geq n$, $P(B_m|B_{m+1}) = 1 - \pi_{m+1}$ and

$$P(A_n|B_m) = \pi_n \Pi_{n<j\leq m}(1 - \pi_j) = p_n/(p_1+\cdots+p_m).$$

Thus

$$P(A_n) = p_n(1 - \Sigma_{k>m}P(A_k))/(p_1+\cdots+p_m).$$

Letting $m \longrightarrow \infty$ gives $P(A_n) = p_n$ for n large. Then, since $p_1 > 0$, $P(B_n) > 0$ for all n and the above holds for all n (by induction downward). Thus, Theorem 2 is proved.

Letting $A_n = \mathcal{A}_{t(n)}$ and $A_n = \{\mathcal{J} = t(n)\}$ Theorem 1 is also proved.

Example. It may happen that for every law μ on the closed interval $[0,\infty]$, there is a stopping time with law μ, even though $\mathcal{A}_0$ is trivial. Let $T = [0,1]$ and $X_t(\omega) := \omega t$ where ω is uniformly distributed on $[0,1]$. Let $\omega \longrightarrow g(\omega)$ have law μ. The identity $\omega \longrightarrow \omega$ is measurable from $(\Omega, \bigcap_{t>0}\mathcal{A}_t)$ into R, so g is a stopping time.

Proposition. There is a stopping time τ with any law μ on $[s,\infty]$ iff both a) $s \geq NA(X)$ and b) for any $p \in (0,1)$ there is an event $A \in \bigcap_{t>s}\mathcal{A}_t$ with $P(A) = p$.

Proof. By Theorem 1, a) is necessary. To show b) necessary, pick a law μ with $p = \mu\{s\}$ and let $A = \{\tau = s\}$. Conversely, given a law μ with $\mu\{s\} = p < 1$, choose A as in b) and apply Theorem 1 to $\mu'(\cdot) = \mu(\cdot|(s,\infty])$ and $P'(\cdot) = P(\cdot|A^c)$. This proves the proposition.

If C is a σ-algebra generated by atoms of size 2^{-n}, $n = 1,2,\ldots,$ then C contains A with $P(A) = p$ for each $p \in (0,1)$, although C is purely atomic.

REFERENCE

Halmos, P. (1950), Measure Theory (Princeton, Van Nostrand).

Footnote

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